Geometric Extremization for AdS/CFT and Black Hole Entropy

Jerome Gauntlett

Couzens, JPG, Martelli, Sparks

JPG, Martelli, Sparks x 3

(Ferrero, JPG, Ipina, Martelli, Sparks x 2)

SCFTs with abelian R-symmetry

$$\{Q, Q\} \sim P \qquad\qquad \{S, S\} \sim K$$

$$\{Q, S\} \sim M + D + R$$

The R-symmetry encodes important exact results for physical observables. E.g.

- For chiral primary operators $\Delta(\mathcal{O}) = nR(\mathcal{O})$
- Central charges/free energies can be obtained from R

The R-symmetry can be obtained by variational techniques…

$\mathcal{N} = 1, d = 4$ a-maximization [Intriligator,Wecht 03]

$$a(R_T) = \frac{9}{32} Tr R_T^3 - \frac{3}{32} Tr R_T \quad \text{and} \quad a = a(R_*)$$

$\mathcal{N} = 2, d = 3$ F-extremization [Jafferis 10]

$F_{S^3}(R_T)$ and $F_{S^3} = F_{S^3}(R_*)$

$\mathcal{N} = (0,2), d = 2$ c-extremization [Benini,Bobev 12]

$$c_R(R_T) = 3 Tr \gamma_3 R_T^2 \qquad c_R = c_R(R_*)$$

$\mathcal{N} = 2, d = 1$ Is there a general extremization principle for susy QM??

$\mathscr{I}$ -extremization conjecture: [Benini,Hristov,Zaffaroni 15]

Holographic dual for these extremization principles?

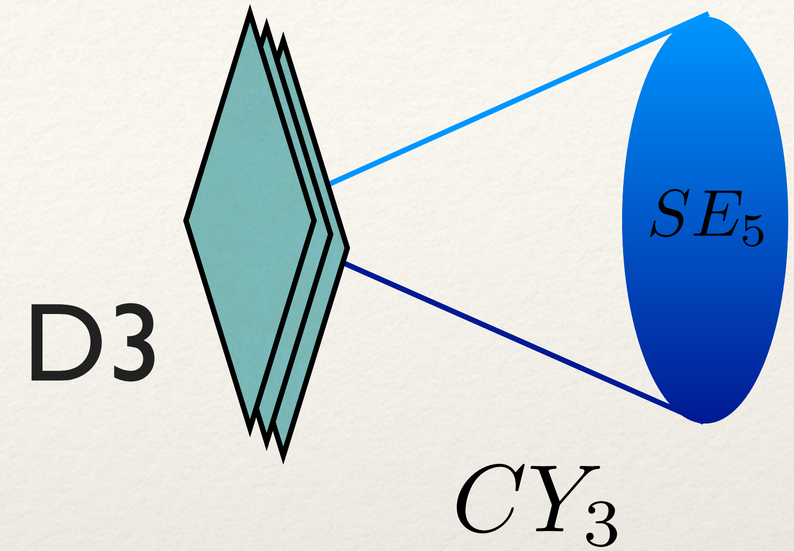
Well established for Sasaki-Einstein solutions:

Type IIB

$$ds^2_{10} = L^2[ds^2(\mathrm{AdS}_5) + ds^2(SE_5)]$$

$$F_5 = -L^4\left[vol_{\mathrm{AdS}_5} + vol_{SE_5}\right]$$

Dual to N=1SCFT in d=4



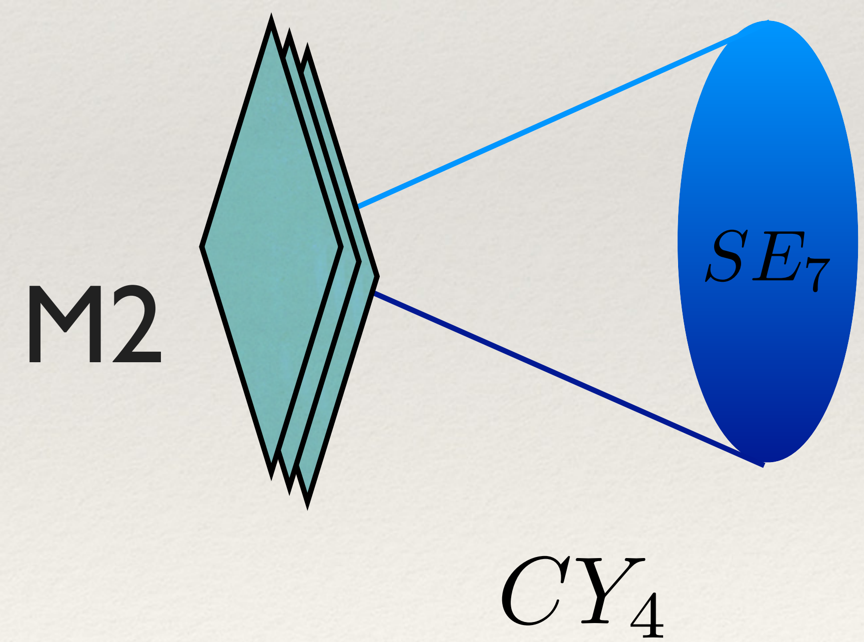
$dr^2 + r^2 ds^2(SE_5)$

D=11

$$ds^2_{11} = L^2[ds^2(AdS_4) + ds^2(SE7)]$$

$$G = L^3 vol_{\mathrm{AdS}_4}$$

Dual to N=2 SCFT in d=3



Fact: SE have canonical Killing vector ξ

$\mathcal{N}=1, d=4$ SCFT dual to $AdS_5 \times SE_5$: $a \propto \frac{1}{Vol(SE_5)}$

$\mathcal{N}=2, d=3$ SCFT dual to $AdS_4 \times SE_7$: $F_{S^3} \propto \frac{1}{\sqrt{Vol(SE_7)}}$

R can be obtained using volume minimization: [Martelli,Sparks,Yau 05]

Go off-shell:

Consider Sasaki metrics - cone is Kahler with (n+1,0) form $\Psi \neq 0$

Choose holomorphic ξ with $\mathcal{L}_\xi \Psi = \frac{i}{c}\Psi$

Extremize $Vol(Sas)(\xi)$

- Can study dual CFT without knowing the Sasaki-Einstein metric

Very powerful for identifying dual SCFTs

- Geometry: results apply to arbitrary SE_{2n+1}

Recent progress:

c-extremization for AdS_3 solutions dual to $\mathcal{N} = (0,2), d = 2$

Identify infinite classes of AdS/CFT examples

New principle for AdS_2 solutions dual to $\mathcal{N} = 2, d = 1$

Includes a dual of $\mathscr{I}$- extremization as a special case and hence microstates of infinite classes of AdS4 black holes

Type IIB

$$ds^2_{10} = L^2 e^{-B/2}[ds^2(\mathrm{AdS}_3) + ds^2(Y_7)]$$

$$F_5 = -L^4\,[vol_{\mathrm{AdS}_3} \wedge F + *_7 F]$$

[Kim 05]

Dual d=2 SCFT has (0,2) supersymmetry

D=11

$$ds^2_{11} = L^2 e^{-2B/3}\left[ds^2(\mathrm{AdS}_2) + ds^2(Y_9)\right]$$

$$G_4 = L^3 vol_{\mathrm{AdS}_2} \wedge F$$

[Kim,Park 06]

Dual SCQM has 2 supersymmetries with R-symmetry

Also can arise as near horizon limits of magnetically charged supersymmetric black holes in $AdS_4 \times SE_7$

[Gauntlett,Kim 07]

Both special cases of GK geometry $(Y_{2n+1}, g_{\mu\nu}, B, F)$

Infinite classes of explicit $AdS_3 \times Y_7$ and $AdS_2 \times Y_9$ solutions have been known for a while

[Gauntlett,MacConamhna,Mateos,Waldram 06]
[Gauntlett,Kim,Waldram 06]
[Donos,Gauntlett,Kim 08]

(roughly analogous to Sasaki-Einstein $Y^{p,q}$)

Until recently dual field theories essentially unknown…

Plan

- Introduce GK geometry
- Go off-shell and derive new geometric extremization principles
- Utilise toric geometry to further analyse special classes
- SCFTs on spindles

GK Geometry $(Y_{2n+1}, g_{\mu\nu}, B, F)$

[Gauntlett,Kim 07]

Action: $F = dA$

$$S = \int_{Y_{2n+1}} e^{(1-n)B} \left[R_{2n+1} - \frac{2n}{(n-2)^2} + \frac{n(2n-3)}{2} (dB)^2 + \frac{1}{4} e^{2B} F^2 \right] vol_{2n+1}$$

- Equations of motion: $\delta S = 0$
- Supersymmetry - existence of certain Killing spinors
- Flux quantisation - on codimension two cycles

GK Geometry (Y_{2n+1}, B, F) $n \geq 3$

Supersymmetry implies:

- Killing vector ξ (R-symmetry) $||\xi||^2 = 1$

- Define one-form η dual to Killing vector: $\xi^a \eta_a = 1$

- Metric:

$$ds^2_{2n+1} = \eta^2 + e^B ds^2_{2n}$$

⟵ Kahler J, ρ

$\rho_{ij} = J_i{}^k R_{kj}$

$$d\eta = c\rho \qquad e^B = \frac{c^2}{2} R > 0 \qquad F = -\frac{1}{c} J + d\left(e^{-B}\eta\right)$$

$c = \frac{1}{2}(n-2)$

- Supersymmetric solution if

$$\delta S = 0 \quad \Leftrightarrow \quad \Box R = \frac{1}{2} R^2 - R_{ij} R^{ij}$$

GK Geometry (Y_{2n+1}, B, F) $n \geq 3$

Off-Shell GK Geometry

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$$\delta S = 0 \quad \Leftrightarrow \quad \Box R - \frac{1}{2} R^2 - R_{ij} R^{ij}$$

Off-shell GK Geometry

- Consider cone metric on $C(Y_{2n+1}) \equiv \mathbb{R}_{>0} \times Y_{2n+1}$

$$ds^2_{2n+2} = dr^2 + r^2 ds^2(Y_{2n+1})$$

- Cone has an integrable complex structure

- R symmetry vector ξ is holomorphic

- No-where vanishing (n+1,0) form Ψ with

$$d\Psi = 0 \qquad \mathcal{L}_\xi \Psi = \frac{i}{c}\Psi \qquad c = \tfrac{1}{2}(n-2)$$

Geometric extremal problem

[Couzens,Gauntlett,Martelli,Sparks 18]

- Complex cone $C(Y_{2n+1})$ with (n+1,0) form Ψ
- Choose holomorphic $\xi \neq 0$ and $\mathcal{L}_\xi \Psi = \frac{i}{c}\Psi$
- Consider an off shell GK geometry on Y_{2n+1}

- Impose constraint: $$\int_{Y_{2n+1}} \eta \wedge \rho^2 \wedge \frac{J^{n-2}}{(n-2)!} = 0$$
- Impose flux quantization on codimension 2 cycles: $$\int_{\Sigma_A} \eta \wedge \rho \wedge \frac{J^{n-2}}{(n-2)!} = N_A$$
- Extremise action: $$S(\xi) = \int_{Y_{2n+1}} \eta \wedge \rho \wedge \frac{J^{n-1}}{(n-1)!}$$

For $AdS_3 \times Y_7$

$$S(\xi_*) = \frac{3L}{2G_3} = c_{SCFT}$$

For $AdS_2 \times Y_9$ define "entropy function"

$$S(\xi_*) = \frac{1}{4G_2}$$

Generically expect $S(\xi_*) = \ln Z,$

For black hole horizons $S(\xi_*) = S_{BH}$

Special Cases and Toric Geometry

Type IIB $AdS_3 \times Y_7$ with $Y_5 \hookrightarrow Y_7 \to \Sigma_g$

Physical picture:

- Start with $AdS_5 \times Y_5$ and SE metric on Y_5

  Dual to d=4 N=1 SCFT

  Isometries of Y_5 give rise to global (mesonic) symmetries

  3-cycles of Y_5 give rise to global (baryonic) symmetries

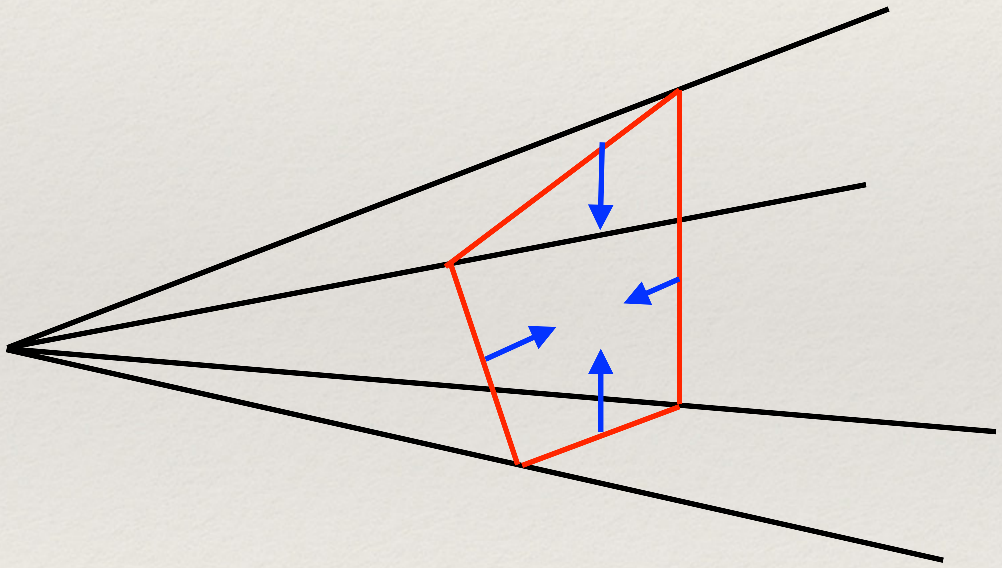
- Compactify d=4 SCFT on Σ_g and add magnetic fluxes n_i , M_a for the global symmetries, including topological twist for susy

- IF we flow to d=2 SCFT in IR then expect it is dual to $AdS_3 \times Y_7$ with Y_7 fibred as above …. can we match c?

Focus on $AdS_5 \times Y_5$ with the complex cone $C(Y_5)$ admitting a toric Kahler cone metric: [Martelli,Sparks,Yau 05]

- Three holomorphic Killing vectors ∂_{φ_i} generate $U(1)^3$

- There is an associated polyhedral cone with d facets specified by inward pointing normal vectors $\vec{v}_a \in \mathbb{Z}^3$



$\vec{v}_a$ specifies which $U(1)$ collapses along that facet

- The extremization problem for the $AdS_3 \times Y_7$ solutions with $Y_5 \hookrightarrow Y_7 \to \Sigma_g$ becomes algebraic in the $\vec{v}_a$!

[Gauntlett,Martelli,Sparks 18]

Master volume for Y_5 fibre:

$$\mathcal{V}(\vec{b};\{\lambda_a\}) = \frac{(2\pi)^3}{2}\sum_{a=1}^{d}\lambda_a \frac{\lambda_{a-1}(\vec{v}_a,\vec{v}_{a+1},\vec{b}) - \lambda_a(\vec{v}_{a-1},\vec{v}_{a+1},\vec{b}) + \lambda_{a+1}(\vec{v}_{a-1},\vec{v}_a,\vec{b})}{(\vec{v}_{a-1},\vec{v}_a,\vec{b})(\vec{v}_a,\vec{v}_{a+1},\vec{b})}$$

Extremization problem

$A \sim$ Kahler class for Σ_g

$\lambda_a \sim$ Kahler class for Y_5

$$S_{\text{SUSY}}(\vec{b};\{\lambda_a\};A) = -A\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a} - 4\pi\sum_{i=1}^{3} n_i \frac{\partial\mathcal{V}}{\partial b_i}$$

$$0 = A\sum_{a,b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b} - 2\pi n_1\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a} + 4\pi\sum_{a=1}^{d}\sum_{i=1}^{3} n_i\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i}$$

$$N = -\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a}$$

Flux on Y_5

$$M_a = \frac{1}{2\pi}A\sum_{b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b} + 2\sum_{i=1}^{3} n_i\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i}$$

Flux on $\Sigma_g \times (\Sigma_a \subset Y_5)$

Results

- For arbitrary toric Y_5 and $Y_5 \hookrightarrow Y_7 \to \Sigma_g$

  Can calculate c_{sugra} for the $AdS_3 \times Y_7$ solutions as a function of the geometric twists and fluxes using geometric extremization

$$S = S(\vec{b}, g, v_a, n_i, N, M_a)$$

- Can compare with known dual quiver gauge theories using field theory c-extremization procedure

Find exact agreement (even off-shell)!

[JPG,Martelli,Sparks 19]

[Hosseini,Zaffaroni 19]

This provides an identification of an infinite classes of d=4 quiver field theories compactified on Σ_g with these $AdS_3 \times Y_7$ solutions!

Open issues: provided that they both exist…

- Geometry: there can be obstructions to the existence of Y_7 eg examples with $c < 0$

- Field theory: the field theory may not flow in the IR to a SCFT of the type we are considering

D=11 version

- Analogous story for $AdS_2 \times Y_9$ solutions with
  with $Y_7 \hookrightarrow Y_9 \rightarrow \Sigma_g$ and Y_7 toric [JPG,Martelli,Sparks 19]
  [Hosseini,Zaffaroni 19]

- Using toric data can calculate an off shell entropy function
  as a function of geometric twists and fluxes

- This can be identified with the entropy of a magnetically charged black hole in $AdS_4 \times Y_7$ (provided that they exist)

- Field theory: off-shell calculation of topological index $\mathscr{I}$ for certain quiver gauge theories compactified on Σ_g calculated in
  [Hosseini,Zaffaroni 16

Find exact agreement, even off-shell!

[JPG,Martelli,Sparks 19][Hosseini,Zaffaroni 1
[Kim,Kim19]

Gives microscopic state count for the entropy for asymptotically AdS black holes

Another special class

[Fererro,JPG,Ipina,Martelli,Sparks 20]

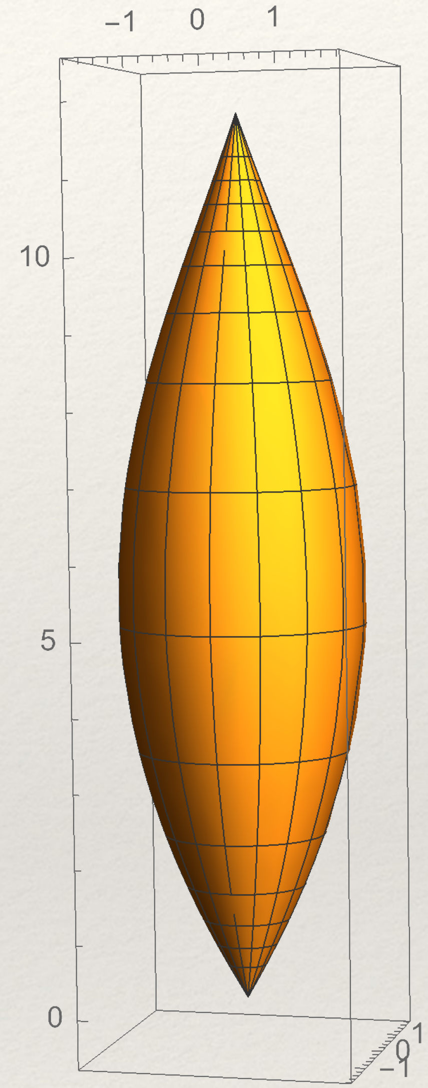
- Consider $AdS_3 \times Y_7$ and $AdS_2 \times Y_9$ with

$$S^3/\mathbb{Z}_q \to Y_7 \to KE_4 \qquad S^3/\mathbb{Z}_q \to Y_9 \to KE_6$$

explicit solutions known

- Remarkably geometry can be recast as

$$SE_5 \to Y_7 \to \mathbb{WCP}_{n_-,n_+} \qquad SE_7 \to Y_9 \to \mathbb{WCP}_{n_-,n_+}$$



- Describe D=5,4 SCFTs compactified on a spindle!

- Latter case we have full D=4 black hole solution: an accelerating charged, rotating PD metric!

- Supersymmetry without usual topological twist!

- Isometry of the spindle mixes with R-symmetry!

c.f. [Hosseini,Hristov, Tachikawa,Zaffaroni 20]

Summary and outlook

- Geometric dual of c-extremization for type IIB $AdS_3 \times Y_7$
- Geometric extremization for SCQM dual to D=11 $AdS_2 \times Y_9$
    - What is the field theory story? does it exist for finite N?
    - When arise as black hole horizons, entropy via extremization
    - Add rotation? [Couzens,Marcus,Stemerdink,van De Heisteeg, 20]
- Interesting sub-class of examples

$$Y_5 \hookrightarrow Y_7 \rightarrow \Sigma_g$$
$$Y_7 \hookrightarrow Y_9 \rightarrow \Sigma_g$$

    - Toric case: striking agreement with field theory and new microstate counting of entropy of mag. charged AdS4 BHs
    - Obstructions? Geometry/field theory
    - Novel features arise in toric geometry - develop
    - Many new black hole and black string solutions must exist
- SCFTs on spindles, orbifolds,….