Multi-instanton Contributions and Complexified Bion Solutions in Quantum Mechanics

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1 Multi-instanton Contributions in sine-Gordon QM

1.1 Multi-Instanton Contributions

Borel sum of **Perturbation** series $\rightarrow$ Imaginary **ambiguities**

**Instanton (Bion)** contributions $\rightarrow$ Imaginary **ambiguities**

Cancel each other: nonperturbative physics encoded in perturbation series

Resurgence to higher orders of nonperturbative exponentials requires

**Systematic computations** of multi-instanton (bions) contributions

Instanton and antiinstanton attract (non-BPS): moduli integral requires

**BZJ** prescription

(i) $g^2 < 0 \rightarrow$ Analytic continuation to $g^2 > 0$

(ii) $\epsilon$ species of fictitious fermions and subtract $1/\epsilon$ poles


Verified only at the **leading order** of nonperturbative factor (single Bion)

**Universal WKB** results in quantum mechanics:

provides a good test for multiple instanton calculations

G.V.Dunne and M.Ünsal, Phys.Rev.D **89**, 105009 (2014); ···
Our purpose

1. To compute multi-instanton amplitudes in the sine-Gordon quantum mechanics
2. To propose an appropriate subtraction scheme for multi-instanton amplitudes.
3. To compare with the result of the uniform WKB method.

1.2 Path-integral representation of Energy

Schrödinger eq. for Sine-Gordon QM (harmonic oscillator when $g^2 = 0$)

$$H \psi(x) = \left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{8g^2} \sin^2(2gx) \right] \psi(x) = E \psi(x)$$

Euclidian Lagrangian

$$L = \frac{1}{2} \left( \frac{dx}{dt} \right)^2 + V(x), \quad V(x) = \frac{1}{8g^2} \sin^2(2gx)$$

Bloch angle $\theta \in [0, \pi]$ to label energy eigenstate in the band

$$\psi (x + \pi/(2g)) = e^{i\theta} \psi(x)$$
Figure 1: Sin-Gordon potential with $g = 1$. Periodicity is $\pi/(2g)$.

Path-integral representation of the lowest band energy eigenvalue

$$E = \lim_{\beta \to \infty} -\frac{1}{\beta} \log Z, \quad Z = \text{Tr} e^{-\beta H}$$

$$Z = \int_{x(t=-\beta/2)=x(t=\beta/2)} Dx(t) \ e^{-S+iQ\theta}$$

$$Q = \frac{1}{\pi} \int_{-\infty}^{\infty} dt \frac{d(2gx)}{dt}$$

Perturbative $E_{\text{pert}}$ and nonperturbative $\Delta E$ contributions

$$E = E_{\text{pert}}(g^2) + \Delta E$$
Perturbation series is divergent

\[ E_{\text{pert}}(g^2) = \sum_{K=0}^{\infty} a_K (g^2)^K, \quad a_K \sim K! \]

\[ E_{\text{pert}}(g^2) = \int_0^\infty dt e^{-t} B_{\text{pert}}(g^2 t), \quad B_{\text{pert}}(t) = \sum_{K=0}^{\infty} \frac{a_K}{K!} t^K \]

Nonalternating divergent series: Borel resummation gives imaginary part \( E_{\text{pert}}(g^2) \) is a real analytic function of \( g^2 \) → dispersion relation

Dispersion rel. in complex \( g^2 \)-plane gives the coefficient \( a_K \) as

\[ a_K \approx -\frac{1}{\pi} \int_0^\infty dg^2 \frac{\text{Im} E_{\text{pert}}(g^2)}{(g^2)^{K+1}} \]

This imaginary part should be unphysical and cancelled by

the nonperturbative contribution \( \Delta E \) from instantons (Resurgence)

### 1.3 Instantons

BPS equation

\[ \frac{dx}{dt} = \frac{1}{2g} \sin(2gx), \]
anti-BPS equation

\[ \frac{dx}{dt} = -\frac{1}{2g} \sin(2gx), \]

Single instanton solution with \( Q = 1 \)

\[ x_\mathcal{I}(t) = \frac{(2 \arctan e^{t-t_0} + n\pi)}{(2g)}, \quad n \in \mathbb{Z} \]

is BPS for even \( n \), anti-BPS for odd \( n \): No other (anti-)BPS solutions

\( t_0 \) is the translational zero mode (moduli)

Integrating the translational zero mode \( t_0 \), one finds the contribution \( \Delta E^{(1,0)} \)

of single instanton \([\mathcal{I}]\) to the energy (with one-loop determinant)

\[ \Delta E^{(1,0)} = -[\mathcal{I}] = -\left( \frac{e^{-S_I}}{\sqrt{\pi g^2}} \right) e^{i\theta}. \]

Any combination of adjacent (anti-)instantons are always non-BPS

Interaction energy of instanton and (anti-)instanton at separation \( R \)

\[ V_{\mathcal{I}\mathcal{I}}(R) = \frac{2}{g^2} \exp[-R], \quad V_{\mathcal{I}\bar{\mathcal{I}}}(R) = -\frac{2}{g^2} \exp[-R] \]
2 Multi-instanton contributions

2.1 General setting with BZJ prescription

Methods to obtain multi-instanton contributions

1. **Integration over separation** $R$ between instantons
2. Attractive interactions require **analytic continuations** from $g^2 < 0$
3. Divergence at large $R$: Regularize by $\epsilon$ number of **fictitious fermions**
   (fermion zero mode exchange gives attraction at large $R$)
4. **Subtraction of divergence** as $\epsilon \to 0$

Interaction potential between instanton and (anti-)instanton

$$V(R) = \pm \frac{2}{g^2} \exp(-R) + \epsilon R$$

Our proposal for subtraction of multiple moduli integral

1. Enumerate all possible distinct **configurations**
2. Enumerate possible **ordering** of moduli integrations for each config.
3. **Subtract possible poles** like $1/\epsilon$ for the first integration, and then perform the next integration successively, and retain the finite piece.

4. **Average** the results of all orderings and **sum** over distinct configurations

Physics behind the $1/\epsilon$ subtraction:

Logarithm $\rightarrow$ only connected configurations

(dilute gas multi-particle contributions are subtracted)

$$Z = Z_0 + Z_1 + Z_2 + \cdots$$

$$E = \lim_{\beta \to \infty} -\frac{1}{\beta} \log Z = \lim_{\beta \to \infty} -\frac{1}{\beta} \left[ \log Z_0 + \frac{Z_1}{Z_0} + \left( \frac{Z_2}{Z_0} - \frac{Z_1^2}{2Z_0^2} \right) + \cdots \right]$$

### 2.2 2 instantons

There is only one configuration and one ordering: $[\mathcal{II}]$

Two-instanton amplitude in unit of instanton factor $\xi = e^{-S_I}/\sqrt{\pi g^2}$ is

$$[\mathcal{II}] e^{-2i\theta} \xi^{-2} = \int_0^\infty dR \exp \left( -\frac{2}{g^2} e^{-R} - \epsilon R \right)$$
Figure 2: Two instanton configurations $[\mathcal{I}]$. Horizontal lines stand for the vacuum.

$|g^2| \ll 1 \quad \left( \frac{g^2}{2} \right)^\epsilon \Gamma(\epsilon) = - \left( \gamma + \log \frac{2}{g^2} \right) + O \left( \frac{1}{\epsilon} \right) + O(\epsilon)$

$\Delta E^{(2,0)} = e^{2i\theta \xi^2} \left( \gamma + \log \frac{2}{g^2} \right)$

2.3 1 instanton + 1 anti-instanton

Attraction: compute at $g^2 < 0$ then analytic continuation to $g^2 > 0$
There are two configurations with one ordering for each: \([\mathbf{I} \mathbf{I}]\) and \([\bar{\mathbf{I}} \bar{\mathbf{I}}]\).

\[
[\mathbf{I} \mathbf{I}] \xi^{-2} = \int_0^\infty dR \exp \left( -\frac{2}{g^2} e^{-R} - \epsilon R \right)
\]

\[
|g^2| \ll 1 \quad \xrightarrow{\epsilon} \quad \left( \frac{-g^2}{2} \right) \Gamma(\epsilon) \quad -g^2 = e^{\mp i\pi} g^2
\]

\[
- \left( \gamma + \log \frac{2}{g^2} \right) \mp i\pi + O \left( \frac{1}{\epsilon} \right) + O(\epsilon)
\]

\[
\Delta E^{(1,1)} = ([\mathbf{I} \mathbf{I}] + [\bar{\mathbf{I}} \bar{\mathbf{I}}]) = \xi^2 \left[ 2 \left( \gamma + \log \frac{2}{g^2} \right) \pm 2i\pi \right]
\]

Figure 3: One-instanton and one anti-instanton amplitude \(([\mathbf{I} \mathbf{I}], [\bar{\mathbf{I}} \bar{\mathbf{I}}])\).
Imaginary part should be cancelled by perturbative contribution
\[ \text{Im}[\Delta E^{(1,1)}] + \text{Im}[E_{\text{pert}}] = 0 \]
\[ a_k = \frac{-1}{\pi} \int_0^\infty d g^2 \frac{\text{Im}[E_{\text{pert}}(g^2)]}{(g^2)^{k+1}} = -\frac{1}{\pi} \int_0^\infty d(g^2) \frac{2e^{-1/g^2}}{(g^2)^{k+2}} = -\frac{2}{\pi} k! \]

### 2.4 2 instantons + 1 anti-instanton

3 configurations \([\mathcal{I}\mathcal{I}\mathcal{I}], [\mathcal{I}\mathcal{I}\mathcal{I}], [\mathcal{I}\mathcal{I}\mathcal{I}]\)

First configuration \([\mathcal{I}\mathcal{I}\mathcal{I}]\): unique ordering
\[ [\mathcal{I}\mathcal{I}\mathcal{I}] e^{-i\theta} \xi^{-3} = \int_0^\infty dR_1 dR_2 \exp \left[ -\frac{2}{g^2} (e^{-R_1} + e^{-R_2}) - \epsilon(R_1 + R_2) \right] \]
\[ = \frac{3}{2} \left( \gamma + \log \frac{2}{g^2} \right)^2 - \frac{17\pi^2}{12} \pm 3i\pi \left( \gamma + \log \frac{2}{g^2} \right) + O\left(\frac{1}{\epsilon}\right) + O(\epsilon) \]

Second configuration \([\mathcal{I}\mathcal{I}\mathcal{I}]\): two orderings
\[ [\mathcal{I}\mathcal{I}\mathcal{I}] e^{-i\theta} \xi^{-3} = \int_0^\infty dR_1 dR_2 \exp \left[ -\frac{2}{g^2} e^{-R_1} - \frac{2}{\bar{g}^2} e^{-R_2} - \epsilon(R_1 + R_2) \right] \]
1. \(R_2\) then \(R_1\) integral gives
\[ \frac{3}{2} \left( \gamma + \log \frac{2}{g^2} \right)^2 - \frac{5\pi^2}{12} \pm 2i\pi \left( \gamma + \log \frac{2}{g^2} \right) + O\left(\frac{1}{\epsilon}\right) + O(\epsilon) \]
Figure 4: Two-instanton and one–anti-instanton amplitudes ([III], [III], [III]).
2. $R_1$ then $R_2$ integral gives
\[
\frac{3}{2} \left( \gamma + \log \frac{2}{g^2} \right)^2 + \frac{\pi^2}{12} \pm i\pi \left( \gamma + \log \frac{2}{g^2} \right) + O \left( \frac{1}{\epsilon} \right) + O(\epsilon)
\]
Averaging the two orderings gives
\[
\mathcal{III} \bar{\mathcal{I}} e^{-i\theta} \xi^{-3} = \frac{3}{2} \left( \gamma + \log \frac{2}{g^2} \right)^2 - \frac{\pi^2}{6} \pm \frac{3}{2} i\pi \left( \gamma + \log \frac{2}{g^2} \right)
\]
Third configuration gives the same result as second $[\bar{\mathcal{III}} \mathcal{II}] = [\mathcal{III} \bar{\mathcal{I}}]$, By summing over $[\bar{\mathcal{III}} \mathcal{II}], [\mathcal{III} \bar{\mathcal{I}}], [\mathcal{II} \bar{\mathcal{III}}]$, nonperturbative correction is
\[
\Delta E^{(2,1)} = -\frac{9}{2} e^{i\theta} \xi^3 \left[ \left( \gamma + \log \frac{2}{g^2} \right)^2 - \frac{7\pi^2}{18} \pm \frac{4}{3} i\pi \left( \gamma + \log \frac{2}{g^2} \right) \right]
\]
Imaginary part should be cancelled by imaginary part of perturbative contribution on one-instanton background (Large order behavior)
\[
a_k \approx \frac{1}{\pi} \int_0^\infty dg^2 \frac{\text{Im}[\Delta E^{(2,1)} e^{-i\theta} / \xi]}{(g^2)^{k+1}} = \frac{6}{\pi} k! \left( \log 2 + \frac{s(k + 1, 2)}{k!} \right)
\]
Stirling number of first kind : $s(k + 1, 2) = (k + 1)s(k, 2) + k!$

Agreement with perturbative result on instanton → **Resurgence**

### 2.5 Comparison to universal WKB

1. We have computed explicitly all nonpertubative contributions up to **four instantons** and/or **anti-instantons**.

2. We found **complete agreement** with nonpertubative contributions obtained by **universal WKB** method.

3. We have given a **systematic prescription** to compute all multi-instanton contributions, which is directly **applicable to field theory**.

### 2.6 Why are Bions not exact classical solutions?

Why should we include bion configurations that are **not the solution**?

**Answer:** Bions are the solutions of **Complexified QM with fermions**


sine-Gordon QM, double-well QM, ··· : Tin’s talk

**We wish to clarify and add several new aspects**
1. Study \( \mathbb{C}P^{N-1} \) QM (to explore \( \mathbb{C}P^{N-1} \) 2d FT) on \( R \times S^1 \) instead of the sine-Gordon QM

2. Compute \textbf{1-loop determinant} explicitly

3. Evaluate the necessary corrections from \textbf{quasi-moduli integral}

4. Determine \textbf{Lefschetz thimbles} and their weight by using dual thimble


### 3 Complexified \( \mathbb{C}P^{N-1} \) Quantum Mechanics

#### 3.1 \( \mathbb{C}P^{N-1} \) QM from 2d FT

\( \mathbb{C}P^{N-1} \) 2d field theory

\[ S = \frac{1}{g_{2d}^2} \int d^2 x \, G_{i\bar{j}} \partial_{\mu} \varphi^i \partial^{\mu} \bar{\varphi}^j, \quad G_{i\bar{j}} = \frac{\partial^2}{\partial \varphi^i \partial \bar{\varphi}^j} \log(1 + \varphi^k \bar{\varphi}^k) \]

\( \mathbb{Z}_N \)-twisted boundary conditions

\[ \varphi^k(x_1, x_2 + L) = \varphi^k(x_1, x_2) e^{\frac{2\pi ki}{N}} \]
Kaluza-Klein decomposition → Dimensional reduction

\[
\varphi^k(x_1, x_2) = \sum_{n \in \mathbb{Z}} \varphi^k_n(x_1) \exp \left[ i \frac{2\pi}{L} \left( n + \frac{k}{N} \right) x_2 \right] \to \varphi^k_{(0)}(x_1) e^{ikm x_2}
\]

\[
L_{1d} = \frac{1}{g_{1d}^2} G_{kl}^{(0)} \left[ \partial x_1 \varphi^k_{(0)} \partial x_1 \bar{\varphi}_{(0)}^l + klm^2 \varphi^k_{(0)} \bar{\varphi}_{(0)}^l \right], \quad G_{kl}^{(0)} = G_{kl}(\varphi^i = \varphi^i_{(0)})
\]

with \( m = 2\pi/(NL) \) and \( 1/g_{1d}^2 = L/g_{2d}^2 \)

\( \mathbb{C}P^{N-1} \) 2d field theory at small \( L \to \mathbb{C}P^{N-1} \) quantum mechanics

\( \mathbb{C}P^1 = S^2 : \varphi = \tan \frac{\theta}{2} e^{i\phi} \)

Phase modulus is lost if we take the sine-Gordon quantum mechanics

3.2 Introducing Fermions (SUSY for \( \epsilon = 1 \))

\[
S_E = \int d\tau \left[ \frac{1}{g^2} \frac{\partial_t \varphi \partial_t \bar{\varphi}}{1 + \varphi \bar{\varphi}} + V(\varphi \bar{\varphi}) \right]
\]

\[
V(\varphi \bar{\varphi}) \equiv \frac{1}{g^2} \frac{m^2 \varphi \bar{\varphi}}{1 + \varphi \bar{\varphi}} - \epsilon m \frac{1 - \varphi \bar{\varphi}}{1 + \varphi \bar{\varphi}}
\]
2 conserved quantities: Energy $E$, angular momentum $l$

$$E \equiv \frac{1}{g^2} \frac{\partial \tau \varphi \partial \tau \bar{\varphi}}{(1 + \varphi \bar{\varphi})^2} - V(\varphi \bar{\varphi})$$

$$l \equiv \frac{i \partial \tau \varphi \bar{\varphi} - \partial \tau \bar{\varphi} \varphi}{g^2 (1 + \varphi \bar{\varphi})^2}$$

Finite action $\rightarrow$ boundary condition at $\tau \to \pm \infty$

$$\lim_{\tau \to \pm \infty} \varphi = \lim_{\tau \to \pm \infty} \bar{\varphi} = 0 \rightarrow l = 0, \ E = E|_{\varphi=0} = \epsilon m$$

**Real Bion** exact solution (Most general solution)

$$\varphi = e^{i\phi_0} \sqrt{\frac{\omega^2}{\omega^2 - m^2 i \sinh \omega (\tau - \tau_0)}}, \ \omega \equiv m \sqrt{1 + \frac{2\epsilon g^2}{m}},$$

$$\varphi^{-1} = e^{\omega (\tau - \tau_+)} e^{i\phi_+} + e^{-\omega (\tau - \tau_-)} e^{-i\phi_-}$$

$$\tau_{\pm} = \tau_0 \pm \frac{1}{2\omega} \log \frac{4\omega^2}{\omega^2 - m^2}, \ \phi_{\pm} = \phi_0 \mp \frac{\pi}{2}$$

2 real moduli parameters: $\tau_0$: translational moduli, $\phi_0$: $U(1)$ moduli

Value of Lagrangian $L$ and action $S$ for the real bion solution

$$L = 4m\epsilon \left[ \frac{\omega^2 \cosh \omega (\tau - \tau_0)}{\omega^2 + (\omega^2 - m^2) \sinh^2 \omega (\tau - \tau_0)} \right]^2 - m\epsilon$$
(a) $\Sigma(\tau)$ for real bion

(b) $\Sigma(\tau)$ for complex bion

Figure 5: Kink profiles of real and complex bions. The complex bion solution has singularities where $\Sigma(\tau) = \frac{m\varphi\bar{\varphi}}{1+\varphi\bar{\varphi}}$ diverges.

$$S_{rb} = \int_{-\infty}^{\infty} d\tau (L + m\epsilon) = \frac{2\omega}{g^2} + 2\epsilon \log \frac{\omega + m}{\omega - m}$$

Real bion gives a nonperturbative correction to ground state energy

Other contributions should cancel this in SUSY case ($\epsilon = 1$)
3.3 Complexification

SUSY (\(\epsilon = 1\)) requires other solution → Complexification

\[ \tilde{\varphi} \to \tilde{\varphi} \neq \text{complex conjugate of } \varphi \]

Action should be **holomorphic** in both \(\varphi, \tilde{\varphi}\)

\[
S[\varphi, \tilde{\varphi}] = \int d\tau \left[ \frac{1}{g^2} \frac{\partial \tau \varphi \partial \tau \tilde{\varphi}}{(1 + \varphi \tilde{\varphi})^2} + V(\varphi \tilde{\varphi}) \right]
\]

Form of Equation of Motion is unchanged : All the solutions can be generated by **complexified global symmetry** → complexified moduli \(\tau_0, \phi_0 \in \mathbb{C}\)

A new solution by **imaginary time translation** : \(\text{Im}\tau_0\)

\[ \tau_0 \to \tilde{\tau}_0 = \tau_0 + \frac{1}{\omega} \pi i \]

**Complex bion** exact solution

\[ \varphi^{-1} = e^{\omega(\tau - \tau_+)} - i \phi_+ + e^{-\omega(\tau - \tau_-)} - i \phi_- \]

\[ \tilde{\varphi}^{-1} = e^{\omega(\tau - \tau_+)} + i \phi_+ + e^{-\omega(\tau - \tau_-)} + i \phi_- \]

with complexified position and phase moduli

\[ \tau_\pm = \tau_0 \pm \frac{1}{2\omega} \left( \log \frac{4\omega^2}{\omega^2 - m^2} + \pi i \right), \quad \phi_\pm = \phi_0 - \frac{\pi}{2} \]
Figure 6: The integration contour for $S_{cb} - S_{rb}$. Depending on the sign of $\arg g^2$, $S_{cb} - S_{rb}$ is given by the residue at either $\tau^+_{\text{pole}}$ or $\tau^-_{\text{pole}}$. 

(a) $\theta = \arg g^2 > 0$

(b) $\theta = \arg g^2 < 0$
\[ S_{cb} = -4m\epsilon \int_{-\infty}^{\infty} d\tau \left[ \frac{\omega^2 \sinh \omega (\tau - \tau_0)}{\omega^2 - (\omega^2 - m^2) \cosh^2 \omega (\tau - \tau_0)} \right]^2 \]

Lagrangian has a double pole at \( \tau = \tau_{\text{pole}}^\pm \equiv \tau_0 \pm \frac{1}{\omega} \arccosh \sqrt{\frac{\omega^2}{\omega^2 - m^2}} \)

Stokes phenomenon: regularize by \( \text{arg} \ g^2 \neq 0 \): Action for complex bion

\[ S_{cb} = S_{rb} \pm 2\pi i\epsilon, \to e^{-S_{cb}} = e^{-S_{rb}} e^{\pm 2\pi i\epsilon} \]

We should integrate only half of complex moduli (Lefschetz thimble)

2 solutions with 2 real moduli in complexified QM

Exact zero modes are divided into two

\begin{itemize}
  \item \textbf{Re}\( \tau_0, \text{Re}\phi_0 \): should be integrated
  \item \textbf{Im}\( \tau_0, \text{Im}\phi_0 \): should not be integrated, label to distinguish solutions
\end{itemize}

SUSY case \( \epsilon = 1 \): \( e^{-S_{cb}} = e^{-S_{rb}} \to \) No cancellation of vacuum energy?

Cancellation can only be verified by computing \textbf{1-loop determinant}
3.4 One-Loop Determinant on Bion background

Combining the real bion (1) and complex bion \((e^{\pm 2\pi \epsilon i})\) contributions

\[- \lim_{\beta \to \infty} \frac{1}{\beta} \frac{Z_1}{Z_0} = -i(1-e^{\pm 2\pi \epsilon i}) \frac{16\omega^4}{g^2(\omega^2 - m^2)} \exp \left( -\frac{2\omega}{g^2} - 2\epsilon \log \frac{\omega + m}{\omega - m} \right)\]

Valid for \(g^2 \to 0\) with fixed boson-fermion coupling \(\lambda \equiv \epsilon mg^2 \ll m^2\)

For \(0 \leq \epsilon \leq 1\), there are normalizable quasi-zero-modes:

Relative position \(\tau_r\) and relative phase \(\phi_r\)

\[Z_{\text{bion}} \approx \int d\tau_0 d\phi_0 \int d\tau_r d\phi_r \det''\Delta \exp ( -V_{\text{eff}} )\]

\(\det''\Delta\) : determinant excluding exact and quasi zero modes

Large separation of instanton and anti-instanton \(\to\)

\(\det''\Delta \approx\) product of determinant of constituent (anti-)instantons

**Complexify** \(\tau_r, \phi_r\) and determine integration paths (**thimbles**) and their weight (by intersection of **dual thimbles** with original path)
3.5 Lefschetz thimble and quasi moduli integral

Path-integral of complexified theory (in Infinite dimensional function space)

\[ Z = \int \mathcal{D}\varphi \exp(-S[\varphi]) = \sum_{\sigma \in \mathcal{G}} n_{\sigma} Z_{\sigma} \]

Saddle points \( \sigma : \delta S/\delta \varphi = 0 \), at \( \varphi = \varphi_{\sigma} \)

**Thimble** : \( \text{Re} S \) increasing away from saddle point with \( \text{Im} S \) constant

**Dual thimble** : \( \text{Re} S \) decreasing from saddle point with \( \text{Im} S \) constant

**Intersection number** of dual thimble gives a weight \( n_{\sigma} \) of the thimble.

**Gradient flow** equation

\[ \frac{d\varphi}{dt} = G^{-1} \overline{\delta S}_{\delta \varphi}, \quad \lim_{t \to -\infty} \varphi = \varphi_{\text{sol}, \sigma} \]

Infinite dimensional path-integral \( \rightarrow \) truncate to quasi-moduli modes

**Quasi-moduli integral of Sine-Gordon QM**

Sine-Gordon instanton \( \mathcal{I} \) and anti-instanton \( \bar{\mathcal{I}} \) : attractive interaction

\[ [\mathcal{I}\bar{\mathcal{I}}] = \int_{c_{\mathbb{R}}} d\tau e^{-V_{SG}(\tau)}, \quad V_{SG}(\tau) \equiv -\frac{4m}{g^2} e^{-m\tau - i\theta} + 2\epsilon m\tau \]
To regularize the Stokes phenomenon, a phase for $g^2 \rightarrow g^2 e^{i\theta}$ is introduced.

**Gradient flow** equation (1 dimensional $\tau$, instead of infinite dim.)

$$\frac{d\tau}{dt} = \frac{1}{2m} \frac{\partial V_{SG}}{\partial \tau} = -\frac{2m}{g^2} e^{-m\bar{\tau} + i\theta} + \epsilon$$

**Saddle points** : $\partial V_{SG}/\partial \tau = 0$ at

$$\tau = \tau_\sigma \equiv \frac{1}{m} \left[ \log \frac{2m}{\epsilon g^2} + ia_\sigma \right], \quad a_\sigma = (2\sigma - 1)\pi - \theta, \quad \sigma \in \mathbb{Z}$$

**Thimble** : $-\infty < \text{Re}\tau < \infty$, $\text{Im}\tau = -a_\sigma/m = \text{constant}$,

**Dual thimble** : $\text{Im}V_{SG}(\tau) = \text{Im}V_{SG}(\tau = \tau_\sigma) = 2\epsilon(-1 + m\tau_\sigma)$

$$\rightarrow m\text{Re}\tau = \log \left[ \frac{2m}{\epsilon g^2} \sin(m\text{Im}\tau + a_\sigma) \right], \quad (-a_\sigma - \pi \leq m\tau I \leq -a_\sigma + \pi)$$

**Intersection number** $n_\sigma$ of $\sigma$-thimbles

$$(n_0, n_1) = \begin{cases} (0, 1) & \text{for } \theta = +0 \\ (1, 0) & \text{for } \theta = -0 \end{cases}$$

$$[\mathcal{I} \bar{\mathcal{I}}] = \begin{cases} Z_{\sigma=1} & \text{for } \theta = +0 \\ Z_{\sigma=0} & \text{for } \theta = -0 \end{cases}, \quad Z_{\sigma} = \frac{1}{m} e^{-2\pi\epsilon(2\sigma - 1)} \left( \frac{g^2}{4m} e^{i\theta} \right)^{2\epsilon} \Gamma(2\epsilon)$$

Exact agreement with BZJ prescription
Figure 7: Real axis intersects with dual thimble of $\sigma = 1$ for $\theta = +0$ (left), and that of $\sigma = 0$ for $\theta = -0$ (right)
Quasi-moduli integral of $\mathbb{CP}^1$ QM

Fractional instanton and anti-instanton interaction: depends on $\phi$

$$[\mathcal{I}\bar{\mathcal{I}}] = \int_{-\pi}^\pi d\phi \int_{-\infty}^{\infty} d\tau \ e^{-V(\tau, \phi)}, \ V(\tau, \phi) = -\frac{4m}{g^2} \cos \phi \ e^{-m\tau} + 2\epsilon m\tau$$

Periodic variable $-\pi \leq \phi < \pi$ : add half infinite contours

E.Witten, [arXiv:1001.2933 [hep-th]] …

Redefining variables as $\tau_+ = \tau + \frac{i}{m} \phi, \ \tau_- = \tau - \frac{i}{m} \phi$, $V(\tau, \phi) = \frac{V_{SG}(\tau_+) + V_{SG}(\tau_-)}{2}$
Combining 1-loop determinant with quasi-moduli integral

Non-perturbative correction to the ground state energy

$$- \lim_{\beta \to \infty} \frac{1}{\beta} \frac{Z_1}{Z_0} \approx - \frac{8m^4}{\pi g^4} [I\bar{I}] e^{-\frac{2m}{g^2}}$$

$$= -2m \left( \frac{g^2}{2m} \right)^{2(\epsilon-1)} \frac{\sin \epsilon \pi}{\pi} \Gamma(\epsilon)^2 e^{-\frac{2m}{g^2}} \times \left\{ \begin{array}{ll} e^{\pi i \epsilon} & \text{for } \theta = -0 \\ e^{-\pi i \epsilon} & \text{for } \theta = +0 \end{array} \right.$$ 

Agrees with BZJ prescription for $\epsilon \to 0$

Vanishing correction for SUSY limit ($\epsilon = 1$)

Precise agreement for near susy case at leading order in $\epsilon - 1$

$$E = E_{\text{pert}} + E_{\text{bion}}, \quad E_{\text{pert}} = (g^2 - m) \delta \epsilon + O(\delta \epsilon^2),$$

$$E_{\text{bion}} = -2m e^{-\frac{2m}{g^2}} \delta \epsilon + O\left(e^{-\frac{4m}{g^2}}, \delta \epsilon^2\right)$$
4 Conclusions

1. To compute multi-instanton amplitudes, we have proposed a generalization of BZJ prescription and an appropriate subtraction scheme for multi-instanton amplitudes.

2. Our results agree with the result of the uniform WKB method.

3. $\mathbb{C}P^{N-1}$ quantum mechanics instead of the sine-Gordon quantum mechanics should be used to find the compactification limit of $\mathbb{C}P^{N-1}$ field theory on $\mathbb{R} \times S^1$.

4. Nonperturbative contributions in $\mathbb{C}P^1$ quantum mechanics can be given by real and complex bion exact solutions, provided one introduces fermions, and complexify the theory.

5. We have explicitly evaluated the 1-loop determinant on the background of the bion solutions.

6. For most of cases of interest including purely bosonic and supersymmetric cases, 1-loop approximation is not accurate and we need to integrate over quasi-zero modes.

7. To determine the integration path and weight for bion saddle points, we determined the Lefschetz thimbles and dual thimbles in the
truncated function space.

8. The nonperturbative contributions from the bion solutions agree with the results of the BZJ prescription.