Symmetries of M-theory backgrounds

based on works
[2012.13263, 2007.01213]

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Intro

Well-known T-duality:

- is a symmetry of the string action;
- Relates string backgrounds with a U(1) isometry;
- transforms branes into each other (say NS5 $\to$ KK5).
- Mirror symmetry relates different CY's

Depending on isometries of the string background and its dual T-duality can be

- abelian — $\mathrm{U}(1)^n$ isometry;
- non-abelian — a general isometry group, dual has no isometry
- Poisson-Lie — both initial and dual have no isometry

Non-abelian T-duality

String sigma-model on a group manifold G (dropped spectator fields)

$$S_1 = T \int_\Sigma \sigma^a \wedge \Big(G_{ab} * + B_{ab} \Big) \sigma^b. \tag{1}$$

Maurer-Cartan forms

$$\begin{aligned} g \in G, \quad g^{-1} dg = \sigma^a T_a, \quad d\sigma^a = \frac{1}{2} f_{bc}{}^a \sigma^b \wedge \sigma^c, \\ \{T_a\} = \text{bas}\, \mathfrak{g}. \end{aligned} \tag{2}$$

Introducing coordinates X^m on G

$$G_{ab}\sigma^a \wedge * \sigma^b = G_{ab} \sigma_m{}^a \sigma_n{}^b dX^m \wedge * dX^n = G_{mn}(X) dX^m \wedge * dX^n. \tag{3}$$

Non-abelian T-duality

1. gauge the isometry $g^{-1}dg \to A^a T_a$,
2. fix extra degrees of freedom

$$
\begin{aligned}
S_2 &= T\int_\Sigma A^a \wedge \Big(G_{ab} * + B_{ab}\Big) A^b + \tilde{X}_a F^a, \\
F^a &= 2dA^a + f_{bc}{}^a A^b \wedge A^c.
\end{aligned}
\tag{4}
$$

0. Integrate out $\tilde{X}_a$ to impose $F^a = 0 \longrightarrow$ return to S_1
3. Integrate out A^a to arrive at

$$
S_3 = T\int_\Sigma d\tilde{X}_a \wedge \Big(\tilde{G}_{ab} * + \tilde{B}_{ab}\Big) dX_b \tag{5}
$$

The dual background is defined as

$$
\tilde{G}_{ab} + \tilde{B}_{ab} = \Big(G_{ab} + B_{ab} + \tilde{X}_c f_{ab}{}^c\Big)^{-1} \tag{6}
$$

[Buscher, de la Ossa, Quevedo, Fradkin, Tseytlin, Borsato, Wulff]

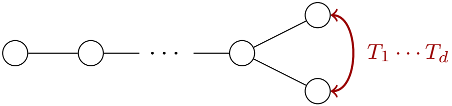
B-shift procedure

NATD can be written as an O(d,d) transformation "B-shift + reflections"

$$\mathcal{H}_{AB} = \begin{bmatrix} G_{ab} + B_a{}^c B_{cb} & B_a{}^b \\ B_a{}^b & G^{ab} \end{bmatrix} \longrightarrow \tilde{\mathcal{H}} = (\mathcal{O}_{\Delta B} T_1 \dots T_d)^T \mathcal{H} \mathcal{O}_{\Delta B} T_1 \dots T_d \quad (7)$$

with

$$\mathcal{O}_{\Delta B} = \begin{bmatrix} \delta_a{}^b & \tilde{X}_c f_{ab}{}^c \\ 0 & \delta_a{}^b \end{bmatrix}, \quad T_1 \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^d \\ \tilde{v}_1 \\ \tilde{v}_2 \\ \vdots \\ \tilde{v}_d \end{bmatrix} = \begin{bmatrix} \tilde{v}_1 \\ v^2 \\ \vdots \\ v^d \\ v^1 \\ \tilde{v}_2 \\ \vdots \\ \tilde{v}_d \end{bmatrix} \quad (8)$$



The transformation $T_1 \cdots T_d$ is an outer Aut of SO(d,d).

Non-abelian T-dualities

One classifies T-dualities as

- **Abelian**: initial and dual σ-models have the same (abelian) isometry;
- **NATD**: initial isometry is destroyed;
- **Poisson-Lie**: both initial and dual sigma-models have no isometry.

For a group G with algebra $\mathfrak{g}$ freely acting on target space M one defines Noether forms $J_a \sim v_a$ (left-inv. vector fields) and requires

$$dJ_a = \tilde{f}_a{}^{bc} J_b \wedge J_c, \tag{9}$$

$\tilde{f}_a{}^{bc}$ – structure constants of some algebra $\tilde{\mathfrak{g}}$. Compatibility constraint

$$\tilde{f}_a{}^{be} f_{ce}{}^d - \tilde{f}_a{}^{de} f_{ce}{}^b - \tilde{f}_c{}^{be} f_{ae}{}^d + \tilde{f}_c{}^{de} f_{ae}{}^b + \tilde{f}_e{}^{bd} f_{ac}{}^e = 0 \tag{10}$$

Such $(\mathfrak{g}, \tilde{\mathfrak{g}})$ forms a classical Drinfeld double.

[Klimcik, Severa]

Classical Drinfeld double

1. an algebra $\mathcal{D}$, dim$\mathcal{D} = 2n$;
2. Manin triple decomposition $(\mathcal{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$, such that $\mathcal{D} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$;
3. symmetric bilinear form η (with certain properties)

Given a basis $\{T_a\} = \mathrm{bas}\,\mathfrak{g}$, $\{\tilde{T}^a\} = \mathrm{bas}\,\tilde{\mathfrak{g}}$ the above means

$$\begin{aligned} [T_a, T_b] &= f_{ab}{}^c T_c, \quad [T_a, \tilde{T}^b] = f_a{}^{bc} T_c - f_{ac}{}^b \tilde{T}^c, \\ [\tilde{T}^a, \tilde{T}^b] &= f_c{}^{ab} \tilde{T}^c, \\ \eta(T_a, \tilde{T}^b) &= \delta_a{}^b \end{aligned} \tag{11}$$

Convenient to combine $T_A = (T_a, \tilde{T}^a)$ and write $[T_A, T_B] = \mathcal{F}_{AB}{}^C T_C$. All constraints are simply

$$\mathcal{F}_{[AB}{}^E \mathcal{F}_{CD]E} = 0. \tag{12}$$

Dualisation procedure

For group manifold sigma-model

1. Drinfeld double $(\mathfrak{g}, \tilde{\mathfrak{g}})$ generated by $T_A = (T_a, \tilde{T}^a)$;
2. Start with $g \in G = \exp \mathfrak{g}$ and right-inv 1-forms $r^a(x) T_a = g^{-1}(x) dg(x)$;
3. Adjoint action $g^{-1}(x) T_A g(x) = M_A{}^B(x) T_B$

$$M_A{}^B(x) = \begin{bmatrix} \delta_a^c & 0 \\ -\pi^{ac}(x) & \delta_c^a \end{bmatrix} \begin{bmatrix} a_c{}^b(x) & 0 \\ 0 & (a^{-1}(x))_b{}^c \end{bmatrix} \qquad (13)$$

4. Generalised vielbein

$$E_A{}^I(x) := \begin{bmatrix} \delta_a^b & 0 \\ -\pi^{ab} & \delta_b^a \end{bmatrix} \begin{bmatrix} e_b^m & 0 \\ 0 & r_m^b \end{bmatrix}. \qquad (14)$$

5. Generalised metric $\mathcal{H}_{IJ} = E_I{}^A E_J{}^B \mathcal{H}_{AB} \Longrightarrow (G_{mn}, B_{mn})$

Dualisation procedure

1. Make a different choice of the maximally isotropic subalgebra

$$T'_A = C_A{}^B T_B \tag{15}$$

2. Things transform

$$\mathcal{F}'_{A'B'C'} = C_{A'}{}^A C_{B'}{}^B C_{C'}{}^C \mathcal{F}_{ABC}, \quad \mathcal{H}'_{A'B'} = C_{A'}{}^A C_{B'}{}^B \mathcal{H}_{AB} \tag{16}$$

3. New structure constants $\mathcal{F}'_{ABC}$ define new $E'_I{}^A$, the dualised background is then

$$\mathcal{H}'_{IJ} = E'_I{}^A E_J{}^B \mathcal{H}'_{AB} \quad \Longrightarrow \quad (G'_{mn}, B'_{mn}). \tag{17}$$

$$\begin{aligned} C_A{}^B &= \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix} : \quad \text{PL T-duality}, \quad \mathfrak{g} \Longleftrightarrow \tilde{\mathfrak{g}} \\ C_A{}^B &\in \mathrm{O(d,d)} : \quad \text{PL T-plurality.} \end{aligned} \tag{18}$$

Geometry

The generalised vielbein $E_I{}^A(x)$ has nice properties

- It satisfies the same algebra $\mathcal{D}$ under generalised Lie derivative of Double Filed Theory

$$\mathcal{L}_{E_A} E_B{}^I = -\mathcal{F}_{AB}{}^C\, E_C{}^I\,, \tag{19}$$

- The bi-vector $\pi^{mn}(x)$ defines (locally) a Poisson-Lie structure

$$\pi^{qm}\,\partial_q \pi^{np} + \pi^{qn}\,\partial_q \pi^{pm} + \pi^{qp}\,\partial_q \pi^{mn} = 0 \tag{20}$$

Thinking of $E_I{}^A(x)$ as of a field of DFT allows to naturally generalise the whole story to 11 dimensions.

Exceptional field theory

As DFT provides T-covariant formulation of 10d supergravity, ExFT provides U-covariant formulation of 11d supergravity.

- $T_A = (T_a, \tilde{T}^a) \Longleftarrow$similar$\Longrightarrow$ gen. momentum $P_M = (p_m, \omega^m)$
- U-duality exchanges momentum with winding modes of membranes

$$P_M = (p_m, \omega^{m_1 m_2}, \omega^{m_1 \dots m_5}, \dots) \tag{21}$$

- Algebra of generalised vielbeins $\mathcal{L}_{E_A} E_B = \mathcal{F}_{AB}{}^C E_C$, structure constants satisfy linear and quadratic constraints of gauged supergravity

$$\begin{aligned} &\mathbb{P}_{\mathcal{R}} \mathcal{F} = 0, \\ &\mathcal{F}_{AC}{}^E \mathcal{F}_{EB}{}^D - \mathcal{F}_{AC}{}^E \mathcal{F}_{EB}{}^D = \mathcal{F}_{AB}{}^E . \mathcal{F}_{AC}{}^D \end{aligned} \tag{22}$$

Exceptional Drinfeld algebra

The idea is straightforwards: consider such $T_A = (T_a, \tilde{T}^{ab}, \tilde{T}^{a_1 \dots a_5}, \dots)$ that

$$[T_A, T_B] = \mathcal{F}_{AB}{}^C T_C, \tag{23}$$

where $\mathcal{F}_{AB}{}^C$ is the same as the embedding tensor of gauged supergravity, containing only

$$\mathcal{F}_{AB}{}^C \sim (f_{ab}{}^c, \tilde{f}_a{}^{bcd}). \tag{24}$$

The isometry condition $\eta(T_A, T_B) = \eta_{AB}$ restricts geometric subalgebra

$$\mathfrak{g} \otimes \mathfrak{g}\Big|_{\mathcal{R}_2} = 0. \tag{25}$$

$\mathcal{R}_2$ – the irrep $\mathbf{27}$ of E$_{6(6)}$ OR $\mathbf{16}$ of SO(5,5) OR $\mathbf{5}$ of SL(5) OR $\mathbf{1}$ of O(d,d).

[Sakatani, Malek, Thompson, Blair, Zhidkova]

EDA for $n \leq 6$

$$
\begin{aligned}
T_a \circ T_b &= f_{ab}{}^c\, T_c\,,\\
T_a \circ T^{b_1 b_2} &= f_a{}^{b_1 b_2 c}\, T_c + 2\, f_{ac}{}^{[b_1}\, T^{b_2]c} + 3\, Z_a\, T^{b_1 b_2}\,,\\
T_a \circ T^{b_1 \cdots b_5} &= -\, f_a{}^{b_1 \cdots b_5 c}\, T_c - 10\, f_a{}^{[b_1 b_2 b_3}\, T^{b_4 b_5]} - 5\, f_{ac}{}^{[b_1}\, T^{b_2 \cdots b_5]c}\\
&\quad + 6\, Z_a\, T^{b_1 \cdots b_5}\,,\\
T^{a_1 a_2} \circ T_b &= -f_b{}^{a_1 a_2 c}\, T_c + 3\, f_{[c_1 c_2}{}^{[a_1}\, \delta^{a_2]}_{b]}\, T^{c_1 c_2} - 9\, Z_c\, \delta^{[c}_b\, T^{a_1 a_2]}\,,\\
T^{a_1 a_2} \circ T^{b_1 b_2} &= -2\, f_c{}^{a_1 a_2 [b_1}\, T^{b_2]c} - f_{c_1 c_2}{}^{[a_1}\, T^{a_2] b_1 b_2 c_1 c_2} + 3\, Z_c\, T^{a_1 a_2 b_1 b_2 c}\,,\\
T^{a_1 a_2} \circ T^{b_1 \cdots b_5} &= 5\, f_c{}^{a_1 a_2 [b_1}\, T^{b_2 \cdots b_5]c}\,,\\
T^{a_1 \cdots a_5} \circ T_b &= f_b{}^{a_1 \cdots a_5 c}\, T_c + 10\, f_b{}^{[a_1 a_2 a_3}\, T^{a_4 a_5]} + 20\, f_c{}^{[a_1 a_2 a_3}\, \delta^{a_4}_b\, T^{a_5]c}\\
&\quad + 5\, f_{bc}{}^{[a_1}\, T^{a_2 \cdots a_5]c} + 10\, f_{c_1 c_2}{}^{[a_1}\, \delta^{a_2}_b\, T^{a_3 a_4 a_5] c_1 c_2}\\
&\quad - 36\, Z_c\, \delta^{[c}_b\, T^{a_1 \cdots a_5]}\,,\\
T^{a_1 \cdots a_5} \circ T^{b_1 b_2} &= 2\, f_c{}^{a_1 \cdots a_5 [b_1}\, T^{b_2]c} - 10\, f_c{}^{[a_1 a_2 a_3}\, T^{a_4 a_5] b_1 b_2 c}\,,\\
T^{a_1 \cdots a_5} \circ T^{b_1 \cdots b_5} &= -5\, f_c{}^{a_1 \cdots a_5 [b_1}\, T^{b_2 \cdots b_5]c}\,.
\end{aligned}
$$

Isometry condition

DDA: $\mathfrak{g} \otimes \mathfrak{g}\Big|_{\mathbf{1}} = 0$ defines the possible choices of $\mathfrak{g}$

$$\begin{aligned} &\eta^{AB} T_A \otimes T_B = 0, \quad \text{no O(d,d) singlets in the basis,} \\ &T_a \otimes T_b \quad \text{cannot be contracted into a singlet of O(d,d)} \\ &\tilde{T}^a \otimes \tilde{T}^b \quad \text{cannot be contracted into a singlet of O(d,d),} \\ &\tilde{T}^a \otimes T_b \quad \text{can be contracted into a singlet of O(d,d)} \end{aligned} \tag{26}$$

EDA for n=4, $T_A = T_{[ij]} = (T_a, T^{ab})$: $\mathfrak{g} \otimes \mathfrak{g}\Big|_{\mathbf{5}} = 0$

$$\begin{aligned} &\epsilon^{ijklm} T_{ij} \otimes T_{kl} = 0, \\ &T_{5a} \otimes T_{5b} \quad \text{cannot make the } \mathbf{5} \text{ of SL(5), can choose } \mathrm{bas}\, \mathfrak{g} = \{T_{5a}\}, \\ &T_{\alpha\beta} \otimes T_{\gamma\delta} \quad \text{cannot make the } \mathbf{5},\ \alpha, \beta = 1, 2, 3. \text{ IIB section} \end{aligned} \tag{27}$$

Section condition

Section condition of DFT and ExFT ensures consistency of algebra of generalised diffeos

- DFT: $\eta^{AB}\partial_A\Phi_1\partial_B\Phi_2 = 0$, all fields are allowed to depend on a half of the coordinates $\mathbb{X}^M = (x^m, x_m)$
- ExFT : $d^{iAB}\partial_A\Phi_1\partial_B\Phi_2 = 0$, d^{iAB} – invariant tensor of E$_d$

1. M-theory section $E_d \to GL(d)$

$$\mathcal{R}_2 \to \mathbf{d} \oplus \dots \tag{28}$$

2. IIB section $E_d \to GL(d-1) \times SL(2)$

$$\mathcal{R}_2 \to (\mathbf{d-1, 2}) \oplus (\mathbf{d-1, 1}) \oplus \dots \tag{29}$$

[Hull, Hohm, Berman, Thompson, Kleinschmidt, Cederwall, Samtleben]

Nambu-Lie U-duality

Goes along the same lines

1. Pick up the geometric subalgebra $\mathfrak{g}$, construct ${E_M}^A$ using its adjoint action on the EDA $\mathcal{D}$
2. Find different choice of the geometric subalgebra $\mathfrak{g}'$

$$T'_A = {C_A}^B T_B \tag{30}$$

3. Construct ${E'_M}^A$, that corresponds to the new geometric subalgebra

Nambu-Lie U-duality

Step **2** is the most complicated:

- $C_A{}^B$ should preserve the structure of EDA.
- No classification is available so far.

We find that for $n = 5$ (SO(5,5) theory) with **vanishing**

$$\Big(f_{ab}{}^{b}, \quad f_{a5}{}^{b}, \quad f_{b}{}^{ba5}, \quad f_{a}{}^{bcd}, \quad f_{5}{}^{abc}, \quad Z_a\Big) = 0\,, \quad (a,b=1,\dots,4) \qquad (31)$$

the following preserves the EDA structure

$$\underbrace{f'_{ab}{}^{c} = -f_{c}{}^{ab5}\,, \quad f'_{a}{}^{bc5} = -f_{bc}{}^{a}}_{\text{non-abelian T-duality}}\,, \quad \underbrace{f'_{ab}{}^{5} = -\tfrac{1}{2!}\,\epsilon^{abcd5}\,f_{cd}{}^{5}}_{\text{NA U-duality}}\,. \qquad (32)$$

Toy example

NAUD based on SO(5,5) EDA. Start with $\mathfrak{g}$ defined by

$$f_{23}{}^{1} = 1\,, \quad f_{34}{}^{5} = 1\,, \quad f_{24}{}^{3} = c_0 \quad (|c_0| < 1). \tag{33}$$

The initial (flat) geometry

$$g_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1-c_0^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & (1+c_0)\,y & 0 \\ 1 & 0 & (1+c_0)\,y & c_0\,(1+c_0)\,y^2+z^2 & z \\ 0 & 0 & 0 & z & 1 \end{pmatrix}, \quad g_{\mu\nu} = \text{diag}[1,\dots,1]. \tag{34}$$

Dual structure constants, defining $\mathfrak{g}'$

$$f'_{12}{}^{5} = -1\,, \quad f'_{1}{}^{235} = -1\,, \quad f'_{3}{}^{245} = -c_0\,. \tag{35}$$

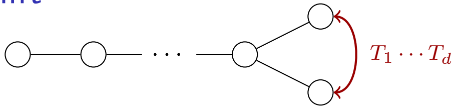
Toy example

The dual 11D background

$$
\begin{aligned}
g_{ij} &= \frac{1}{(1-c_0^2+x^2)^{\frac{2}{3}}}\begin{pmatrix} -c_0^2 z^2 - \frac{y^2}{4} & \frac{xy}{4} & -c_0 xz & -c_0^2+x^2+1 & -\frac{y}{2} \\ \frac{xy}{4} & 1-\frac{x^2}{4} & 0 & 0 & \frac{x}{2} \\ -c_0 xz & 0 & 1-c_0^2 & 0 & 0 \\ 1-c_0^2+x^2 & 0 & 0 & 0 & 0 \\ -\frac{y}{2} & \frac{x}{2} & 0 & 0 & -1 \end{pmatrix}, \\
C_3 &= \frac{1}{1-c_0^2+x^2}\left(\tfrac{xy}{2}\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z + x\,\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}v - c_0\,z\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}v\right), \\
g_{\mu\nu} &= \frac{(1-c_0^2+x^2)^{\frac{1}{3}}}{(1-c_0^2)^{\frac{1}{4}}}\,\mathrm{diag}(1,1,1,1,1,1)\,.
\end{aligned}
\tag{36}
$$

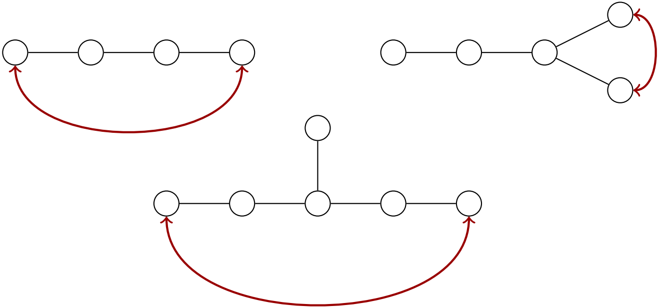
This is an M*-background (two time directions and wrong sing of kinetic terms). Hence, the transformation acts as a “timelike” NAUD.

C-shift



non-abelian T-duality

Similar external automorphisms exist for abelian U-duality groups



C-shift

The B-shift can be generalised to 11D backgrounds

1. Undress fields $g_{mn} = \sigma_m^a(x)\sigma_n^b(x)g_{ab}$, $C_{mnk} = \sigma_m^a(x)\sigma_n^b(x)\sigma_k^c(x)C_{abc}$
2. Perform C-shift $\tilde{\mathcal{H}} = \mathcal{O}_{\Delta C}^{-1}\mathcal{H}\mathcal{O}_{\Delta C}$

$$\begin{aligned} \mathcal{O}_{\Delta C}(\tilde{x}) \in E_{d(d)}, \\ \Delta C_{abc} = \tilde{x}_{d[a}f_{bc]}{}^d. \end{aligned} \tag{37}$$

3. Turn all $\tilde{x}_{ab}$ in $\tilde{\mathcal{H}}_{AB}$ into geometric coordinates
4. The initial $E_M{}^A$ and the dual $\tilde{E}_A{}^B$ generate the same generalised fluxes $\mathcal{F}_{AB}{}^C$ (under some conditions) $\Longrightarrow$ the dual is a solution

Summary

- A general rule for choosing alternative “isotropic” $\mathfrak{g}$ inside EDA’s with $n = 5$;
- C-shift procedure
- Some toy-model examples;

TO-DO

- Does the procedure always give “timelike” NAUD’s
- More sensible examples based on non-trivial 11D backgrounds
- NAUD for coset spaces and more general backgrounds;
- The E_7 story
- Examples of 3-vector non-abelian generalised Yang-Baxter deformations

Thank you!