Multi-instanton Contributions and Complexified Bion Solutions in Quantum Mechanics

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JHEP 09 (2015) 157 [arXiv:1507.00408] ; arXiv:1607.04205 ; PTEP 2015 033B02 (2015)

[arXiv:1409.3444] ; JHEP 06 (2014) 164 [arXiv:1404.7225]

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1 Multi-instanton Contributions in sine-Gordon QM

1.1 Multi-Instanton Contributions

Borel sum of **Perturbation** series $\longrightarrow$ Imaginary **ambiguities**

Instanton (**Bion**) contributions $\longrightarrow$ Imaginary **ambiguities**

Cancel each other : nonperturbative physics encoded in perturbation series

Resurgence to higher orders of nonperturbative exponentials requires

Systematic computations of multi-instanton (bions) contributions

Instanton and antiinstanton attract (non-BPS) : moduli integral requires

BZJ prescription

(i) $\boldsymbol{g^2 < 0}$ $\longrightarrow$ Analytic continuation to $\boldsymbol{g^2 > 0}$

(ii) $\boldsymbol{\epsilon}$ species of fictitious fermions and subtract $\boldsymbol{1/\epsilon}$ poles

E.B.Bogomolny, Phys.Lett.**B91** 431 (1980); J.Zinn-Justin, Nucl.Phys.**B192** 125 (1981); $\cdots$

Verified only at the **leading order** of nonperturbative factor (single Bion)

Universal WKB results in quantum mechanics:

provides a good test for multiple instanton calculations

G.V.Dunne and M.Ünsal, Phys.Rev.D **89**, 105009 (2014); $\cdots$

Our purpose

1. To compute multi-instanton amplitudes in the sine-Gordon quantum mechanics
2. To propose an appropriate subtraction scheme for multi-instanton amplitudes.
3. To compare with the result of the uniform WKB method.

1.2 Path-integral representation of Energy

Schrödinger eq. for Sine-Gordon QM (harmonic oscillator when $g^2 = 0$)

$$H\psi(x) = \left[-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{8g^2}\sin^2(2gx)\right]\psi(x) = E\,\psi(x)$$

Euclidian Lagrangian

$$L = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x), \qquad V(x) = \frac{1}{8g^2}\sin^2(2gx)$$

Bloch angle $\theta \in [0, \pi]$ to label energy eigenstate in the band

$$\psi\left(x + \pi/(2g)\right) = e^{i\theta}\psi(x)$$

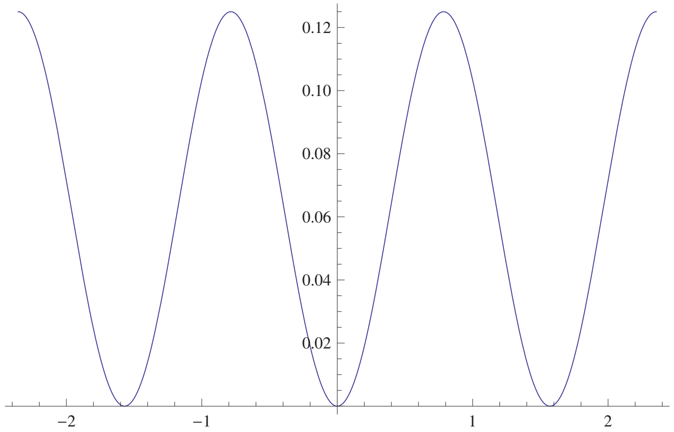


Figure 1: Sin-Gordon potential with $g = 1$. Periodicity is $\pi/(2g)$.

Path-integral representation of the lowest band energy eigenvalue

$$E = \lim_{\beta\to\infty} \frac{-1}{\beta} \log Z, \qquad Z = \mathrm{Tr} e^{-\beta H}$$

$$Z = \int_{x(t=-\beta/2)=x(t=\beta/2)} Dx(t)\, e^{-S+iQ\theta}$$

$$Q = \frac{1}{\pi} \int_{-\infty}^{\infty} dt \frac{d(2gx)}{dt}$$

Perturbative E_{pert} and nonperturbative ΔE contributions

$$E = E_{\mathrm{pert}}(g^2) + \triangle E$$

Perturbation series is divergent

$$E_{\text{pert}}(g^2) = \sum_{K=0}^{\infty} a_K (g^2)^K, \qquad a_K \sim K!$$

$$\mathbb{E}_{\text{pert}}(g^2) = \int_0^{\infty} dt e^{-t} B_{\text{pert}}(g^2 t), \quad B_{\text{pert}}(t) = \sum_{K=0}^{\infty} \frac{a_K}{K!} t^K$$

Nonalternating divergent series: Borel resummation gives imaginary part

$\mathbb{E}_{\text{pert}}(g^2)$ is a real analytic function of g^2 $\longrightarrow$ **dispersion relation**

Dispersion rel. in complex g^2-plane gives the coefficient a_K as

$$a_K \approx \frac{-1}{\pi} \int_0^{\infty} dg^2 \, \frac{[\mathrm{Im}\mathbb{E}_{\text{pert}}(g^2)]}{(g^2)^{K+1}}$$

This imaginary part should be unphysical and cacelled by

the nonperturbative contribution ΔE from instantons (**Resurgence**)

1.3 Instantons

BPS equation

$$\frac{dx}{dt} = \frac{1}{2g} \sin(2gx),$$

anti-BPS equation

$$\frac{dx}{dt} = -\frac{1}{2g}\sin(2gx),$$

Single instanton solution with $Q = 1$

$$x_{\mathcal{I}}(t) = (2\arctan e^{t-t_0} + n\pi)/(2g), \quad n \in \mathbb{Z}$$

is BPS for even n, anti-BPS for odd n: No other (anti-)BPS solutions

t_0 is the translational **zero mode** (**moduli**)

Integrating the translational zero mode t_0, one finds the contribution $\triangle E^{(1,0)}$ of single instanton $[\mathcal{I}]$ to the energy (with one-loop determinant)

$$\triangle E^{(1,0)} = -[\mathcal{I}] = -\left(\frac{e^{-S_I}}{\sqrt{\pi g^2}}\right) e^{i\theta}.$$

Any combination of adjacent (anti-)instantons are always **non-BPS**

Interaction energy of instanton and (anti-)instanton at separation R

$$V_{\mathcal{I}\mathcal{I}}(R) = \frac{2}{g^2}\exp[-R], \qquad V_{\mathcal{I}\bar{\mathcal{I}}}(R) = -\frac{2}{g^2}\exp[-R]$$

2 Multi-instanton contributions

2.1 General setting with BZJ prescription

Methods to obtain mutli-instanton contributions

1. **Integration over separation $\boldsymbol{R}$** between instantons
2. Attractive interactions require **analytic continuations** from $\boldsymbol{g^2 < 0}$
3. Divergence at large $\boldsymbol{R}$: Regularize by $\boldsymbol{\epsilon}$ number of **fictitious fermions** (fermion zero mode exchange gives attraction at large $\boldsymbol{R}$)
4. **Subtraction of divergence** as $\boldsymbol{\epsilon \longrightarrow 0}$

Interaction potential between instanton and (anti-)instanton

$$\boldsymbol{V(R) = \pm\frac{2}{g^2}\exp(-R) + \epsilon R}$$

Our proposal for subtraction of multiple moduli integral

1. Enumerate all possible distinct **configurations**
2. Enumerate possible **ordering** of moduli integrations for each config.

3. **Subtract possible poles** like $\mathbf{1/\epsilon}$ for the first integration, and then perform the next integration successively, and retain the finite piece.

4. **Average** the results of all orderings and **sum** over distinct configurations

Physics behind the $\mathbf{1/\epsilon}$ subtraction:

Logarithm $\longrightarrow$ only connected configurations

(dilute gas multi-particle contributions are subtracted)

$$\boldsymbol{Z = Z_0 + Z_1 + Z_2 + \cdots}$$

$$\boldsymbol{E = \lim_{\beta\to\infty} \frac{-1}{\beta} \log Z = \lim_{\beta\to\infty} \frac{-1}{\beta} \left[\log Z_0 + \frac{Z_1}{Z_0} + \left(\frac{Z_2}{Z_0} - \frac{Z_1^2}{2Z_0^2} \right) + \cdots \right]}$$

2.2 2 instantons

There is only one configuration and one ordering: $[\boldsymbol{\mathcal{I}\mathcal{I}}]$

Two-instanton amplitude in unit of instanton factor $\boldsymbol{\xi = e^{-S_I}/\sqrt{\pi g^2}}$ is

$$\boldsymbol{[\mathcal{I}\mathcal{I}] e^{-2i\theta} \xi^{-2} \; = \int_0^\infty dR \exp\left(-\frac{2}{g^2} e^{-R} - \epsilon R \right)}$$

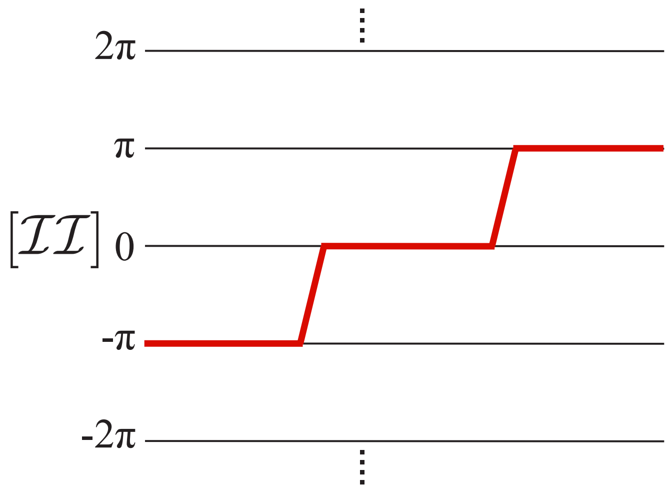


Figure 2: Two instanton configurations $[\mathcal{I}\mathcal{I}]$. Horizontal lines stand for the vacuum.

$$\xrightarrow{|g^2|\ll 1} \left(\frac{g^2}{2}\right)^{\epsilon}\Gamma(\epsilon) = -\left(\gamma + \log\frac{2}{g^2}\right) + O\left(\frac{1}{\epsilon}\right) + O(\epsilon)$$

$$\triangle E^{(2,0)} = e^{2i\theta}\xi^2\left(\gamma + \log\frac{2}{g^2}\right)$$

2.3 1 instanton + 1 anti-instanton

Attraction : compute at $g^2 < 0$ then analytic continuation to $g^2 > 0$

There are two configurations with one ordering for each: $[\mathcal{I}\bar{\mathcal{I}}]$ and $[\bar{\mathcal{I}}\mathcal{I}]$

$$[\mathcal{I}\bar{\mathcal{I}}]\xi^{-2} = \int_0^\infty dR\, \exp\left(-\frac{2}{-g^2}e^{-R} - \epsilon R\right)$$

$$\xrightarrow{|g^2|\ll 1} \left(\frac{-g^2}{2}\right)^{\epsilon} \Gamma(\epsilon) \xrightarrow{-g^2=e^{\mp i\pi}g^2} -\left(\gamma + \log\frac{2}{g^2}\right) \mp i\pi + O\left(\frac{1}{\epsilon}\right) + O(\epsilon)$$

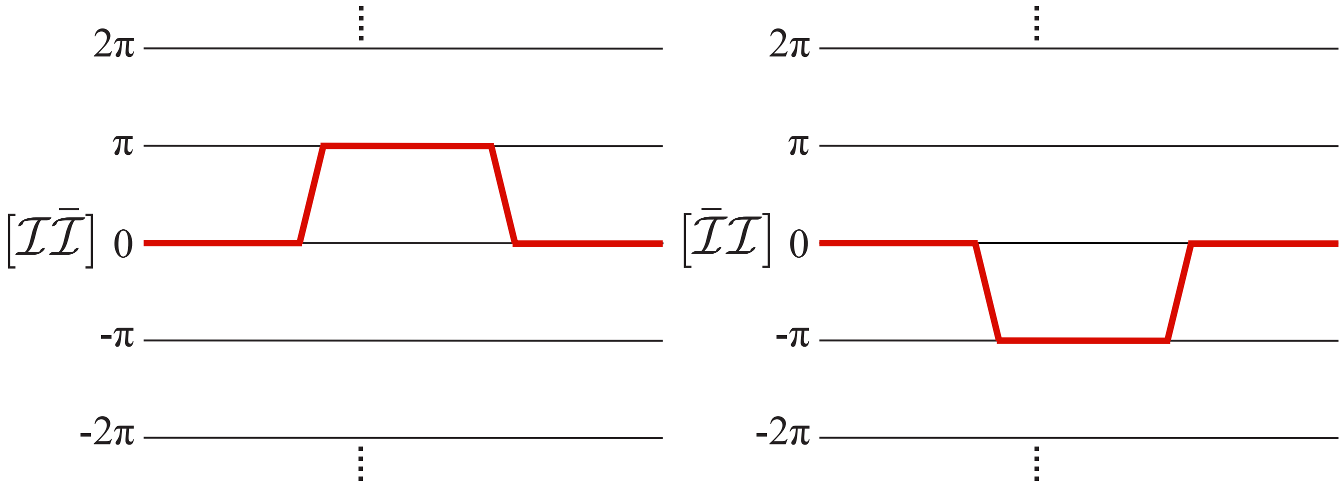


Figure 3: One-instanton and one anti-instanton amplitude ($[\mathcal{I}\bar{\mathcal{I}}]$, $[\bar{\mathcal{I}}\mathcal{I}]$).

Same contribution from $[\bar{\mathcal{I}}\mathcal{I}]$

By summing over $[\mathcal{I}\bar{\mathcal{I}}], [\bar{\mathcal{I}}\mathcal{I}]$, nonperturbative correction is

$$\triangle E^{(1,1)} = ([\mathcal{I}\bar{\mathcal{I}}] + [\bar{\mathcal{I}}\mathcal{I}]) = \xi^2 \left[2\left(\gamma + \log\frac{2}{g^2}\right) \pm 2i\pi\right]$$

Imaginary part should be cancelled by perturbative contribution

$$\mathrm{Im}[\triangle E^{(1,1)}] + \mathrm{Im}[E_{\mathrm{pert}}] = 0$$

$$a_k = \frac{-1}{\pi}\int_0^\infty dg^2 \, \frac{\mathrm{Im}[E_{\mathrm{pert}}(g^2)]}{(g^2)^{k+1}} = -\frac{1}{\pi}\int_0^\infty d(g^2)\, \frac{2e^{-1/g^2}}{(g^2)^{k+2}} = -\frac{2}{\pi}\, k!$$

2.4 2 instantons + 1 anti-instanton

3 configurations $[\mathcal{I}\bar{\mathcal{I}}\mathcal{I}], [\mathcal{I}\mathcal{I}\bar{\mathcal{I}}], [\bar{\mathcal{I}}\mathcal{I}\mathcal{I}]$

First configuration $[\mathcal{I}\bar{\mathcal{I}}\mathcal{I}]$: unique ordering

$$[\mathcal{I}\bar{\mathcal{I}}\mathcal{I}]e^{-i\theta}\xi^{-3} = \int_0^\infty dR_1 dR_2 \exp\left[-\frac{2}{-g^2}(e^{-R_1} + e^{-R_2}) - \epsilon(R_1 + R_2)\right]$$

$$= \frac{3}{2}\left(\gamma + \log\frac{2}{g^2}\right)^2 - \frac{17\pi^2}{12} \pm 3i\pi\left(\gamma + \log\frac{2}{g^2}\right) + O\left(\frac{1}{\epsilon}\right) + O(\epsilon)$$

Second configuration $[\mathcal{I}\mathcal{I}\bar{\mathcal{I}}]$: two orderings

$$[\mathcal{I}\mathcal{I}\bar{\mathcal{I}}]e^{-i\theta}\xi^{-3} = \int_0^\infty dR_1 dR_2 \, \exp\left[-\frac{2}{g^2}e^{-R_1} - \frac{2}{-\tilde{g}^2}e^{-R_2} - \epsilon(R_1 + R_2)\right]$$

1. R_2 then R_1 integral gives

$$\frac{3}{2}\left(\gamma + \log\frac{2}{g^2}\right)^2 - \frac{5\pi^2}{12} \pm 2i\pi\left(\gamma + \log\frac{2}{g^2}\right) + O\left(\frac{1}{\epsilon}\right) + O(\epsilon)$$

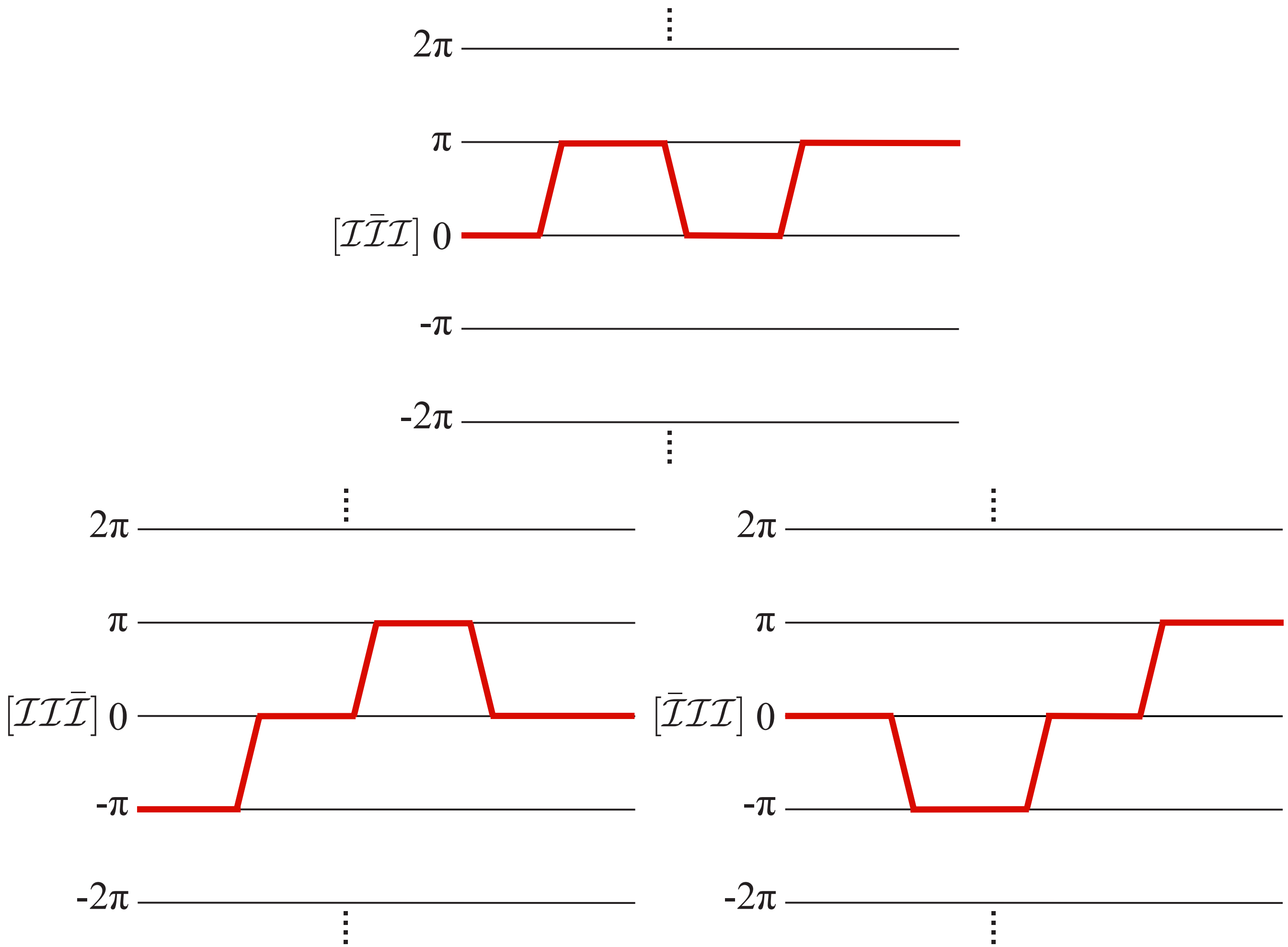


Figure 4: Two-instanton and one–anti-instanton amplitudes ($[\mathcal{I}\bar{\mathcal{I}}\mathcal{I}], [\mathcal{I}\mathcal{I}\bar{\mathcal{I}}], [\bar{\mathcal{I}}\mathcal{I}\mathcal{I}]$).

2. $\boldsymbol{R_1}$ then $\boldsymbol{R_2}$ integral gives

$$\frac{3}{2}\left(\gamma+\log\frac{2}{g^2}\right)^2+\frac{\pi^2}{12}\pm i\pi\left(\gamma+\log\frac{2}{g^2}\right)+O\left(\frac{1}{\epsilon}\right)+O(\epsilon)$$

Averaging the two orderings gives

$$[\mathcal{I}\mathcal{I}\bar{\mathcal{I}}]e^{-i\theta}\xi^{-3}=\frac{3}{2}\left(\gamma+\log\frac{2}{g^2}\right)^2-\frac{\pi^2}{6}\pm\frac{3}{2}i\pi\left(\gamma+\log\frac{2}{g^2}\right)$$

Third configuration gives the same result as second $[\bar{\mathcal{I}}\mathcal{I}\mathcal{I}]=[\mathcal{I}\mathcal{I}\bar{\mathcal{I}}]$,

By summing over $[\bar{\mathcal{I}}\mathcal{I}\mathcal{I}],[\mathcal{I}\mathcal{I}\bar{\mathcal{I}}],[\mathcal{I}\mathcal{I}\bar{\mathcal{I}}]$, nonperturbative correction is

$$\triangle E^{(2,1)}=-\frac{9}{2}e^{i\theta}\xi^3\left[\left(\gamma+\log\frac{2}{g^2}\right)^2-\frac{7\pi^2}{18}\pm\frac{4}{3}i\pi\left(\gamma+\log\frac{2}{g^2}\right)\right]$$

Imaginary part should be cancelled by imaginary part of perturbative contribution on **one-instanton background** (**Large order** behavior)

$$a_k\approx\frac{1}{\pi}\int_0^\infty dg^2\,\frac{\mathrm{Im}[\triangle E^{(2,1)}e^{-i\theta}/\xi]}{(g^2)^{k+1}}=\frac{6}{\pi}k!\left(\log 2+\frac{s(k+1,2)}{k!}\right)$$

Stirling number of first kind : $s(k+1,2)=(k+1)s(k,2)+k!$

J.Zinn-Justin, Nucl.Phys.**B192** 125 (1981)

Agreement with perturbative result on instanton $\longrightarrow$ **Resurgence**

2.5 Comparison to universal WKB

1. We have computed explicitly all nonpertubative contributions up to **four instantons** and/or **anti-instantons**.
2. We found **complete agreement** with nonpertubative contributions obtained by **universal WKB** method.
3. We have given a **systematic prescription** to compute all muliti-instanton contributions, which is directly **applicable to field theory**.

2.6 Why are Bions not exact classical solutions ?

Why should we include bion configurations that are **not the solution** ?

Answer: Bions are the solutions of **Complexified** QM with **fermions**

A.Behtash, G.V.Dunne, T.Schafer, T.Sulejmanpasic and M.Unsal, Phys.Rev.Lett.**116**, 011601 (2016); arXiv:1510.03435 [hep-th]; E.Witten, [arXiv:1001.2933 [hep-th]] $\cdots$

sine-Gordon QM, double-well QM, $\cdots$: Tin's talk

We wish to clarify and add several new aspects

1. Study $\mathbb{C}\boldsymbol{P}^{\boldsymbol{N-1}}$ **QM** (to explore $\mathbb{C}\boldsymbol{P}^{\boldsymbol{N-1}}$ 2d FT) on $\boldsymbol{R} \times \boldsymbol{S}^{\mathbf{1}}$) instead of the sine-Gordon QM
2. Compute **1-loop determinant** explicitly
3. Evaluate the neccessary corrections from **quasi-moduli integral**
4. Determine **Lefschetz thimbles** and their weight by using dual thimble

T.Fujimori, S.Kamata, T.Misumi, M.Nitta and N.Sakai, arXiv:1607.04205 [hep-th]

3 Complexified $\mathbb{C}\boldsymbol{P}^{\boldsymbol{N-1}}$ Quantum Mechanics

3.1 $\mathbb{C}\boldsymbol{P}^{\boldsymbol{N-1}}$ QM from 2d FT

$\mathbb{C}\boldsymbol{P}^{\boldsymbol{N-1}}$ 2d field theory

$$\boldsymbol{S} = \frac{1}{g_{\mathrm{2d}}^2}\int d^2x\, G_{i\bar{j}}\partial_\mu\varphi^i\partial^\mu\bar{\varphi}^j\,, \qquad G_{i\bar{j}} = \frac{\partial^2}{\partial\varphi^i\partial\bar{\varphi}^j}\log(1+\varphi^k\bar{\varphi}^k)$$

$\mathbb{Z}_N$-twisted boundary conditions

$$\varphi^k(x_1, x_2+L) = \varphi^k(x_1, x_2)\, e^{\frac{2\pi k i}{N}}$$

Kaluza-Klein decomposition $\longrightarrow$ **Dimensional reduction**

$$\varphi^k(x_1,x_2) \;=\; \sum_{n\in\mathbb{Z}} \varphi^k_{(n)}(x_1)\;\exp\left[i\frac{2\pi}{L}\left(n+\frac{k}{N}\right)x_2\right] \longrightarrow \varphi^k_{(0)}(x_1)\,e^{ikmx_2}$$

$$L_{\rm 1d} = \frac{1}{g^2_{\rm 1d}} G^{(0)}_{k\bar{l}}\left[\partial_{x_1}\varphi^k_{(0)}\partial_{x_1}\bar{\varphi}^l_{(0)} + klm^2\varphi^k_{(0)}\bar{\varphi}^l_{(0)}\right], \qquad G^{(0)}_{k\bar{l}} = G_{k\bar{l}}(\varphi^i = \varphi^i_{(0)})$$

with $m = 2\pi/(NL)$ and $1/g^2_{\rm 1d} = L/g^2_{\rm 2d}$

$\mathbb{C}P^{N-1}$ 2d field theory at small $L \longrightarrow$ **$\mathbb{C}P^{N-1}$ quantum mechaniics**

$\mathbb{C}P^1 = S^2$: $\varphi = \tan\frac{\theta}{2}e^{i\phi}$

Phase modulus is lost if we take the sine-Gordon quantum mechanics

3.2 Introducing Fermions (SUSY for $\epsilon = 1$)

$$S_E \;=\; \int d\tau\left[\frac{1}{g^2}\frac{\partial_\tau\varphi\partial_\tau\bar{\varphi}}{(1+\varphi\bar{\varphi})^2} + V(\varphi\bar{\varphi})\right]$$

$$V(\varphi\bar{\varphi}) \;\equiv\; \frac{1}{g^2}\frac{m^2\varphi\bar{\varphi}}{(1+\varphi\bar{\varphi})^2} - \epsilon m\frac{1-\varphi\bar{\varphi}}{1+\varphi\bar{\varphi}}$$

2 conserved quantities : Energy E, angular momentum l

$$E \equiv \frac{1}{g^2}\frac{\partial_\tau\varphi\partial_\tau\bar{\varphi}}{(1+\varphi\bar{\varphi})^2} - V(\varphi\bar{\varphi})$$

$$l \equiv \frac{i}{g^2}\frac{\partial_\tau\varphi\bar{\varphi} - \partial_\tau\bar{\varphi}\varphi}{(1+\varphi\bar{\varphi})^2}$$

Finite action $\longrightarrow$ boundary condition at $\tau \to \pm\infty$

$$\lim_{\tau\to\pm\infty}\varphi = \lim_{\tau\to\pm\infty}\bar{\varphi} = 0 \;\;\longrightarrow\;\; l = 0,\;\; E = E|_{\varphi=0} = \epsilon m$$

Real Bion exact solution (**Most general** solution)

$$\varphi = e^{i\phi_0}\sqrt{\frac{\omega^2}{\omega^2-m^2}}\frac{1}{i\sinh\omega(\tau-\tau_0)}, \quad \omega \equiv m\sqrt{1+\frac{2\epsilon g^2}{m}},$$

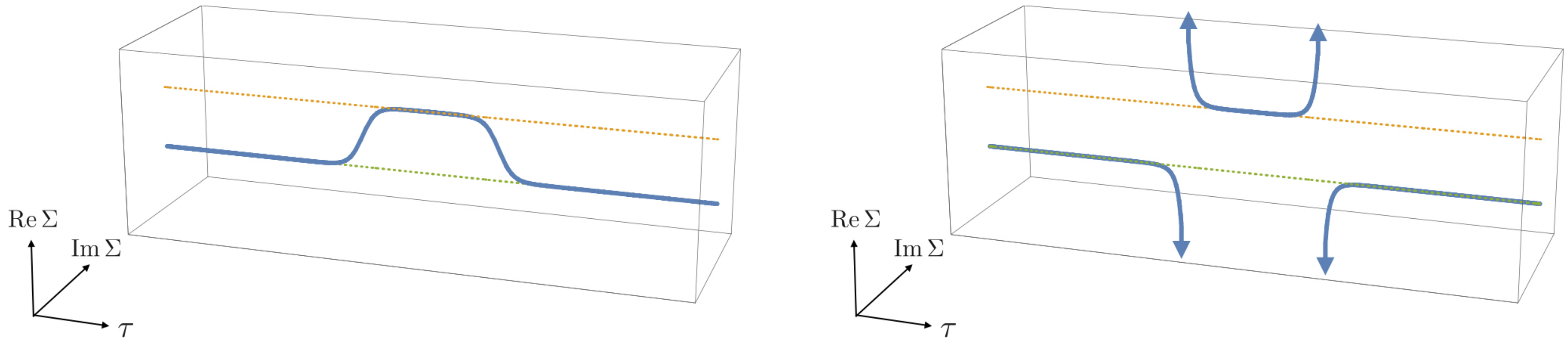
$$\varphi^{-1} = e^{\omega(\tau-\tau_+)-i\phi_+} + e^{-\omega(\tau-\tau_-)-i\phi_-}$$

$$\tau_\pm = \tau_0 \pm \frac{1}{2\omega}\log\frac{4\omega^2}{\omega^2-m^2}, \qquad \phi_\pm = \phi_0 \mp \frac{\pi}{2}$$

2 real moduli parameters : τ_0: translational moduli, ϕ_0: $U(1)$ moduli

Value of Lagrangian L and action S for the real bion solution

$$L = 4m\epsilon\left[\frac{\omega^2\cosh\omega(\tau-\tau_0)}{\omega^2+(\omega^2-m^2)\sinh^2\omega(\tau-\tau_0)}\right]^2 - m\epsilon$$



(a) $\boldsymbol{\Sigma(\tau)}$ for real bion (b) $\boldsymbol{\Sigma(\tau)}$ for complex bion

Figure 5: Kink profiles of real and complex bions. The complex bion solution has singularities where $\boldsymbol{\Sigma(\tau) = \frac{m\varphi\tilde{\varphi}}{1+\varphi\tilde{\varphi}}}$ diverges.

$$\boldsymbol{S_{\mathrm{rb}} = \int_{-\infty}^{\infty} d\tau (L + m\epsilon) = \frac{2\omega}{g^2} + 2\epsilon \log \frac{\omega + m}{\omega - m}}$$

Real bion gives a nonperturbative correction to ground state energy

Other contributions should cancel this in SUSY case ($\boldsymbol{\epsilon = 1}$)

3.3 Complexification

SUSY ($\epsilon = 1$) requires other solution $\longrightarrow$ **Complexification**

$$\bar{\varphi} \;\longrightarrow\; \tilde{\varphi} \;\neq\; \text{complex conjugate of } \varphi$$

Action should be **holomorphic** in both $\varphi, \tilde{\varphi}$

$$S[\varphi, \tilde{\varphi}] \;=\; \int d\tau \left[\frac{1}{g^2}\frac{\partial_\tau\varphi\partial_\tau\tilde{\varphi}}{(1+\varphi\tilde{\varphi})^2} + V(\varphi\tilde{\varphi})\right]$$

Form of Equation of Motion is unchanged : All the solutions can be generated by **complexified global symmetry** $\longrightarrow$ complexified moduli $\tau_0, \phi_0 \in \mathbb{C}$

A new solution by **imaginary time translation** : $\mathrm{Im}\tau_0$

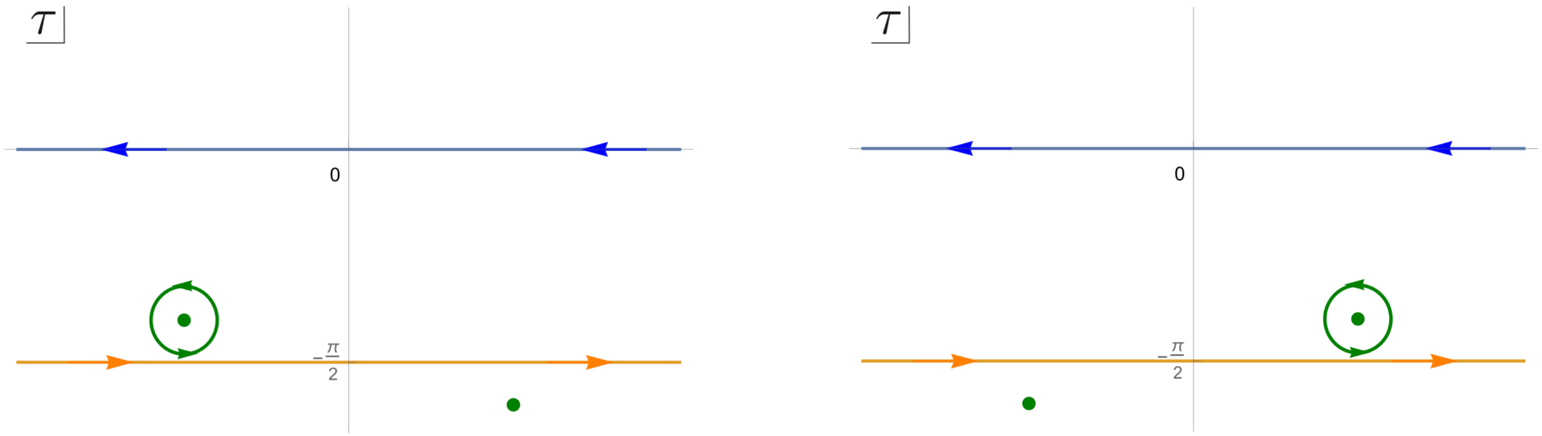
$$\tau_0 \;\longrightarrow\; \tilde{\tau}_0 = \tau_0 + \frac{1}{\omega}\frac{\pi i}{2}$$

Complex bion exact solution

$$\varphi^{-1} = e^{\omega(\tau-\tau_+)-i\phi_+} + e^{-\omega(\tau-\tau_-)-i\phi_-}, \quad \tilde{\varphi}^{-1} = e^{\omega(\tau-\tau_+)+i\phi_+} + e^{-\omega(\tau-\tau_-)+i\phi_-}$$

with complexified position and phase moduli

$$\tau_\pm = \tau_0 \pm \frac{1}{2\omega}\left(\log\frac{4\omega^2}{\omega^2-m^2} + \pi i\right), \quad \phi_\pm = \phi_0 - \frac{\pi}{2}$$



(a) $\boldsymbol{\theta = \arg g^2 > 0}$ (b) $\boldsymbol{\theta = \arg g^2 < 0}$

Figure 6: The integration contour for $\boldsymbol{S}_{\mathbf{cb}} - \boldsymbol{S}_{\mathbf{rb}}$. Depending on the sign of $\mathbf{arg}\,\boldsymbol{g}^{\mathbf{2}}$, $\boldsymbol{S}_{\mathbf{cb}} - \boldsymbol{S}_{\mathbf{rb}}$ is given by the residue at either $\boldsymbol{\tau}^{+}_{\mathbf{pole}}$ or $\boldsymbol{\tau}^{-}_{\mathbf{pole}}$.

$$\boldsymbol{S_{\mathrm{cb}} = -4m\epsilon \int_{-\infty}^{\infty} d\tau \left[\frac{\omega^2 \sinh \omega(\tau - \tau_0)}{\omega^2 - (\omega^2 - m^2) \cosh^2 \omega(\tau - \tau_0)}\right]^2}$$

Lagrangian has a double pole at $\boldsymbol{\tau = \tau^{\pm}_{\mathrm{pole}} \equiv \tau_0 \pm \frac{1}{\omega}\mathrm{arccosh}\sqrt{\frac{\omega^2}{\omega^2 - m^2}}}$

Stokes phenomenon: regularize by $\boldsymbol{\arg g^2 \neq 0}$: Action for complex bion

$$\boldsymbol{S_{\mathrm{cb}} = S_{\mathrm{rb}} \pm 2\pi i\epsilon, \rightarrow e^{-S_{\mathrm{cb}}} = e^{-S_{\mathrm{rb}}} e^{\pm 2\pi i\epsilon}}$$

We should integrate only half of complex moduli (Lefschetz thimble)

$\mathbf{2}$ solutions with $\mathbf{2}$ real moduli in complexified QM

Exact zero modes are divided into two

$\boldsymbol{\mathrm{Re}\tau_0, \mathrm{Re}\phi_0}$: should be integrated

$\boldsymbol{\mathrm{Im}\tau_0, \mathrm{Im}\phi_0}$: should not be intgrated, label to distinguish solutions

SUSY case $\boldsymbol{\epsilon = 1}$: $\boldsymbol{e^{-S_{\mathrm{cb}}} = e^{-S_{\mathrm{rb}}} \longrightarrow}$ No cancellation of vacuum energy ?

Cancellation can only be verified by computing **1-loop determinant**

3.4 One-Loop Determinant on Bion background

Combining the real bion ($\mathbf{1}$) and complex bion ($e^{\pm 2\pi\epsilon i}$) contributions

$$-\lim_{\beta\to\infty}\frac{1}{\beta}\frac{Z_1}{Z_0}=-i(1-e^{\pm 2\pi\epsilon i})\frac{16\omega^4}{g^2(\omega^2-m^2)}\exp\left(-\frac{2\omega}{g^2}-2\epsilon\log\frac{\omega+m}{\omega-m}\right)$$

Valid for $g^2\to 0$ with fixed boson-fermion coupling $\lambda\equiv\epsilon m g^2\ll m^2$

For $0\le\epsilon\le 1$, there are normalizable quasi-zero-modes :

Relative position τ_r and relative phase ϕ_r

$$Z_{\text{bion}}\approx\int d\tau_0 d\phi_0\int d\tau_r d\phi_r\,\det{}''\Delta\,\exp\left(-V_{\text{eff}}\right)$$

$\det''\Delta$: determinant excluding exact and quasi zero modes

Large separation of instanton and anti-instanton $\longrightarrow$

$\det''\Delta\approx$ product of determinant of constituent (anti-)instantons

Complexify τ_r,ϕ_r and determine integration paths (**thimble**s) and their weight (by intersection of **dual thimble**s with original path)

3.5 Lefschetz thimble and quasi moduli integral

Path-integral of complexified theory (in Infinite dimensional function space)

$$Z = \int \mathcal{D}\varphi \, \exp\left(-S[\varphi]\right) = \sum_{\sigma \in \mathfrak{S}} n_\sigma Z_\sigma$$

Saddle points σ : $\delta S / \delta \varphi = 0$, at $\varphi = \varphi_\sigma$

Thimble : $\mathrm{Re}S$ increasing away from saddle point with $\mathrm{Im}S$ constant

Dual thimble : $\mathrm{Re}S$ decreasing from saddle point with $\mathrm{Im}S$ constant

Intersection number of dual thimble gives a weight n_σ of the thimble.

Gradient flow equation

$$\frac{d\varphi}{dt} = G^{-1}\overline{\frac{\delta S}{\delta \varphi}}, \quad \lim_{t \to -\infty} \varphi = \varphi_{\mathrm{sol},\,\sigma}$$

Infinite dimensional path-integral $\longrightarrow$ trancate to quasi-moduli modes

Quasi-moduli integral of Sine-Gordon QM

Sine-Gordon instanton $\mathcal{I}$ and anti-instanton $\bar{\mathcal{I}}$: attractive interaction

$$[\mathcal{I}\bar{\mathcal{I}}] = \int_{\mathcal{C}_{\mathbb{R}}} d\tau \, e^{-V_{\mathrm{SG}}(\tau)}, \quad V_{\mathrm{SG}}(\tau) \equiv -\frac{4m}{g^2} e^{-m\tau - i\theta} + 2\epsilon m \tau$$

To regularize the Stokes phenomenon, a phase for $g^2 \longrightarrow g^2 e^{i\theta}$ is introduced

Gradient flow equation (1 dimensional (τ), instead of infinite dim.)

$$\frac{d\tau}{dt} = \frac{1}{2m}\overline{\frac{\partial V_{\rm SG}}{\partial \tau}} = -\frac{2m}{g^2}e^{-m\bar{\tau}+i\theta} + \epsilon$$

Saddle points : $\partial V_{\rm SG}/\partial\tau = 0$ at

$$\tau = \tau_\sigma \equiv \frac{1}{m}\left[\log\frac{2m}{\epsilon g^2} + i a_\sigma\right], \quad a_\sigma = (2\sigma - 1)\pi - \theta, \quad \sigma \in \mathbb{Z}$$

Thimble : $-\infty < {\rm Re}\tau < \infty$, ${\rm Im}\tau = -a_\sigma/m =$ constant,

Dual thimble : ${\rm Im}V_{\rm SG}(\tau) = {\rm Im}V_{\rm SG}(\tau = \tau_\sigma) = 2\epsilon(-1 + m\tau_\sigma)$

$$\rightarrow m{\rm Re}\tau = \log\left[\frac{2m}{\epsilon g^2}\frac{\sin(m{\rm Im}\tau + a_\sigma)}{m{\rm Im}\tau + a_\sigma}\right], \qquad (-a_\sigma - \pi \le m\tau_I \le -a_\sigma + \pi)$$

Intersection number n_σ of σ-thimbles

$$(n_0, n_1) = \begin{cases} (0,1) & \text{for } \theta = +0 \\ (1,0) & \text{for } \theta = -0 \end{cases}$$

$$[\mathcal{I}\bar{\mathcal{I}}] = \begin{cases} Z_{\sigma=1} & \text{for } \theta = +0 \\ Z_{\sigma=0} & \text{for } \theta = -0 \end{cases}, \quad Z_\sigma = \frac{1}{m}e^{-2\pi i\epsilon(2\sigma-1)}\left(\frac{g^2}{4m}e^{i\theta}\right)^{2\epsilon}\Gamma(2\epsilon)$$

Exact agreement with BZJ prescription

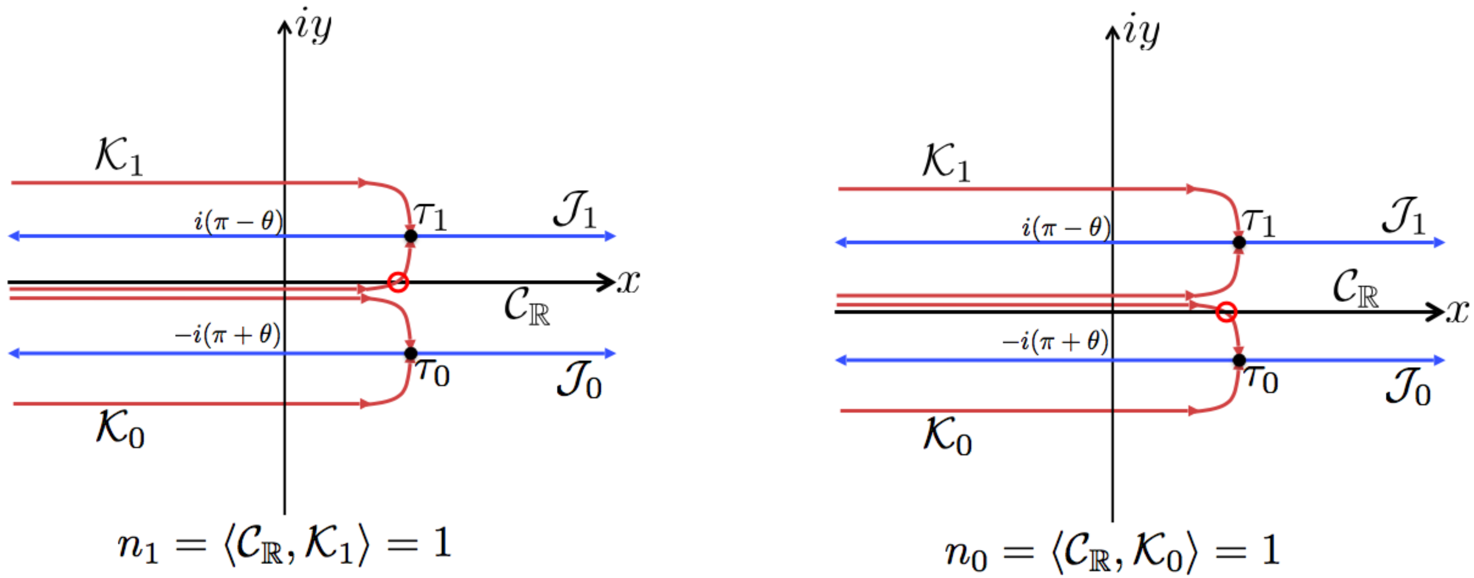
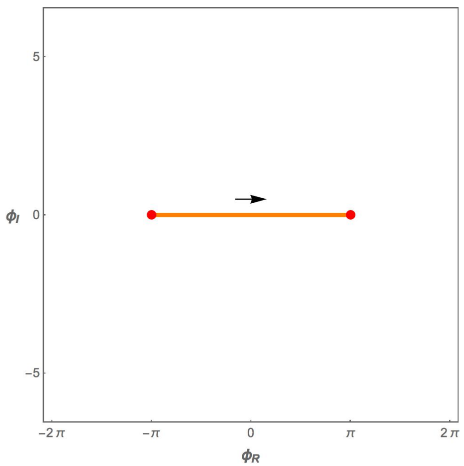
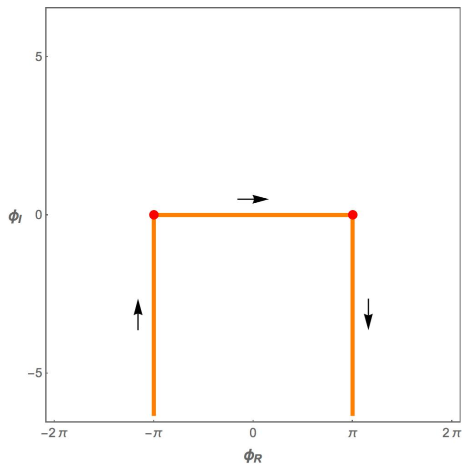


Figure 7: Real axis intersects with dual thimble of $\boldsymbol{\sigma} = \mathbf{1}$ for $\boldsymbol{\theta} = +\mathbf{0}$(left), and that of $\boldsymbol{\sigma} = \mathbf{0}$ for $\boldsymbol{\theta} = -\mathbf{0}$(right)



(a) original contour



(b) extended contour

Quasi-moduli integral of $\mathbb{C}P^1$ QM

Fractional instanton and anti-instanton interaction : depends on ϕ

$$[\mathcal{I}\bar{\mathcal{I}}] = \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} d\tau \, e^{-V(\tau,\phi)}, \;\; V(\tau,\phi) \;=\; -\frac{4m}{g^2}\cos\phi \, e^{-m\tau} + 2\epsilon m\tau$$

Periodic variable $-\pi \leq \phi < \pi$: add half infinite contours

E.Witten, [arXiv:1001.2933 [hep-th]] $\cdots$

Redefining variables as $\tau_+ = \tau + \frac{i}{m}\phi, \;\; \tau_- = \tau - \frac{i}{m}\phi,$

$$V(\tau,\phi) = \frac{V_{\mathrm{SG}}(\tau_+) + V_{\mathrm{SG}}(\tau_-)}{2}$$

Combining $\mathbf{1}$-loop determinant with quasi-moduli integral

Non-perturbative correction to the ground state energy

$$-\lim_{\beta\to\infty}\frac{1}{\beta}\frac{Z_1}{Z_0}\approx -\frac{8m^4}{\pi g^4}[\mathcal{I}\bar{\mathcal{I}}]e^{-\frac{2m}{g^2}}$$

$$= -2m\left(\frac{g^2}{2m}\right)^{2(\epsilon-1)}\frac{\sin\epsilon\pi}{\pi}\Gamma\left(\epsilon\right)^2 e^{-\frac{2m}{g^2}}\times\begin{cases} e^{\pi i\epsilon} & \text{for } \theta=-0 \\ e^{-\pi i\epsilon} & \text{for } \theta=+0 \end{cases}$$

Agrees with BZJ prescription for $\epsilon\longrightarrow 0$

Vanishing correction for SUSY limit ($\epsilon=1$)

Precise agreement for near susy case at leading order in $\epsilon-1$

$$E=E_{\rm pert}+E_{\rm bion},\ E_{\rm pert}=(g^2-m)\delta\epsilon+\mathcal{O}\left(\delta\epsilon^2\right),$$

$$E_{\rm bion}=-2m\,e^{-\frac{2m}{g^2}}\delta\epsilon+\mathcal{O}\left(e^{-\frac{4m}{g^2}},\ \delta\epsilon^2\right)$$

4 Conclusions

1. To compute multi-instanton amplitudes, we have proposed a generalization of **BZJ prescription** and an appropriate **subtraction scheme** for **multi-instanton amplitudes**.

2. Our results agree with the result of the uniform WKB method.

3. $\mathbb{C}\boldsymbol{P}^{\boldsymbol{N-1}}$ **quantum mechanics** instead of the sine-Gordon quantum mechanics should be used to find the compactification limit of $\mathbb{C}\boldsymbol{P}^{\boldsymbol{N-1}}$ field theory on $\boldsymbol{R} \times \boldsymbol{S}^{\boldsymbol{1}}$.

4. Nonperturbative contributions in $\mathbb{C}\boldsymbol{P}^{\boldsymbol{1}}$ quantum mechanics can be given by **real and complex bion exact solutions**, provided one introduces fermions, and complexify the theory.

5. We have explicitly evaluated the **1-loop determinant** on the background of the bion solutions.

6. For most of cases of interest including purely bosonic and supersymmetric cases, $\mathbf{1}$-loop approximation is not accurate and we need to integrate over **quasi-zero modes**.

7. To determine the integration path and weight for bion saddle points, we determined the **Lefschetz thimbles** and **dual thimbles** in the

truncated function space.

8. The nonperturbative contributions from the bion solutions agree with the results of the **BZJ** prescription.