Lyapunov spectra in traversable wormholes and their holographic duals

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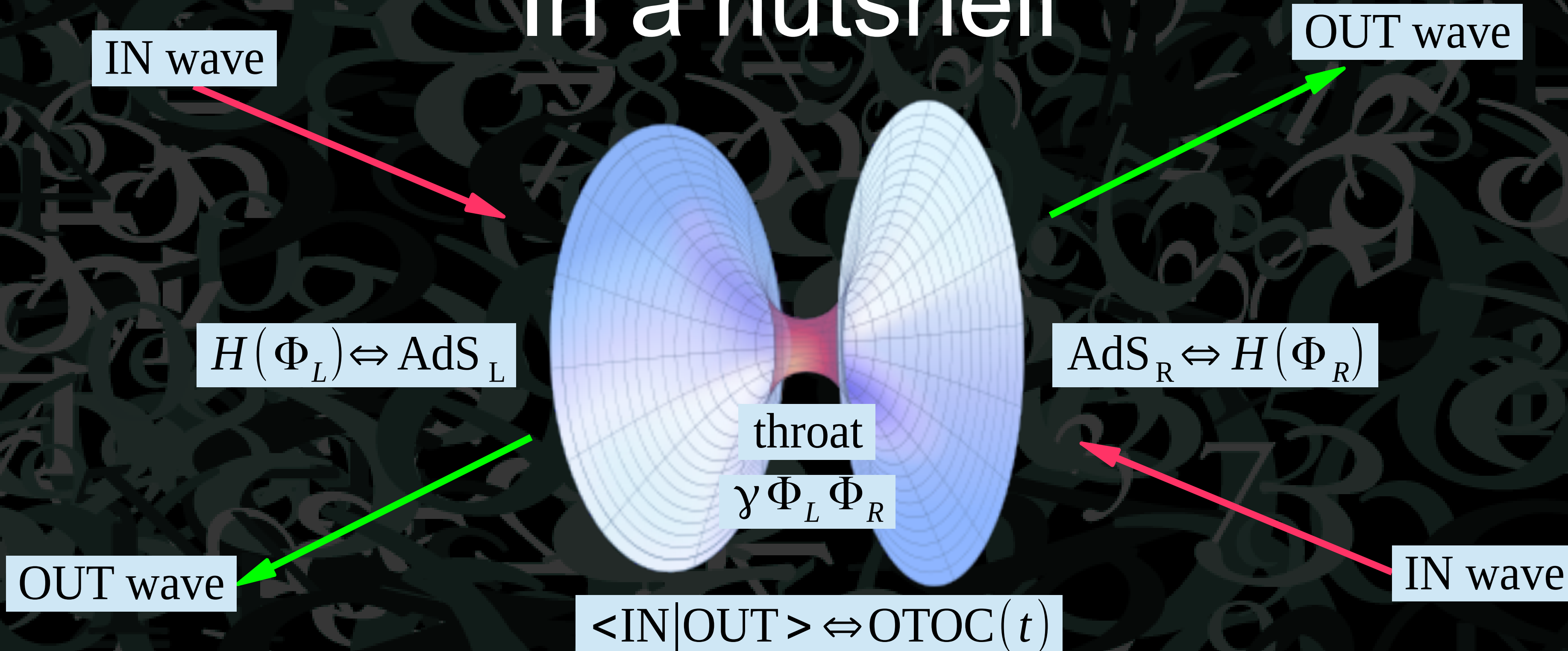
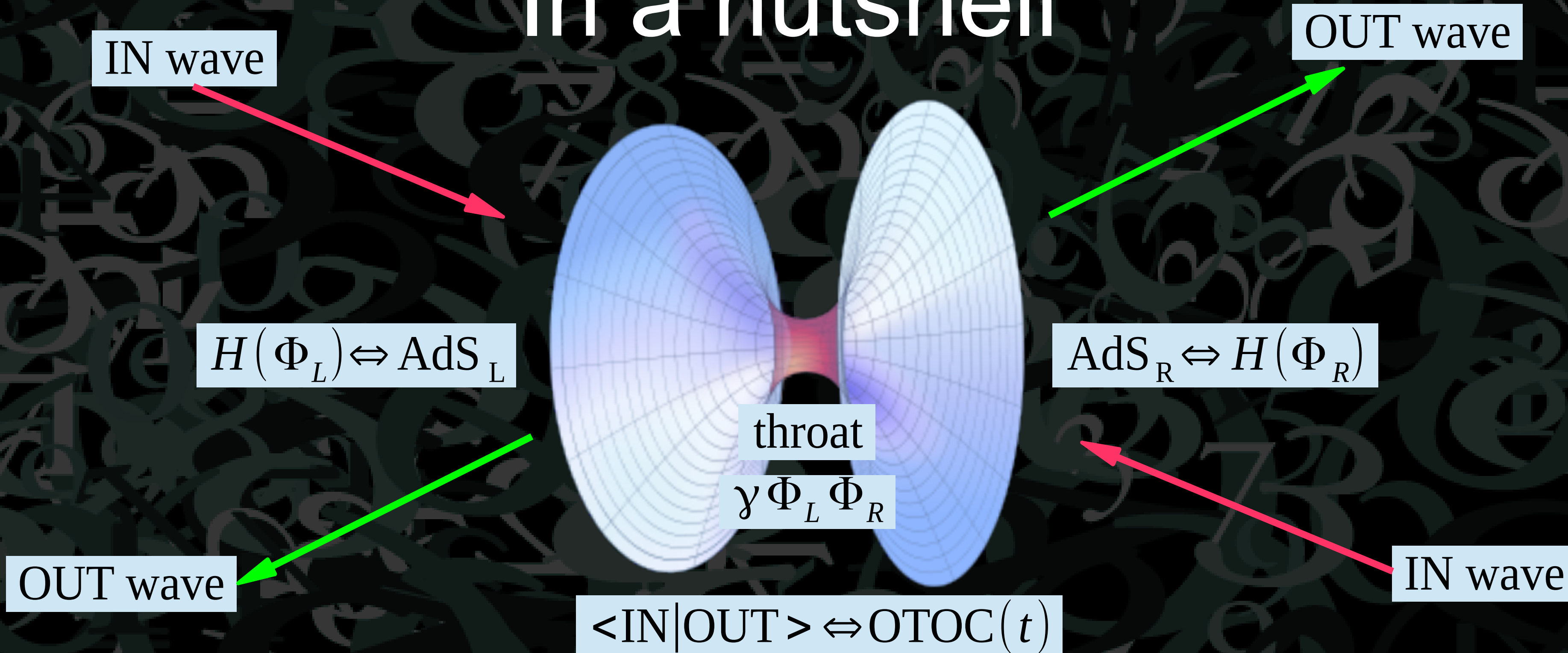
In a nutshell



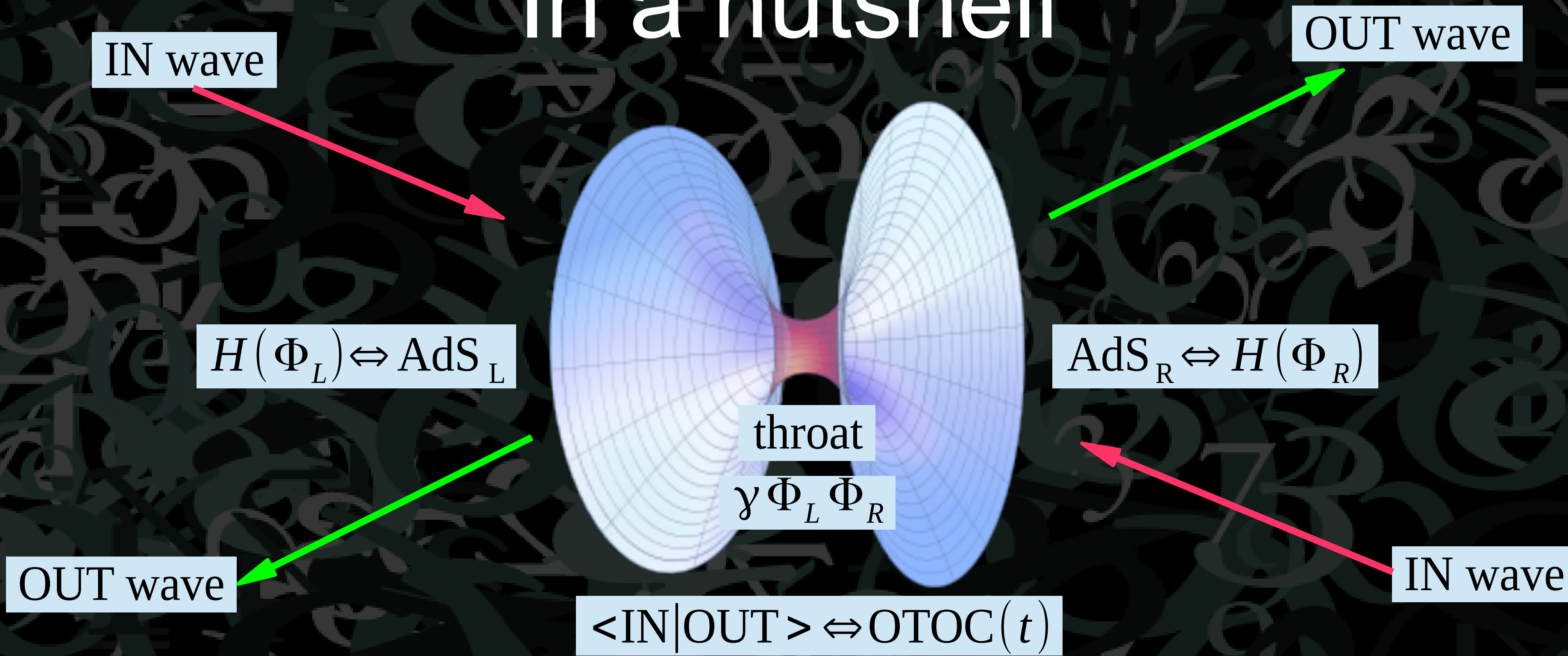
- AdS wormhole (WH) ~ left and right dual CFT; traversability ~ coupling between CFTs
- Four-wave scattering ~ out-of-time-ordered correlator (OTOC)
- OTOC: relaxation rate = Lyapunov exponent; scrambling
- Chaos bound: Lyapunov exponent $\lambda \leq 2\pi T$, saturated for BH

In a nutshell



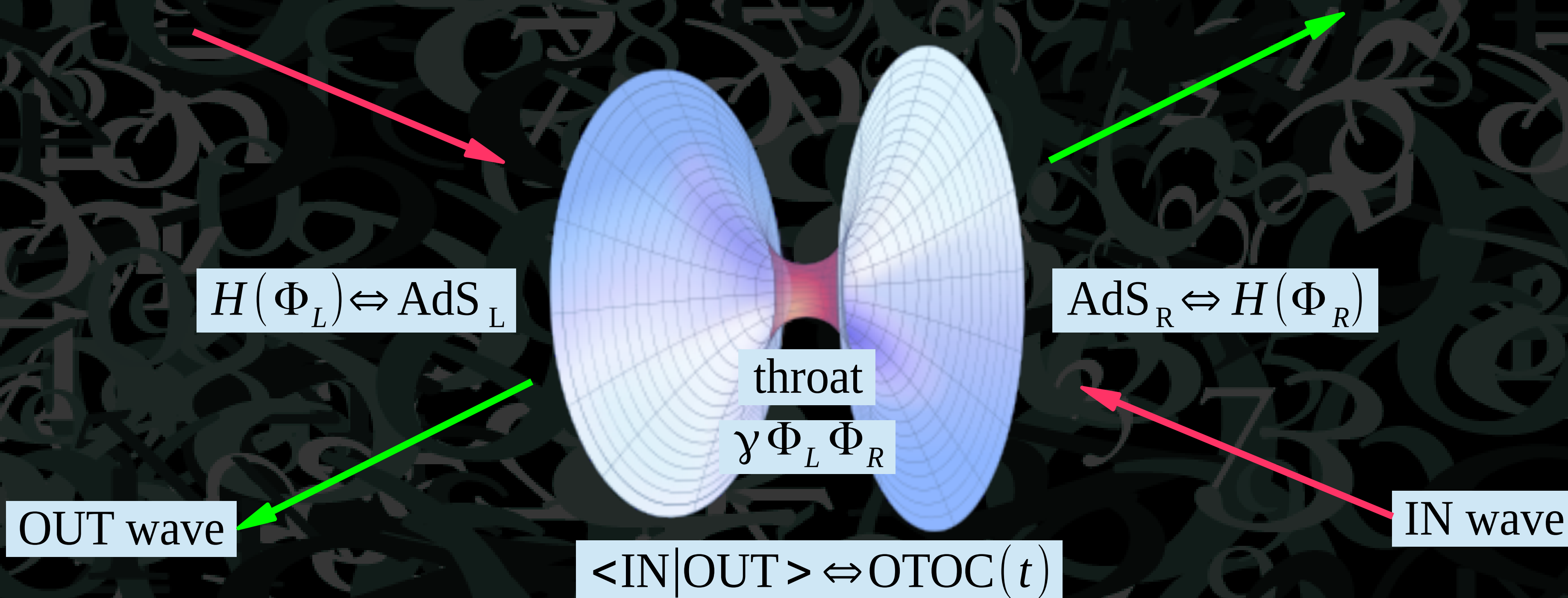
- What happens to OTOC when we perturb BH into a WH?
- Results in the literature for eternal AdS2 wormhole (Maldacena & Qi 2018) from the field theory dual (Garcia-Garcial et al 2019, Gao&Jafferis 2019 ...)
- Here: non-eternal WH and the bulk calculation

In a nutshell



- Eternal AdS traversable WH known in 1+1 dim and non-eternal in 2+1 dim $\Longrightarrow$ can work within Einstein GR
- Dual QFT (not just QM as in AdS2 WH)
- Can access some different regimes than in dual field theory

Conclusions in a nutshell



- #1: non-eternal WH are still strongly chaotic; only very long-living (near-eternal) WH can have slow chaos or no chaos
- #2: catch-22 for the WH teleportation protocol: long-living and large-throat WH increase teleportation fidelity but make it very slow (because of slow OTOC growth)

Outline

- Setup – traversable wormholes and out-of-time ordered correlators (OTOCs)
- Computing OTOCs in wormhole backgrounds – Lyapunov spectra and the dual field theory interpretation
- The phase diagram – maximal chaos, fast chaos, slow chaos and no chaos

Traversable wormhole in AdS

- Making it traversable: need to violate the null energy condition (NEC) $\Longrightarrow$ couple the two CFTs
- Maldacena & Wu 2018: eternal WH dual to two coupled Sachdev-Ye-Kitaev (SYK) models

$$H_{\mathrm{SYK}} = \sum_{i \le j \le k \le l} J_{ijkl} \chi_i \chi_j \chi_k \chi_l$$

- Gao, Jafferis & Wall 2017 (GJW) – time-dependent WH by turning on a double-trace coupling between CFTs:

$$H = H_{\mathrm{QFT}}(\Phi_{\mathrm{L}}) + H_{\mathrm{QFT}}(\Phi_{\mathrm{R}}) + \gamma \Phi_L \Phi_R \Rightarrow \int dU T_{UU} < 0$$

Traversable wormhole in AdS

- Field theory: double-trace coupling

$$H = H_{\mathrm{QFT}}(\Phi_{\mathrm{L}}) + H_{\mathrm{QFT}}(\Phi_{\mathrm{R}}) + \gamma\,\Phi_L \Phi_R \Rightarrow \int dU T_{UU} < 0$$

- In the bulk: excitation backreacts on geometry, removes the horizon and opens the throat:

$$g_{\mu\nu} = \begin{pmatrix} \gamma h & g_{UV} & 0 \\ g_{UV} & \gamma \tilde{h} & 0 \\ 0 & 0 & g_{\varphi\varphi} \end{pmatrix}$$

$$h \equiv h(U, V), \quad \tilde{h} \equiv h(V, U)$$

Kruskal coordinates

$$\exp(2 r_h t) = -U/V$$

$$r/r_h = (1 - UV)/(1 + UV)$$

- For OTOC need the full metric, not just average null energy
- Hard to do explicitly but ...

Traversable wormhole model

- From eternal WH solution we know the throat is AdS_2 and the far region is BTZ
- Quick WH: turn on $\gamma \Phi_L \Phi_R$ at $t=t_0$, turn off at $t=t_1$ and take $t_0 \to t_1$ (Freivogel et al 2019 [1907.13140])
- Slow WH: take $t_0 \to -\infty$ and $t_1 \to \infty$ instead
- Critical conformal dimension $\Delta = 1/2$ of the scalar Φ:
  - $\Delta < 1/2$ smooth WH mouth

$$h(U,V) = -\frac{8\Delta^2}{(1-2\Delta)^2}\frac{1-UV}{1+UV}\frac{1}{(U-U_0)^{2\Delta+1}}$$

  - $\Delta > 1/2$ sharp WH mouth (but of course curvature finite)

$$h(U,V) = -\frac{2U_0}{U^2}\frac{1-UV}{1+UV}\Theta(U-U_0)$$

Matching the WH expansions

- Matching the solutions in the throat (near- AdS_2) and outer (near-BTZ region) (coordinates t, r, φ):

  - outer: $ds^2_{out} \sim -f dt^2 + dr^2/f + r^2 d\varphi^2$
  - throat: $ds^2_{in} = (-dt^2 + d\varphi^2)(1+\rho^2/\gamma r_h^2) + \frac{d\rho^2}{1+\rho^2/\gamma r_h^2} + \frac{2\gamma U_0 r_h^2}{1+\rho^2/\gamma r_h^2} dt\, d\rho$

  $\rho(r) = (r - r_h)/\gamma r_h$

- Klein-Gordon bulk-to-boundary propagator found by mode summation:

$$K(r;t,t';\varphi,\varphi') = \sum_l \int d\omega\, F_{in}(r;\omega,t;l,\varphi)\, F_{out}(r;\omega,t';l,\varphi')\, e^{il(\varphi-\varphi')}$$

$$F_{in}(\rho(r);\omega,t;l,\varphi) = e^{il\varphi - i\omega t - 2i\omega\gamma r_h^2 U_0/3\rho^2} K_{\sqrt{\Delta-1}}\left(\sqrt{l^2-\omega^2}\,\rho\right)$$

$$F_{out}(r;\omega,t;l,\varphi) = e^{i(l\varphi-\omega t)} f(r) r^{-\Delta} {}_2F_1(i(\omega+l)/2R_h - \Delta/2, i(\omega-l)/2R_h - \Delta/2, \Delta, r)$$

$$f(r) = (\omega r - R_h^2)/(r^2 - R_h^2) - \omega \arctan(r/R_h) \qquad R_h = r_h(1 - 2\gamma U_0)$$

Out-of-time-ordered correlators - OTOC

- Out-of-time-ordered correlator (OTOC):

$$\langle |[A(t), B(0)]|^2 \rangle = \langle A^+(t) B^+(0) A(t) B(0) \rangle + \text{h.c.} + \text{TOC}$$

- Several interpretations:

  - thermalization rate (A from the system, B from the bath)

  - quantum Lyapunov exponent ($A \equiv x, \ B \equiv \epsilon p = -i \epsilon d/dx$)

  - quantum teleportation protocol (A from one subsystem and B from the other)

OTOC for black holes in AdS/CFT

- BH: Shenker, Stanford et al 2014, 2015
- Correlator of $A^{+}(t)$, $B^{+}(0)$, $A(t)$ and $B(0)$ @T $\Longleftrightarrow$ scattering amplitude by 2 infalling and 2 outgoing waves:

$$\langle A^{+}(t)B^{+}(0)A(t)B(0)\rangle_T = \int dp_{\mathrm{i}}\int dp_{\mathrm{o}} C\langle \mathrm{IN}(p_{\mathrm{i}},p_{\mathrm{o}})|\mathrm{OUT}(p_{\mathrm{i}},p_{\mathrm{o}})\rangle e^{-iS_{\mathrm{class}}}$$

- Eikonal phase: $S_{\mathrm{class}}=\int dp_{\mathrm{i}}\int dp_{\mathrm{o}}\sqrt{-g}\,\delta g_{\mu\nu}T^{\mu\nu}$
- The key simplification:
infinite redshift at the horizon $\Longrightarrow$ shock-wave solutions

$$\delta g_{UU}\propto\Theta(U),\ \ \delta g_{VV}\propto\Theta(V) \qquad T^{UU}\propto\delta(U),\ \ T^{VV}\propto\delta(V)$$

- Glue the BH solution with mass M to the solution with mass $M+p$

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OTOC in a BH vs. WH

Black hole

- Infinite redshift @horizon: shock wave at leading order

Wormhole

- Large redshift @ $U, V \sim 0$: shock wave + smooth $\delta g^{(1)}_{\mu\nu}$

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- Large redshift @ $U, V \sim 0$: shock wave + smooth $\delta g^{(1)}_{\mu\nu}$
- Phase shift (classical action) comes from the whole space

OTOC in a BH vs. WH

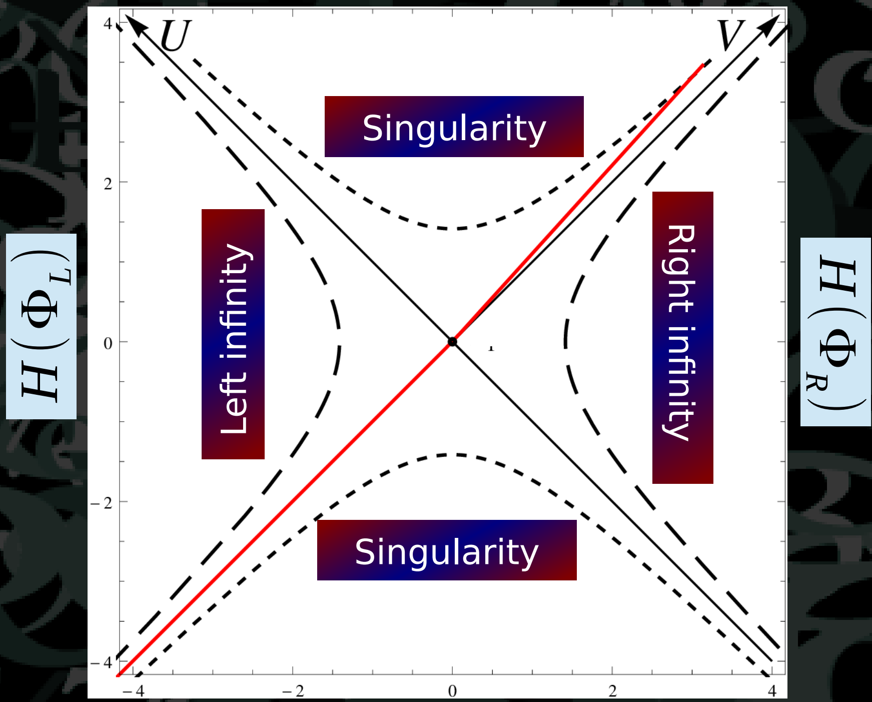
Black hole

- Infinite redshift @horizon: shock wave at leading order
- Phase shift (classical action) negligible away from horizon
- Geodesics always fall into the black hole

Wormhole

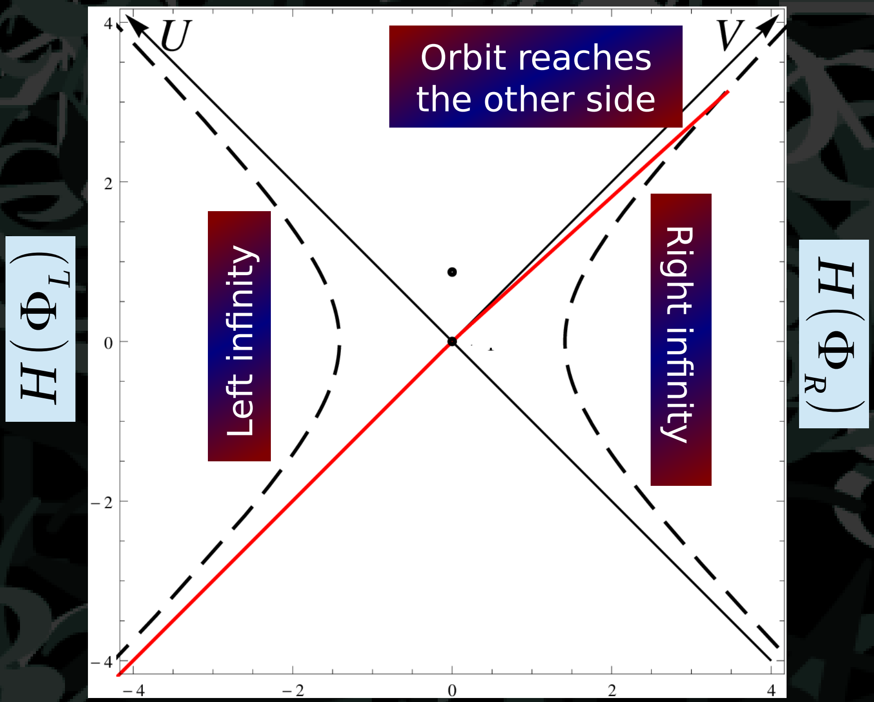
- Large redshift @ $U, V \sim 0$: shock wave + smooth $\delta g^{(1)}_{\mu\nu}$
- Phase shift (classical action) comes from the whole space
- Geodesics may be more complex or go back and forth

Kinematics of the perturbation



Black hole – vacuum solution

Two independent SYKs



Wormhole opened by the boundary coupling $\Phi_L \Phi_R$

Two coupled SYKs

Perturbed wormhole metric

- Perturbatively in wormhole tunnel size γ: keep the shock wave component but add smooth corrections F, f

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}, \quad \delta g_{\mu\nu} = \begin{pmatrix} p_i \Theta(U) & \gamma F(U,V) & 0 \\ \gamma F(U,V) & p_o \Theta(V) & 0 \\ 0 & 0 & \gamma f(U,V) \end{pmatrix}$$

- Stress-energy tensor determined by the geodesic equation (analytical for small U, V):

$$t_{\mu\nu} = \delta(U(\tau) - U)\,\delta(V(\tau) - V)\, g_{\mu\alpha} g_{\nu\beta} \dot{X}^{\alpha} \dot{X}^{\beta} / \dot{U}$$

A

$$t_{UU} = t_{UU}^{\mathrm{BH}} + O(\gamma^2) \qquad t_{VV} = O(\gamma^2) \qquad t_{UV} = O(\gamma)$$

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Scattering amplitude

- Full formalism by Balasubramanian et al 2019 [1908.08955]
- Here: perturbative formalism, start from momentum-space representation and correct perturbatively for momenta non-conservation) by $\partial^U \to p_0^U + \gamma\,\partial^{n_V},\ \ \partial^V \to p_0^V + \gamma\,\partial^{n_U}$
- In and out states for fields A and B with dimensions $\Delta_1,\ \Delta_2$

$$\langle A^+(t)\,B^+(0)\,A(t)\,B(0)\rangle$$

$$\Phi_{1,3} = K(\Delta_1; U, V; t, \varphi)\big|_{U=U(\tau),\, V=V(\tau)} \qquad \Phi_{2,4} = K(\Delta_{2,}\, U, V; t=0, \varphi)\big|_{U=U(\tau),\, V=V(\tau)}$$

$$|\mathrm{IN}\rangle = \partial_{n_U} K(\Delta_2; U, V; 0, \varphi)\, K(\Delta_1; U, V; t, \varphi')|0\rangle$$

$$|\mathrm{OUT}\rangle = \partial_{n_V} K(U, V; 0, \varphi)\, K(U, V; t, \varphi')|0\rangle$$

Eikonal phase

- Classical action:

$$S_{\text{class}} = \int dU \int dV \sqrt{-g}\, \delta g_{\mu\nu} t^{\mu\nu}\big(p_{\text{i}}(U,V), p_{\text{o}}(U,V)\big)$$

$$t_{\mu\nu} \propto p_{\text{i}}\delta + p_{\text{o}}\delta$$

$$\delta g_{\mu\nu} = \begin{pmatrix} p_{\text{i}}\Theta & 0 & 0 \\ 0 & p_{\text{o}}\Theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \gamma F & 0 \\ \gamma F & 0 & 0 \\ 0 & 0 & \gamma f \end{pmatrix}$$

$$S_{\text{class}} \sim p_{\text{i}} + p_{\text{o}} e^{r_h t} + p_{\text{i}} p_{\text{o}} e^{r_h t} + \gamma\left(p_{\text{i}}^2 + p_{\text{o}}^2 e^{2 r_h t}\right)$$

→ wormhole contribution

- Rather unimpressive at first glance: quantitative corrections but still the factor $e^{r_h t}$ implies $\lambda \propto T$
- Perturbative calculations in γ and saddle-point approximations for the integral in the action give $\lambda(T, \gamma, \Delta_{1,2})$

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Regimes of OTOC

- Saddle-point integration and the Schwarz-Pick theorem
- Message #1: sum of exponentials with different exponents:

$$\mathrm{OTOC} = \sum_n e^{\lambda_n t}$$

the relevant exponent: $\lambda = \max_n \lambda_n$ – depends on $\gamma, t_{0,} T$

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- Message #4: for large γ "eternal" WH can give $\lambda = 0$ – polynomial (non-exponential) OTOC

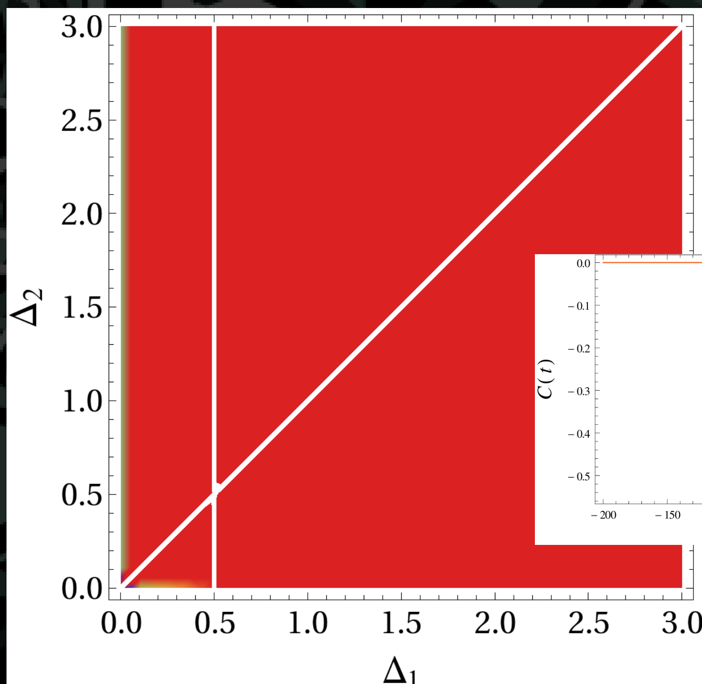
Regimes of OTOC

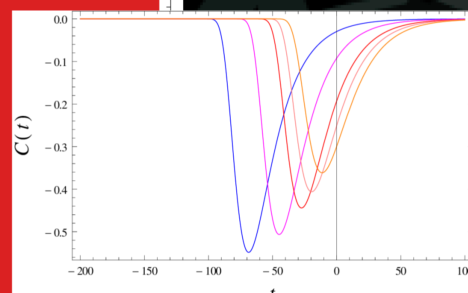
Slow chaos is a nonperturbative effect (need eternal WH)
No chaos is a quantitative effect only (large WH throat)

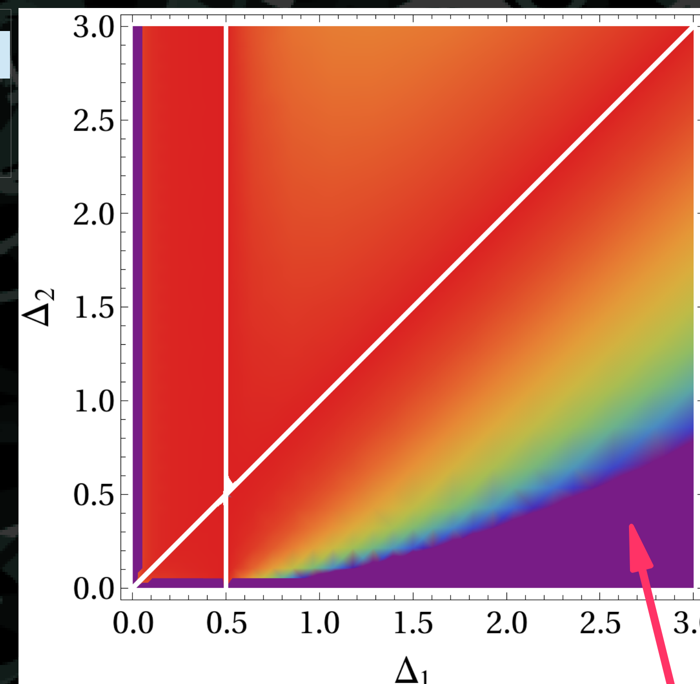
- Message #3: if no BH horizon ever exists ("eternal" WH, for $t_0 \to -\infty, t_1 \to \infty$) then exponentially slow OTOC: $\lambda \propto \exp(-1/T)$
- Message #4: for large γ "eternal" WH can give $\lambda = 0$ – polynomial (non-exponential) OTOC

Phase diagram for OTOC

violet: low λ
red $\lambda = 2\pi T$







Early times $t \sim t_0$

Late times $t \gg t_0$

Reproduced Nosaka&Numusawa [2009.10759]: $\lambda \sim \exp(-1/T)$

Phase diagram for OTOC

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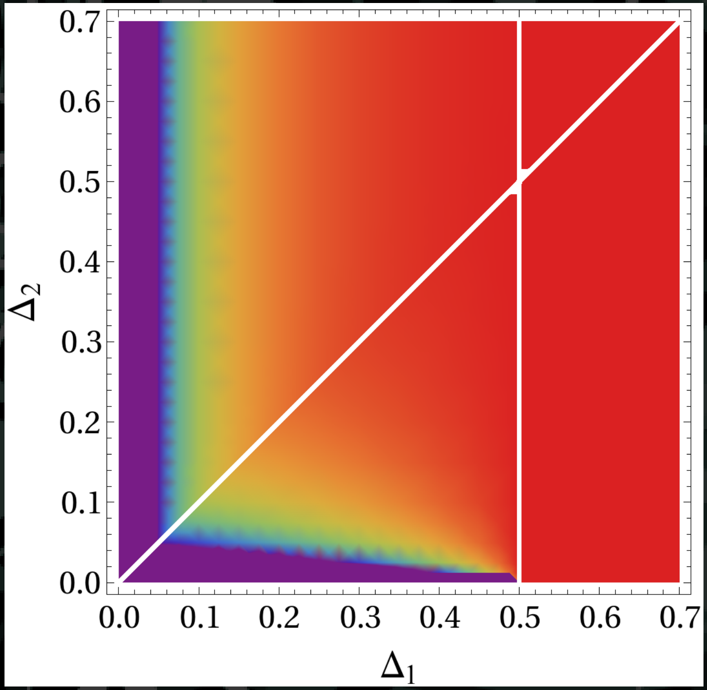
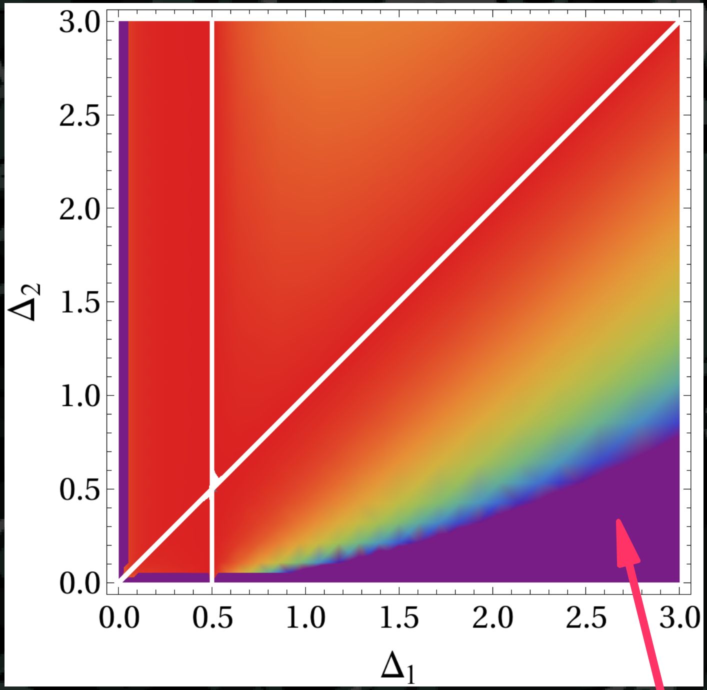


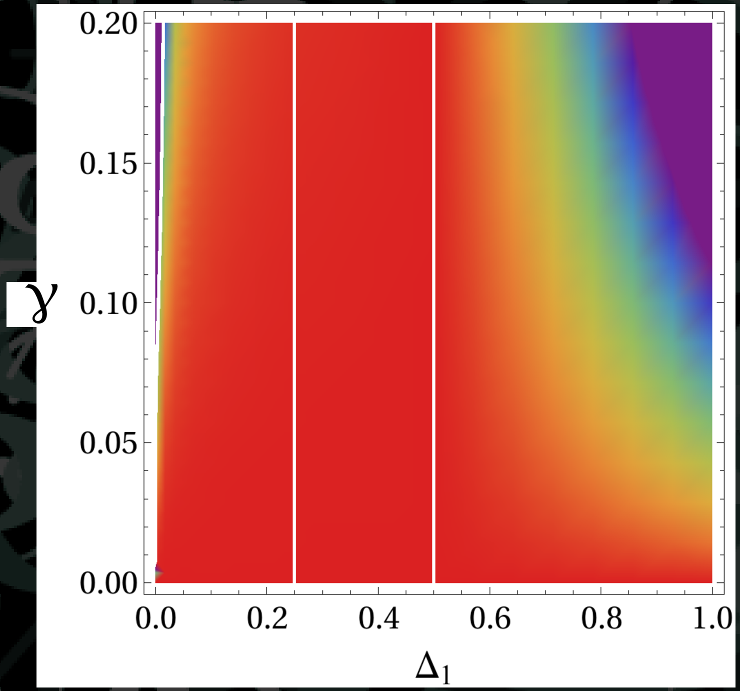


Early times $t \sim t_0$
zoom-in

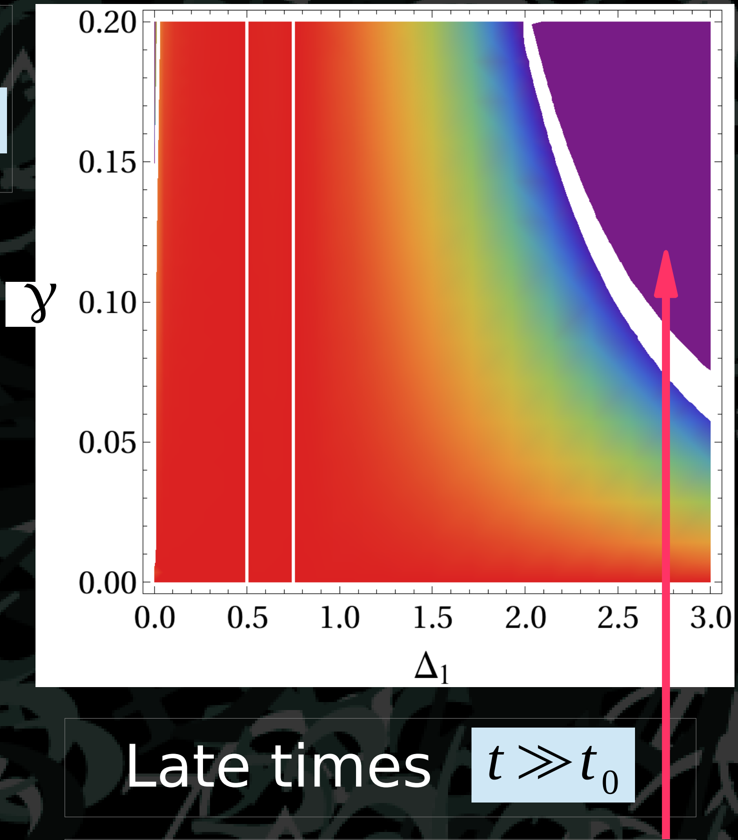
Late times $t \gg t_0$

Reproduced Nosaka&Numusawa [2009.10759]: $\lambda \sim \exp(-1/T)$

Phase diagram for OTOC



violet $\lambda = 0$
red $\lambda = 2\pi T$

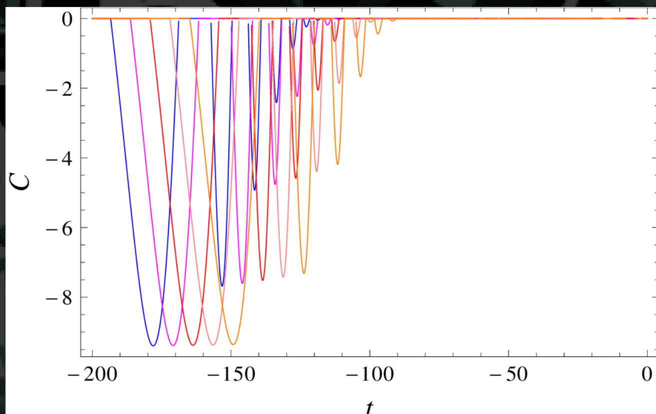


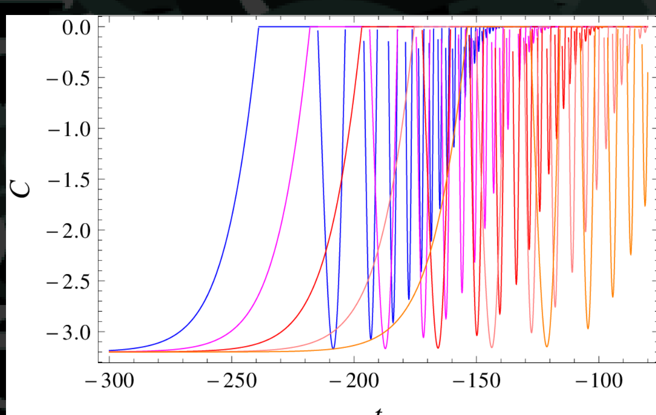
Early times $t \sim t_0$

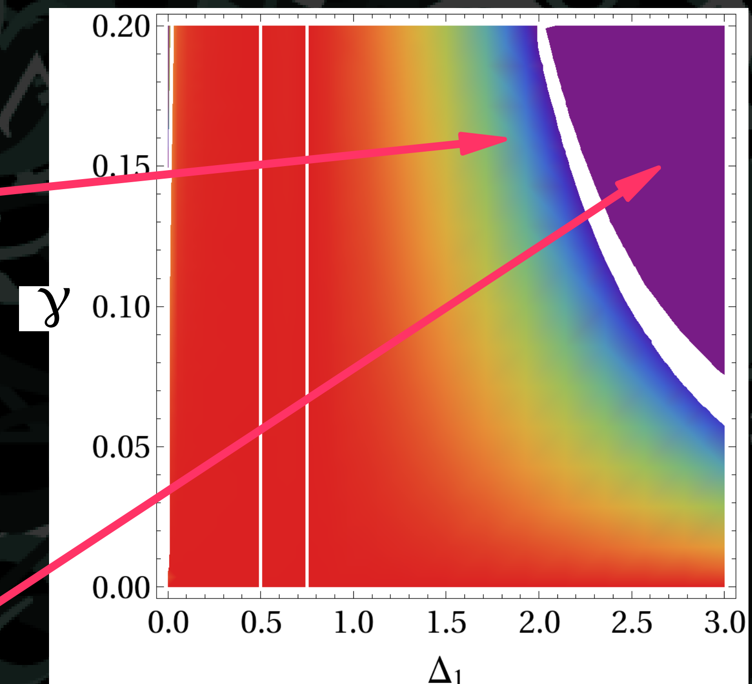
Late times $t \gg t_0$

For $\gamma > \gamma_c$: $\lambda = 0$

Phase diagram for OTOC







Late times $t \gg t_0$

For $\gamma > \gamma_c$: $\lambda = 0$

Appendix: WH teleportation

- What is all this good for? - the meaning of Lyapunov spectra and the relation to teleportation

Quantify the teleportation fidelity by OTOC

- Response of L to manipulation on R (if nontrivial, there is teleportation):

$$\text{response} \equiv \langle U(t) \rangle_T \;=\; \langle e^{-i\epsilon\psi_R(t)} e^{-ig\,\psi_L\psi_R} \psi_L e^{ig\,\psi_L\psi_R} e^{i\epsilon\psi_R(t)} \rangle_T \sim \text{TOC} + g^2 \times \text{OTOC}$$

$$\text{response} \;\approx\; \langle e^{-ig\,\psi_L\psi_R} \psi_L e^{ig\,\psi_L\psi_R} \rangle_T - i\epsilon \langle [\psi_R(t), e^{-ig\,\psi_L\psi_R} \psi_L e^{ig\,\psi_L\psi_R}] \rangle_T + O(\epsilon^2)$$

Manipulate a bit with commutators and BCH

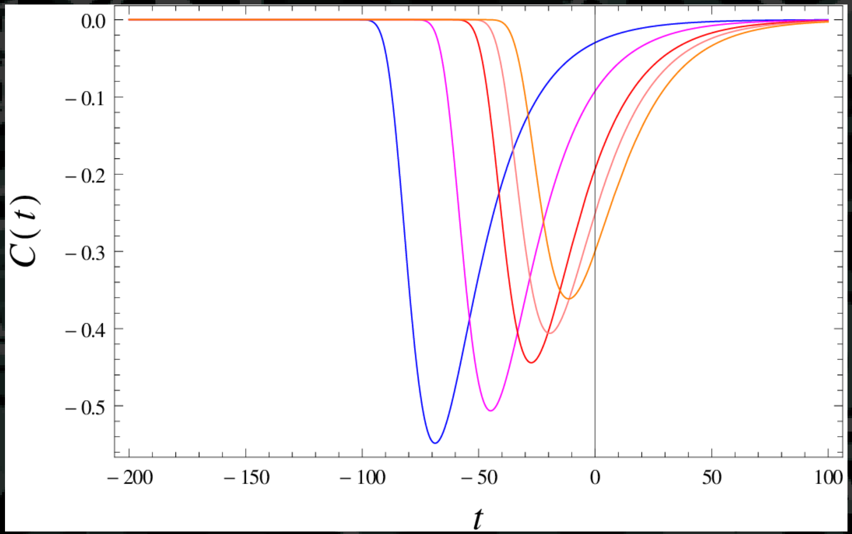
$$g^2 \langle [\psi_R(t), \psi_L(0)] [\psi_R(t), \psi_L(0)] \rangle_T + \ldots$$

OTOC

- Fidelity $\wp$: overlap of the teleported R state with the L state

$$\wp = <0| e^{-i\epsilon\psi_R} U(t) |0>_T = \int d\Delta_\psi \left(\text{TOC}(\Delta_\psi; t) + g^2 \text{OTOC}(\Delta_\psi; t) \right)$$

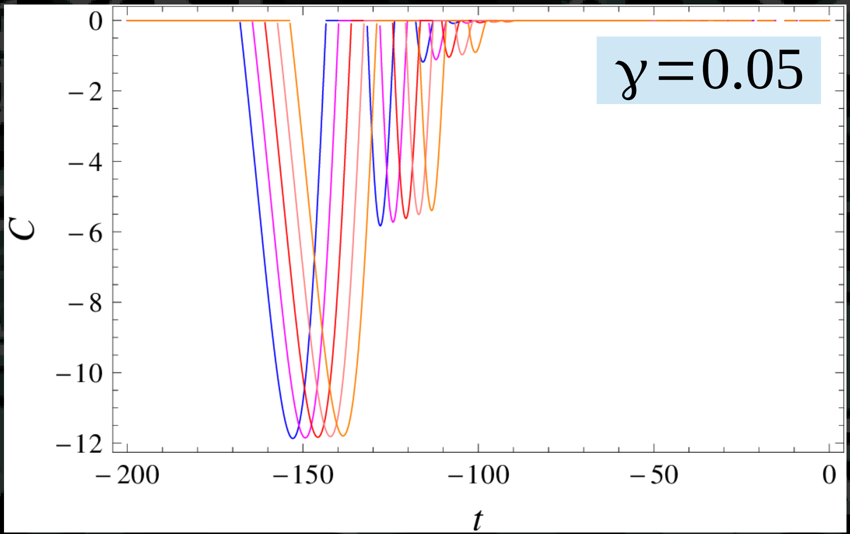
Scrambling for a quick WH



Decay rate linear in T, preserved time order of signals

Teleportation protocol of Gao and Jafferis 2019 [1911.07416]

Scrambling for a slow WH



Decay rate exponentially slow, mixed time order of signals

Reproduced SYK model result of Nosaka and Numusawa 2020 [2009.10759] $\lambda \propto e^{-1/T}$

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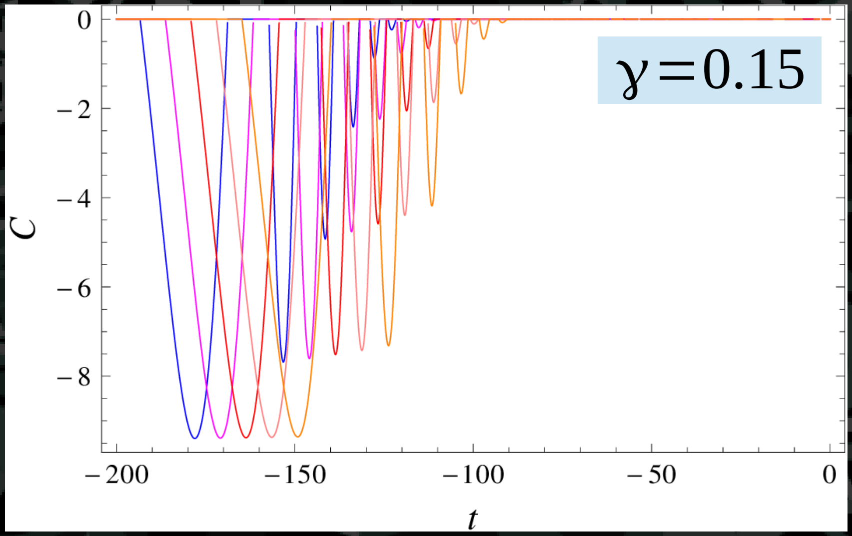
Scrambling for a slow large WH



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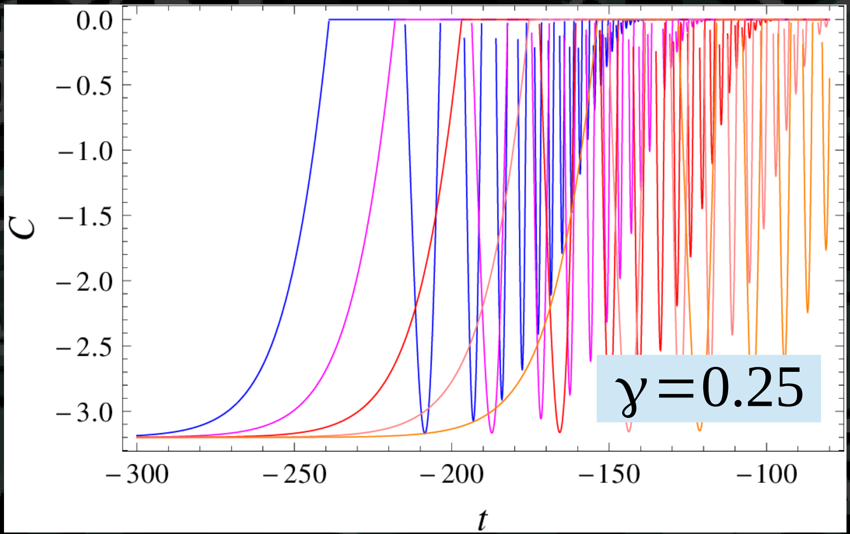
Reproduced SYK model result of Nosaka and Numusawa 2020 [2009.10759] $\lambda \propto e^{-1/T}$

Teleportation protocol of Gao and Jafferis 2019 [1911.07416]

Scrambling for a slow very large WH



Infinitely dense peaks, power-law envelope determines the scrambling time

OTOC ~ $1/t^{2\Delta}$ $\Longrightarrow$ $\lambda \propto e^{-1/T}$

Teleportation protocol of Gao and Jafferis 2019 [1911.07416]

Catch-22: $\wp \sim \mathrm{TOC} + \gamma^2 \exp(2\lambda t) \rightarrow \mathrm{TOC} + \gamma^2 \mathrm{const.}$

Either lo-fi (small γ) or scrambling slooow (large γ)

Conclusions

- Spectrum of Lyapunov exponents for WH (and likely many other complex geometries): there is more to OTOC than just a single exponent!
- If there ever was a BH horizon, it implies exponential OTOC and exponent linear in T!
- Hi-fi wormhole teleportation is fishy!