A Momentum/Complexity Correspondence

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Iberian Strings 2021, IST Lisboa

Based on **1912.05996** / **2006.06607** / **2012.02603**

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Introduction

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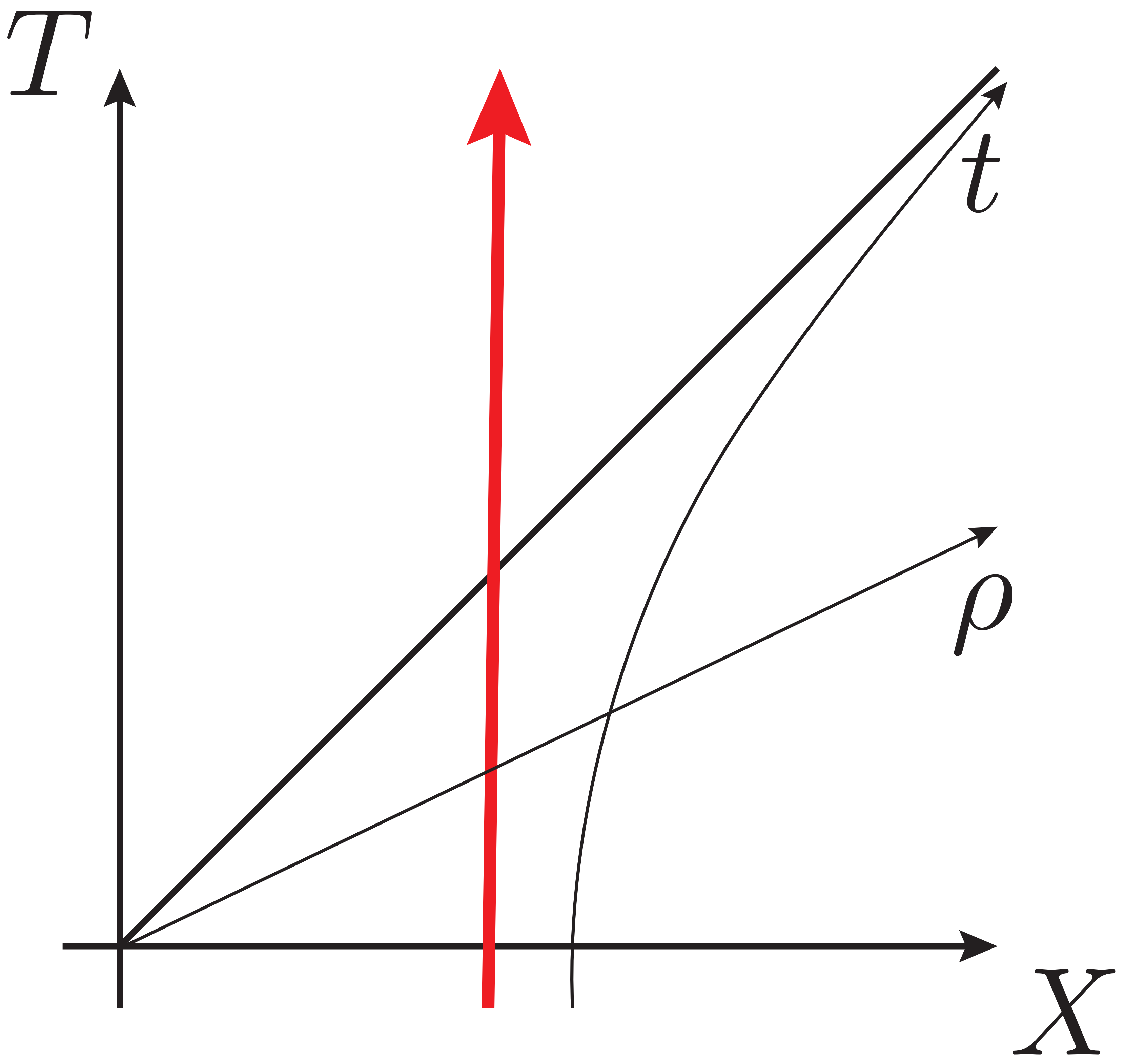

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$$\lambda = \frac{2\pi}{\beta}$$ (maximal Lyapunov exponent)

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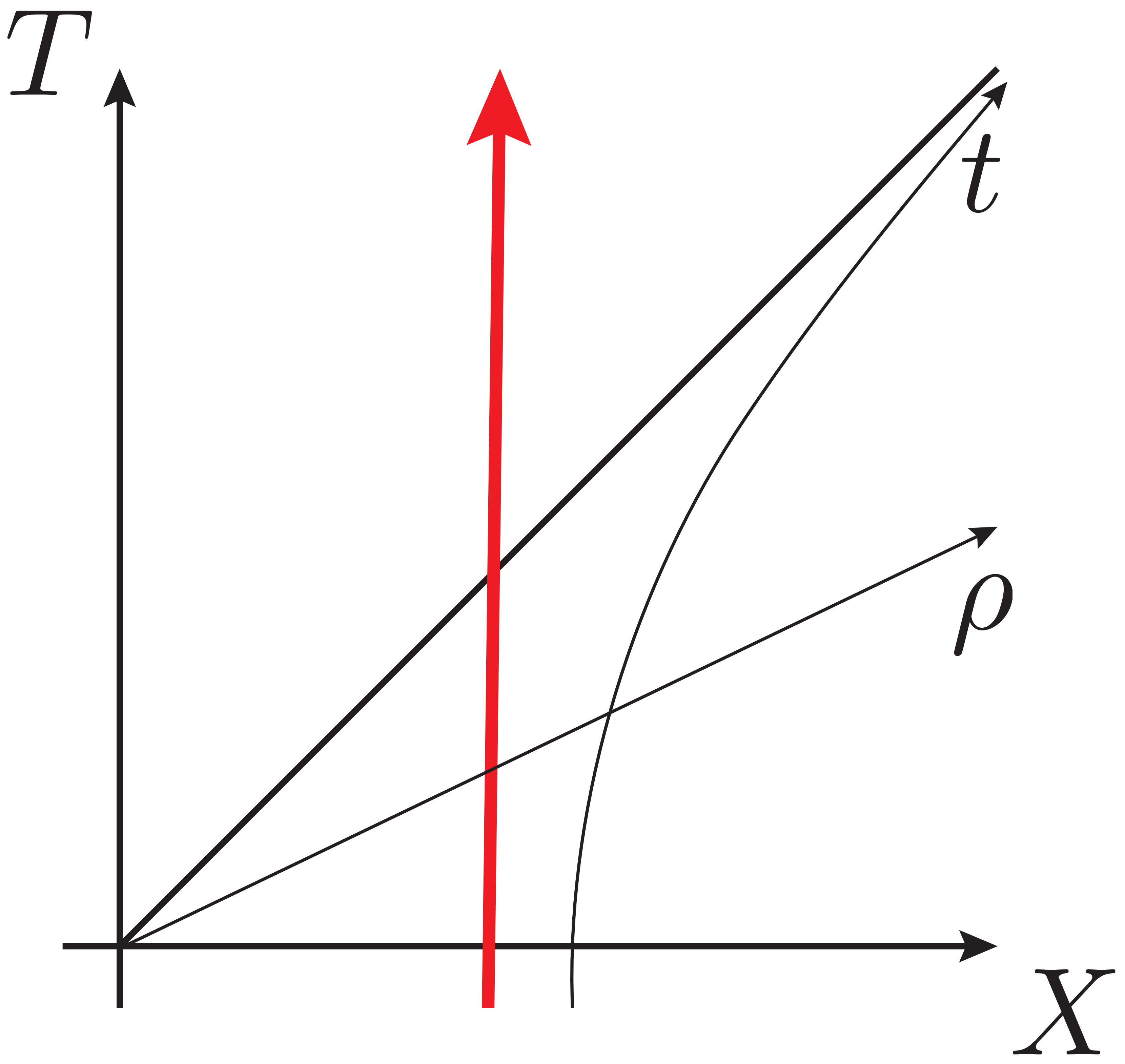

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$$\boxed{P_\rho \leftrightarrow \text{size}}$$

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- The correspondence was refined studying the motion of a particle falling through the throat of a near extremal RN black hole $\leftrightarrow$ Comparison with SYK.

[Brown, Gharibyan, Streicher, Susskind, Thorlacius, Zhao '18] [Susskind '19] [Maldacena, Lin, Zhao '19]

[Roberts, Stanford, Streicher '18] [Qi, Streicher '19]

$$P \approx \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{size}$$

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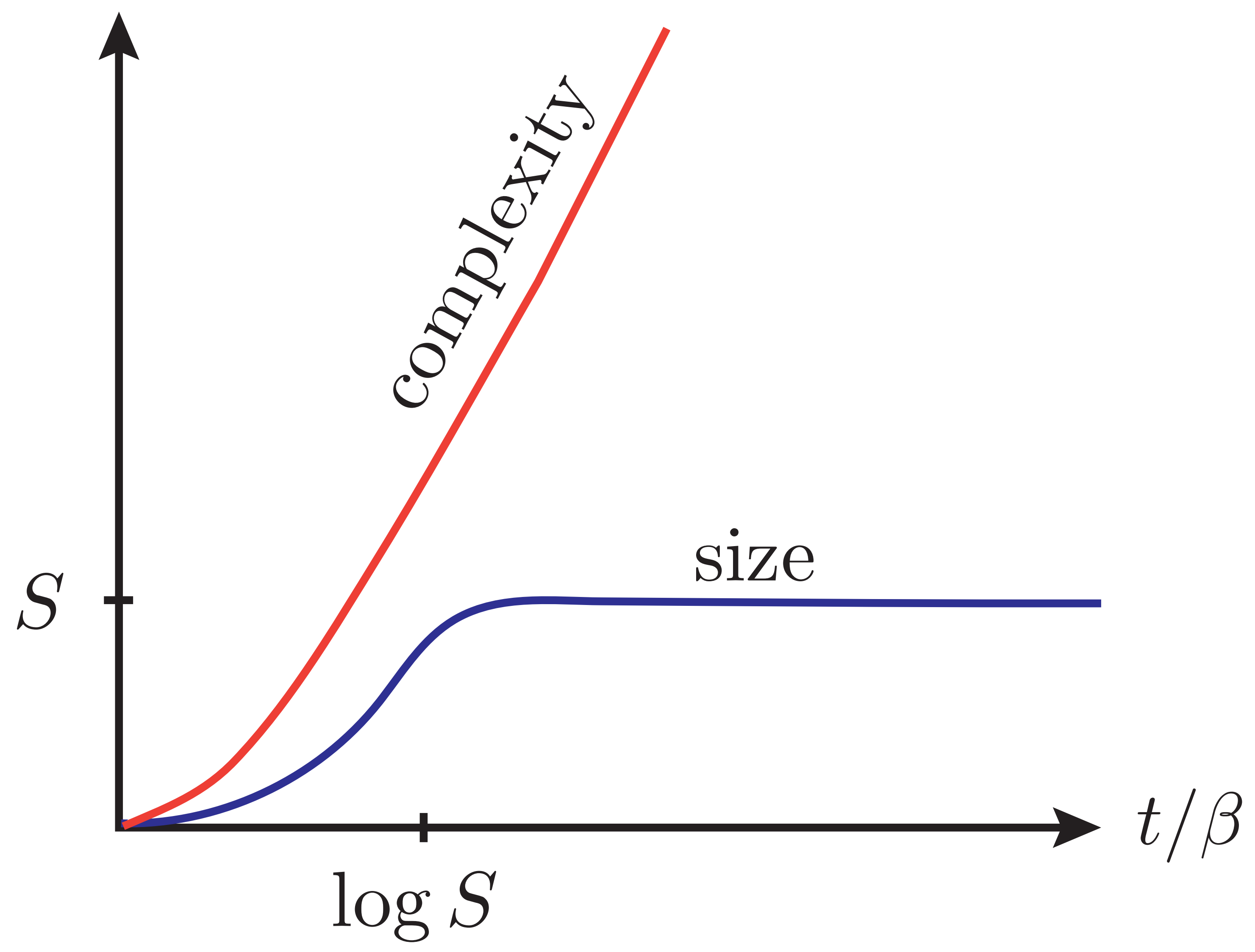

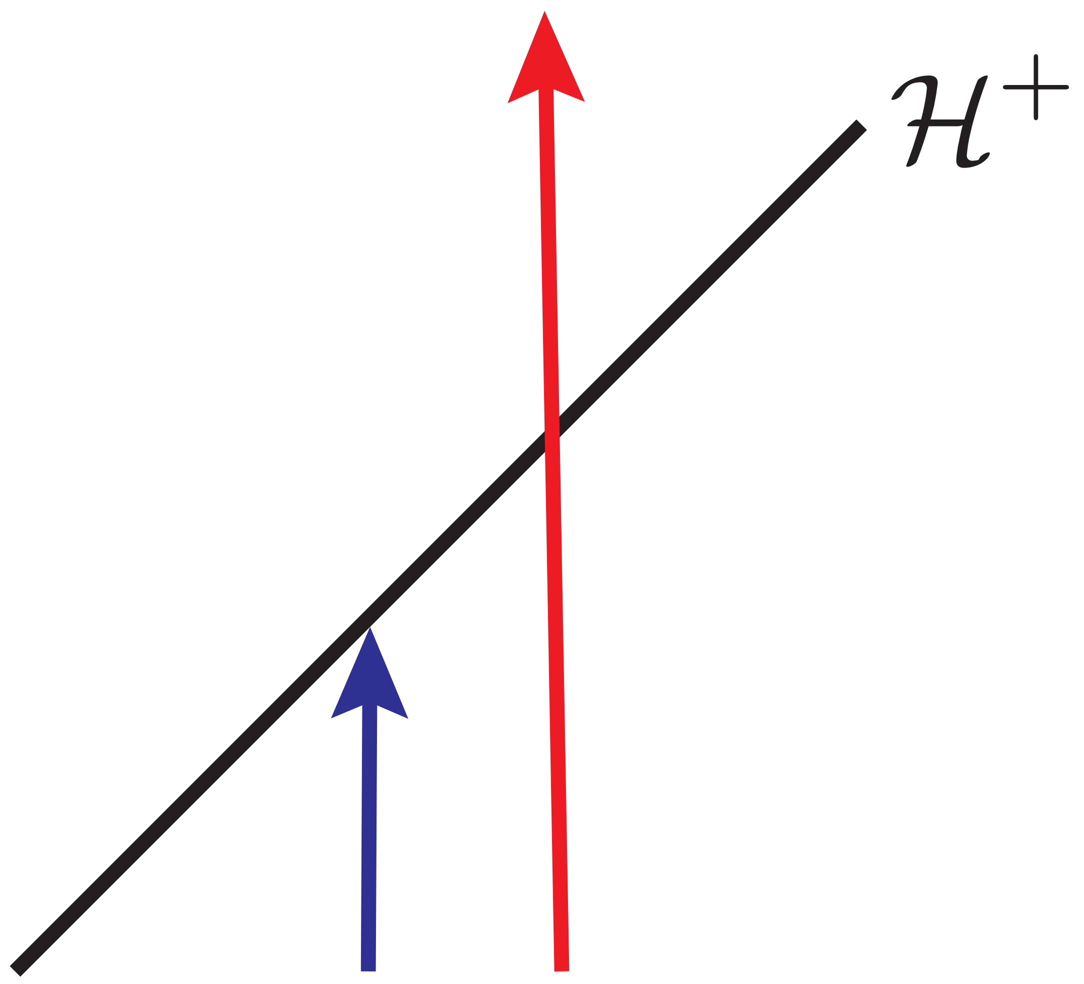

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- If the tendency for complexity to increase is related to the mechanism behind gravity, then there must be a more general correspondence outside the black hole context.

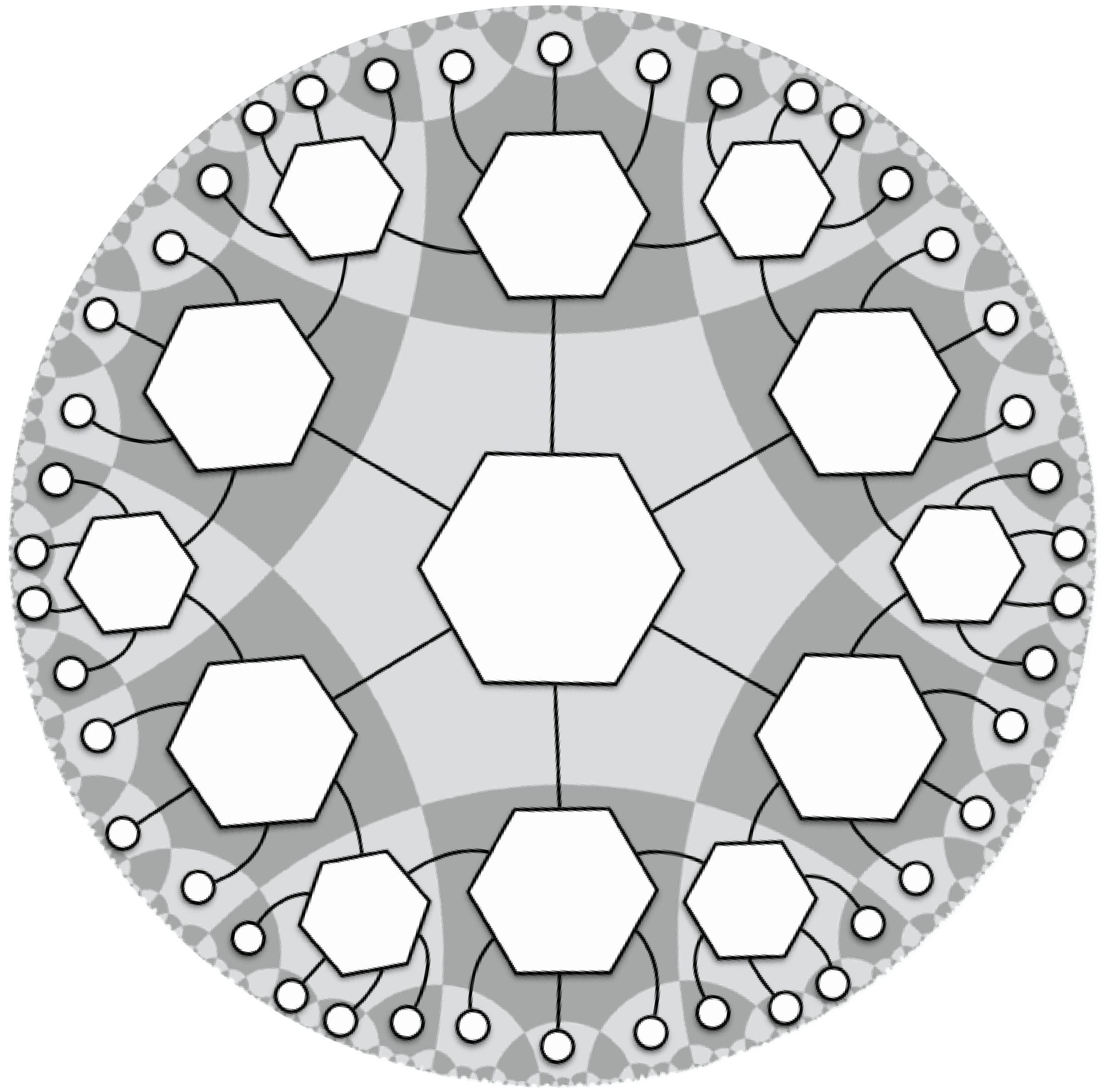
Complexity = Volume

- Consider a holographic CFT on $\mathbf{S}^{d-1}$ and a state $|\Psi_t\rangle$ with geometric dual.

e.g. $$|\Psi_t\rangle = e^{-iHt}\,\mathcal{O}_{\text{simple}}\,|0\rangle$$

- The **complexity** of $|\Psi_t\rangle$ is given by the volume (in units of $G\,\ell$) of the extremal spatial hypersurface Σ_t anchored to the corresponding timeslice of ∂AdS.

$$\mathcal{C}\left(|\Psi_t\rangle\right) \propto \frac{\text{Vol}[\Sigma_t]}{G\ell}$$



Variational Problem

- Extremal volume rate is described by a Hamilton-Jacobi equation.

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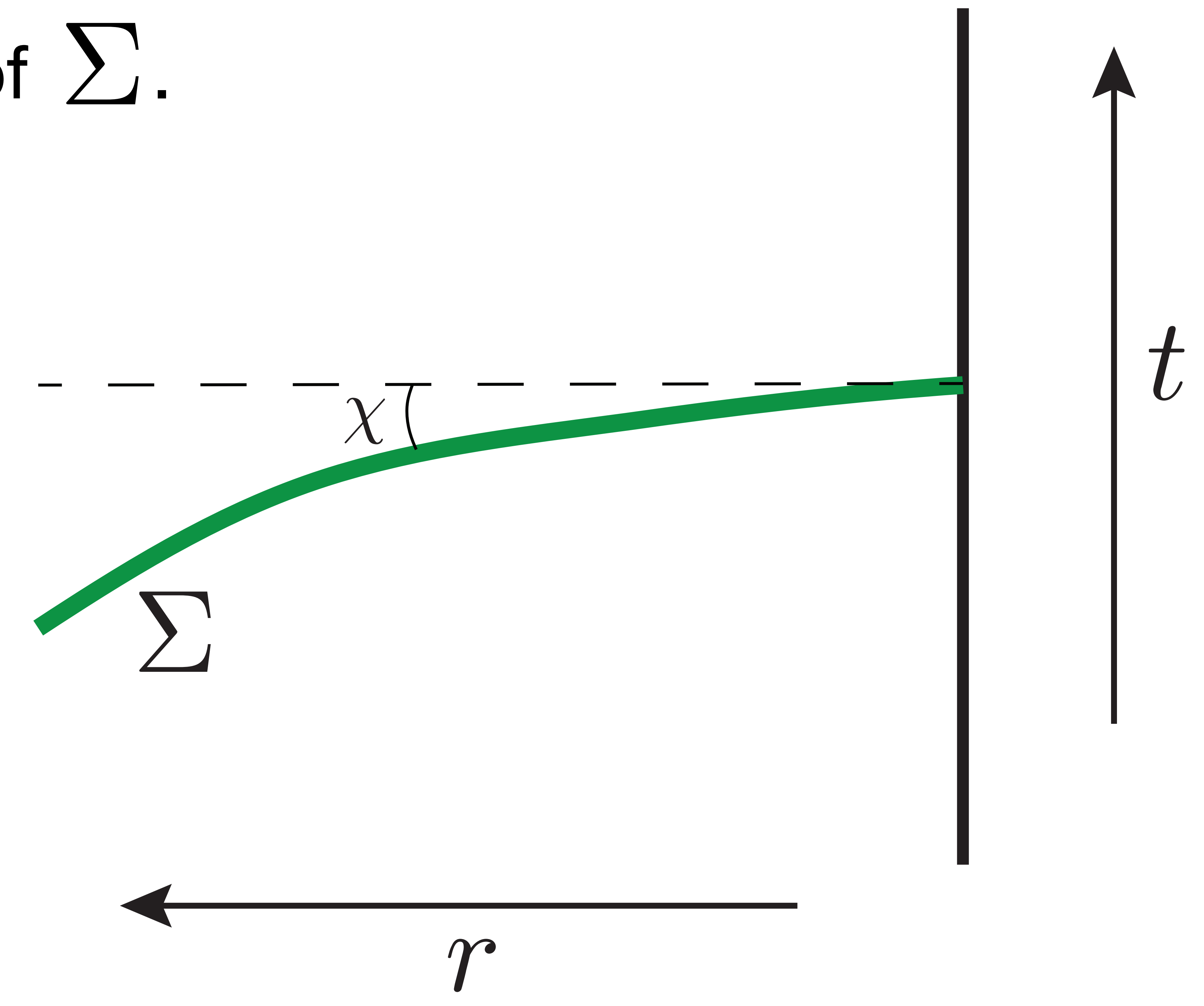
- The (d-1)-form $\boldsymbol{\omega}$ captures the asymptotic bending of Σ.

$$\boldsymbol{\omega} = \left(n_{\partial\Sigma} \cdot \partial_t\right) \boldsymbol{\epsilon}$$



- The strategy is to find an extension $\boldsymbol{j}$ on Σ such that $\mathrm{d}\boldsymbol{j} = \boldsymbol{\mathcal{P}}$.

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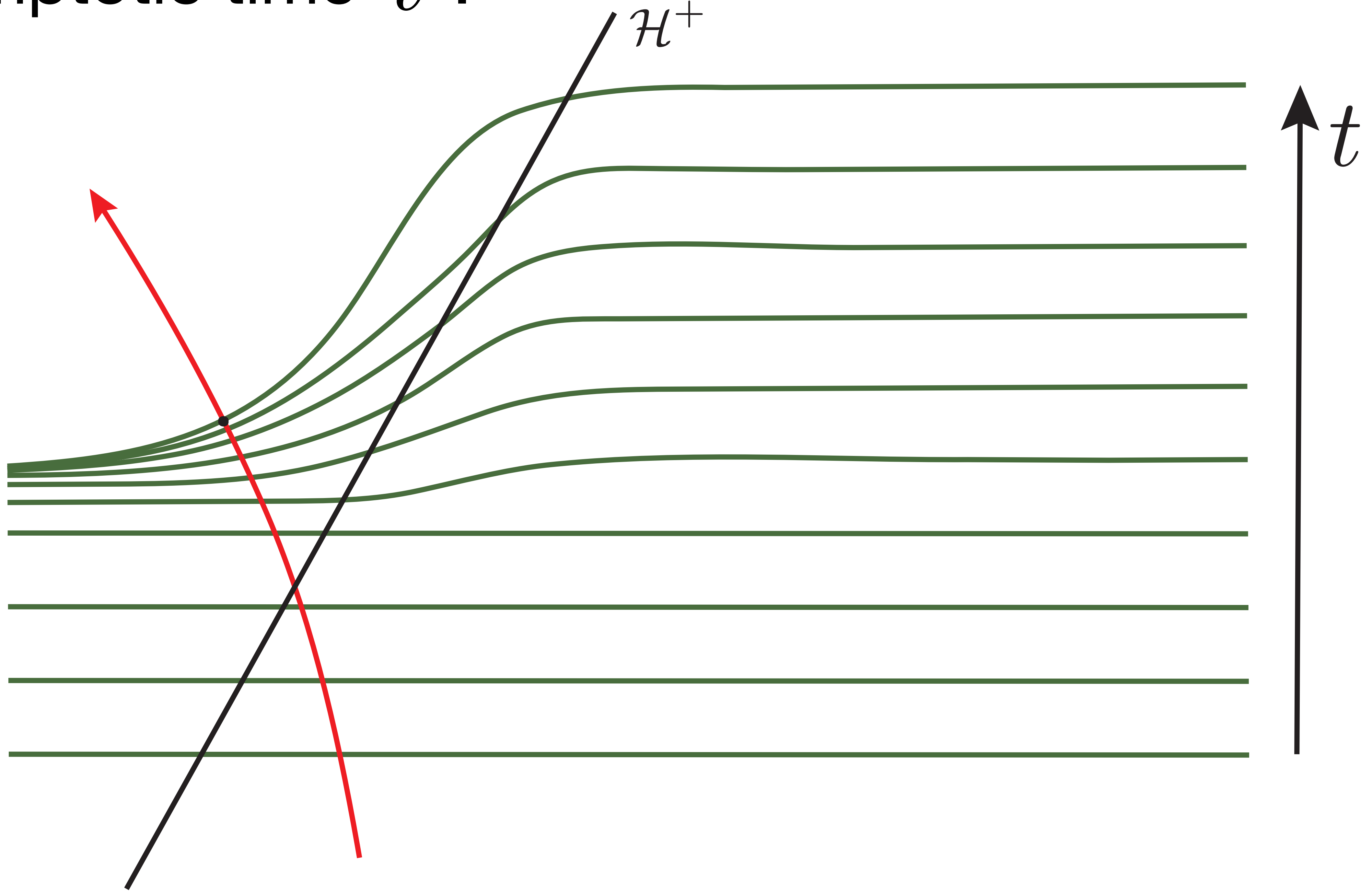
- By extending onto Σ, we implicitly specified the bulk time variable to talk about momentum. This time variable is given by foliating spacetime with extremal slices.

Black Hole Interior

- Extremal slices explore the black hole interior. They accumulate exponentially fast with respect to the asymptotic time t .



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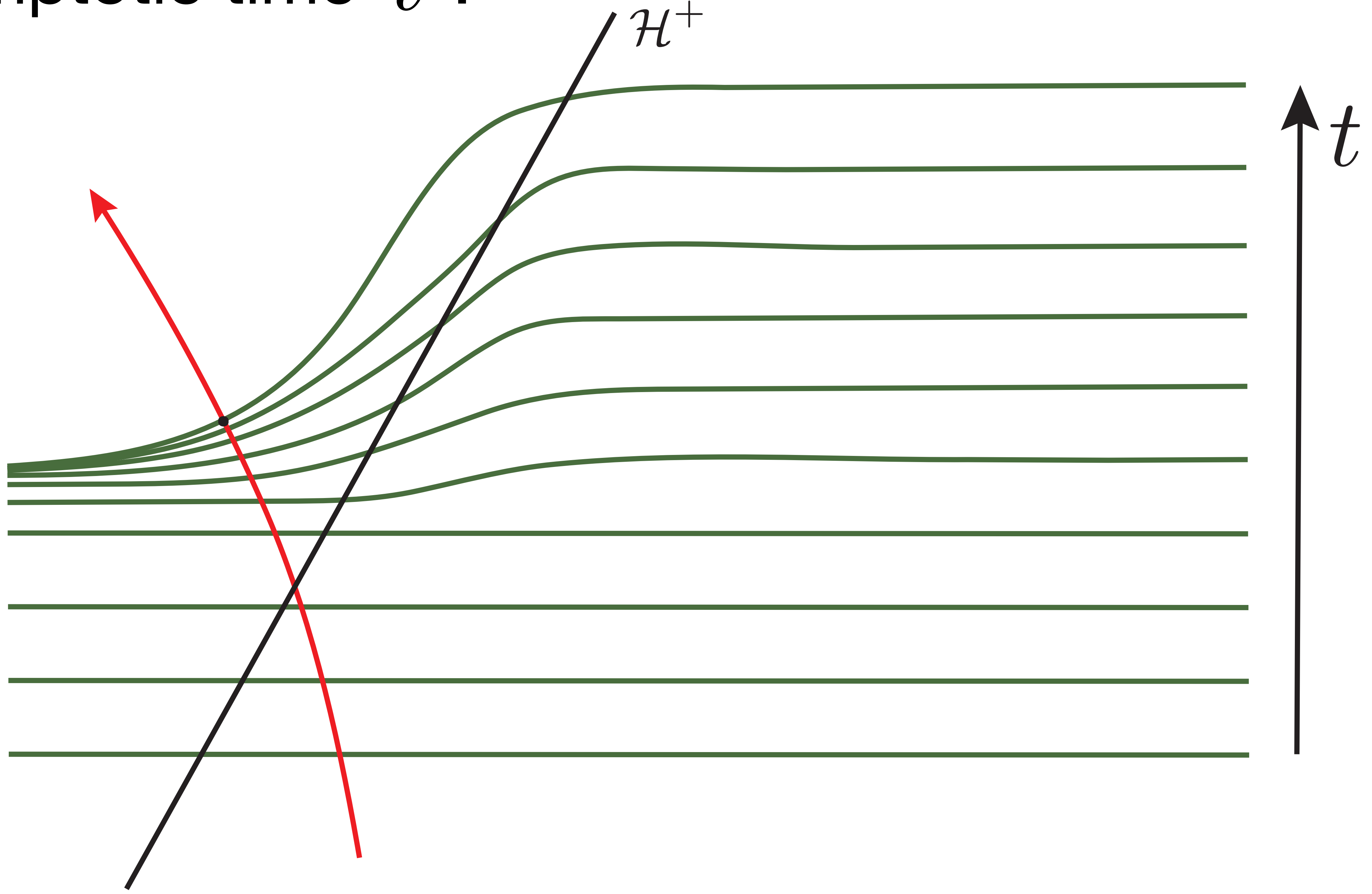

- Using a local notion of momentum that is smooth $P_{\mathcal{C}}(\tau)$, then the quantity $P_{\mathcal{C}}(t)$ must freeze because proper time $\tau(t)$ gets frozen in this foliation.

$$P_{\mathcal{C}} \sim \text{constant}$$

Spatial component of Momentum

- **Theorem**: Given a tangent vector field C^μ which asymptotically is:

  - radial
  - inward pointing
  - of magnitude r

$$C^\mu \sim -r\, e_r^\mu$$

(measures asymptotic radial infall)

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then the **PVC correspondence** holds

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C} = \int_\Sigma \boldsymbol{\mathcal{P}}_C + \int_\Sigma \boldsymbol{R}_C}$$

where

$$\boldsymbol{\mathcal{P}}_C = -N^\mu_\Sigma\, T_{\mu\nu}\, C^\nu\, \boldsymbol{e} \qquad \boldsymbol{R}_C = -\frac{1}{8\pi G}\, K_{\mu\nu}\, \nabla^\mu C^\nu\, \boldsymbol{e}$$

Exact PVC correspondence

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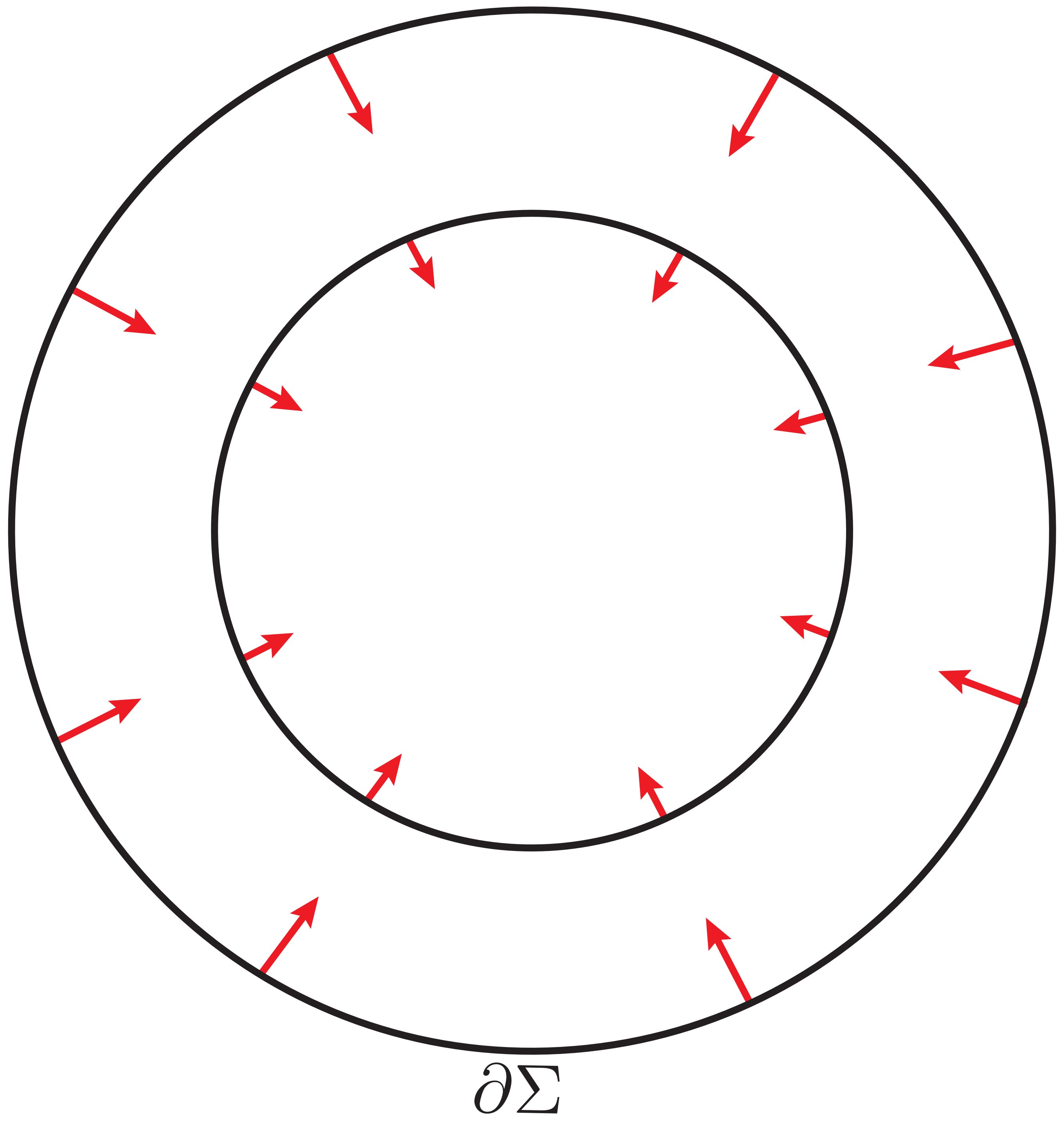
$$\mathrm{d}s^2_\Sigma = \mathrm{d}y^2 + r^2(y)\mathrm{d}\Omega^2_{d-1}$$

$$C = -r(y)\,\partial_y$$



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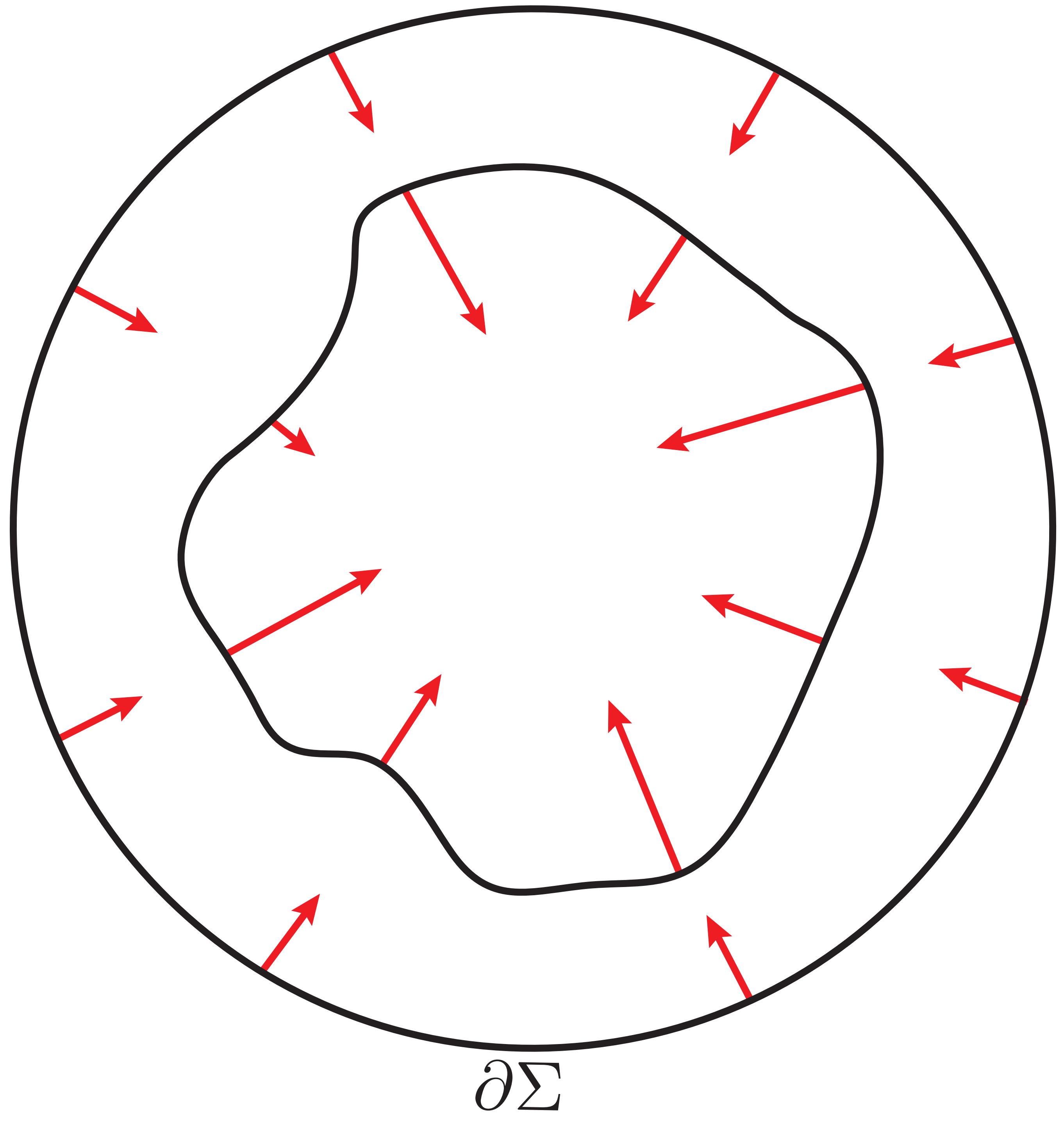
- For any state in 2+1 dimensions, there is a candidate conformal Killing vector.

$$\mathrm{d}s^2_\Sigma = e^{2\omega} \left(\mathrm{d}y^2 + \sinh^2(y)\,\mathrm{d}\phi^2\right)$$

$$C = -\sinh(y)\,\partial_y$$



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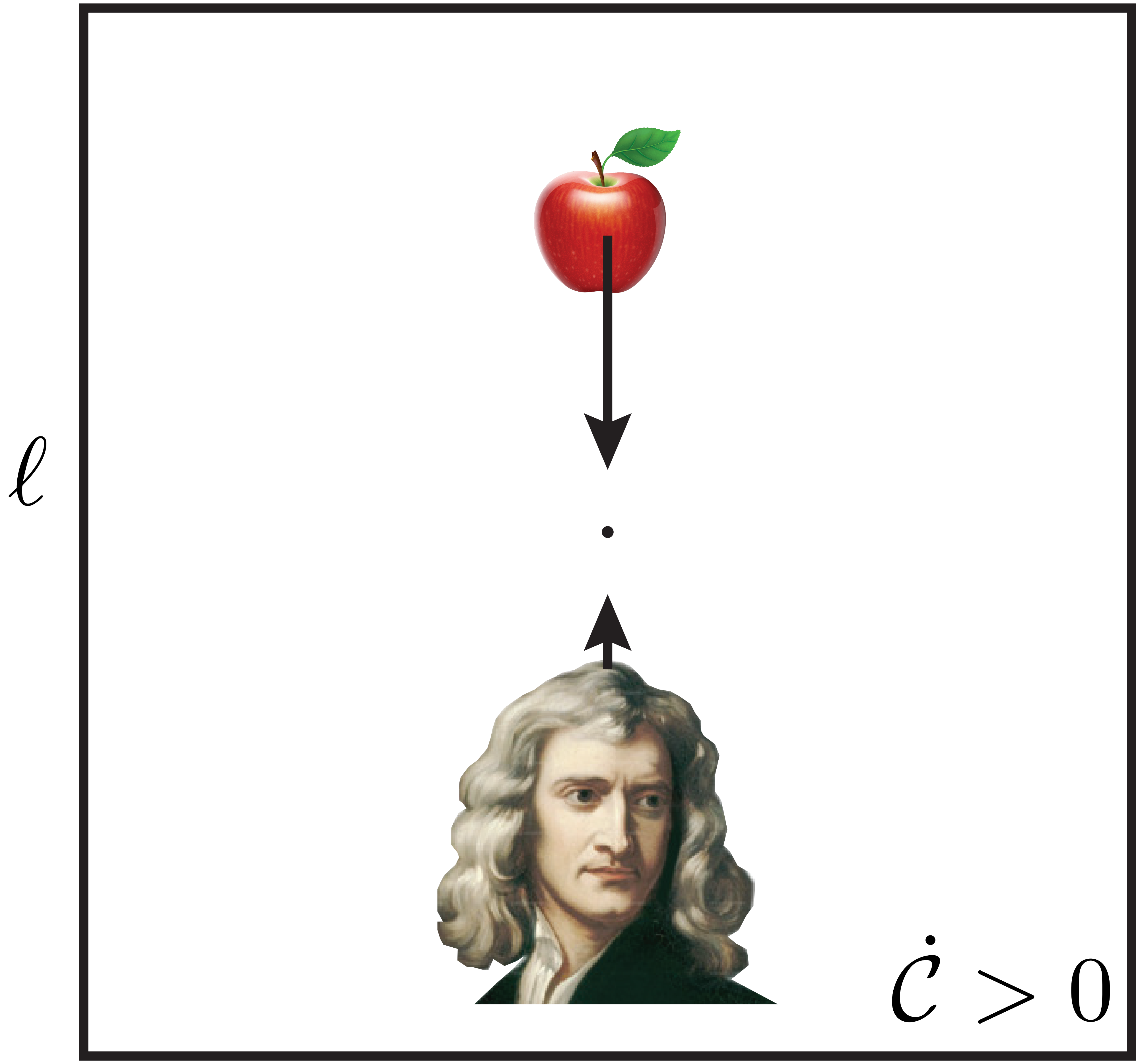
- The relevant momentum in both cases measures **infall** towards the center of the box. This formalizes the idea that gravitational clumping of matter increases complexity.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C} = \int_\Sigma \boldsymbol{\mathcal{P}}_C = P_{\text{infall}}$$

Newtonian limit

- Consider a system of non-relativistic particles in the deep interior of AdS. To leading order in the backreaction, the rate of complexity is given by the infall momentum.



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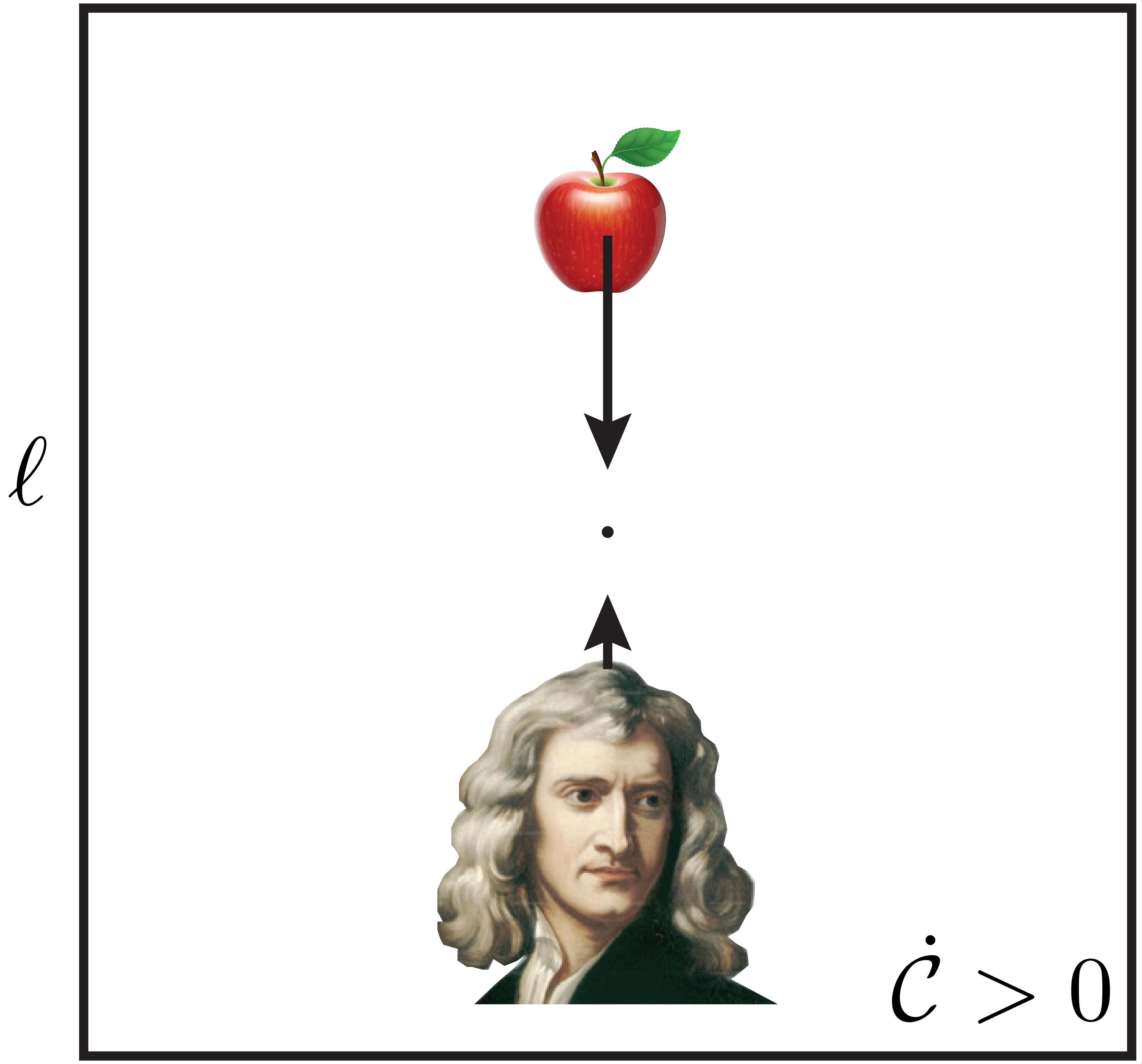

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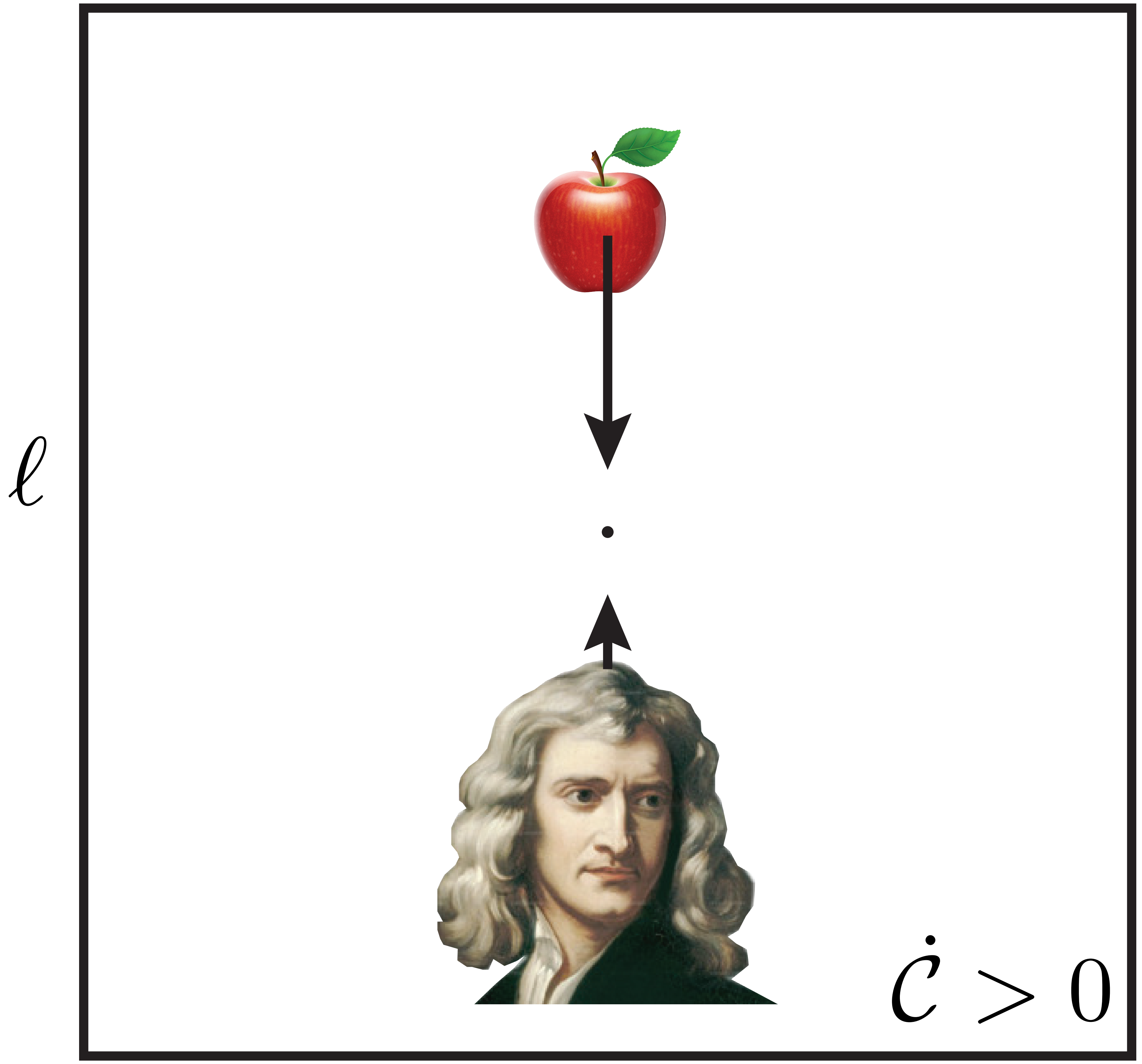

$$P_{\text{infall}} = -\frac{1}{\ell} \sum_i \mathbf{x}_i \cdot \mathbf{p}_i$$

- There is a center of infall set by the box. (Mach's principle)

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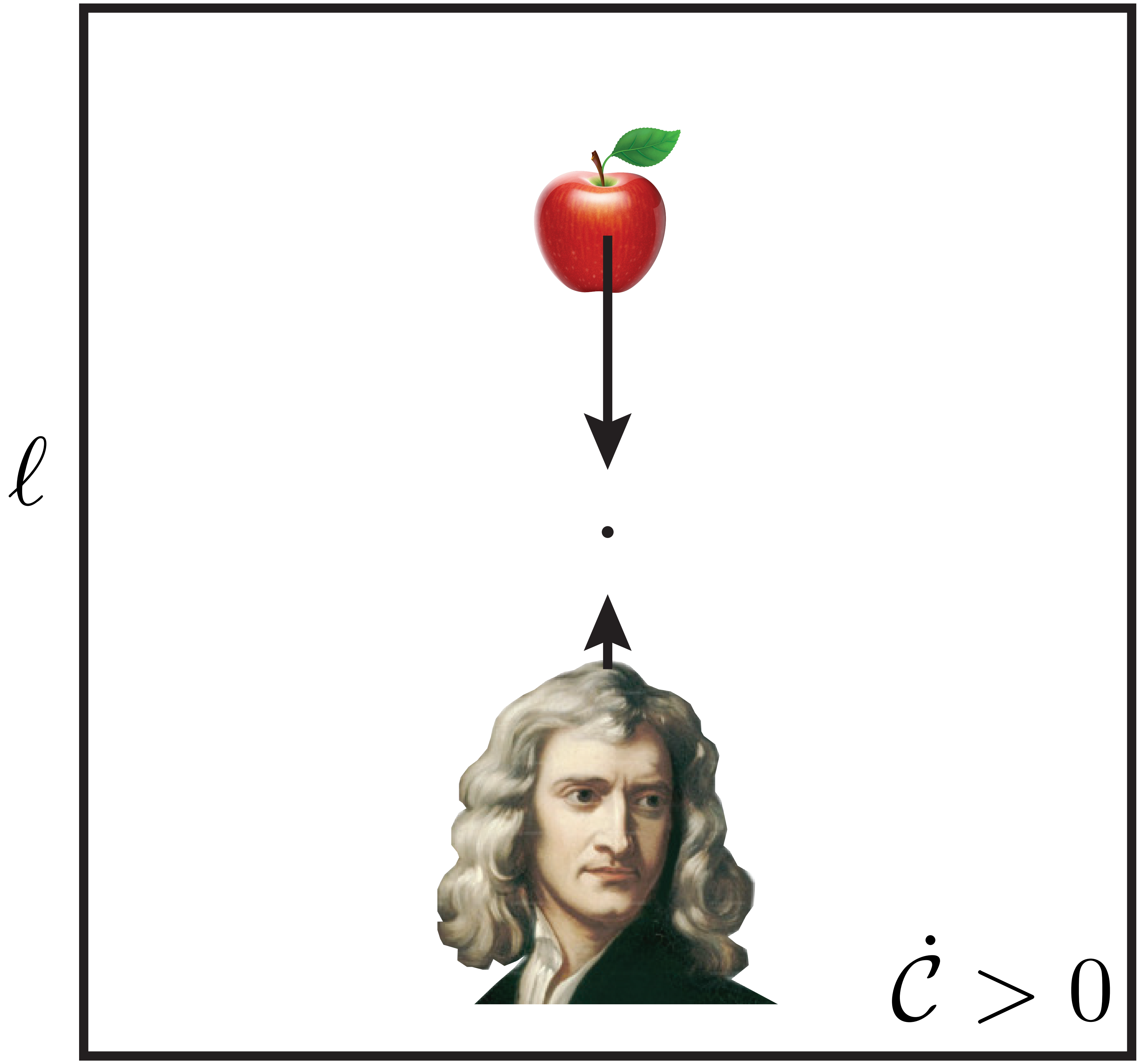

$$P_{\text{infall}} = -\frac{1}{\ell} \sum_i \mathbf{x}_i \cdot \mathbf{p}_i$$

- There is a center of infall set by the box.
- Complexity is a radial moment of inertia.

$$\mathcal{I}_{\text{clump}} = -\frac{1}{2\ell} \sum_i m_i \, \mathbf{x}_i^2$$

$$\mathcal{C} = \mathcal{C}_0 + \mathcal{I}_{\text{clump}}$$

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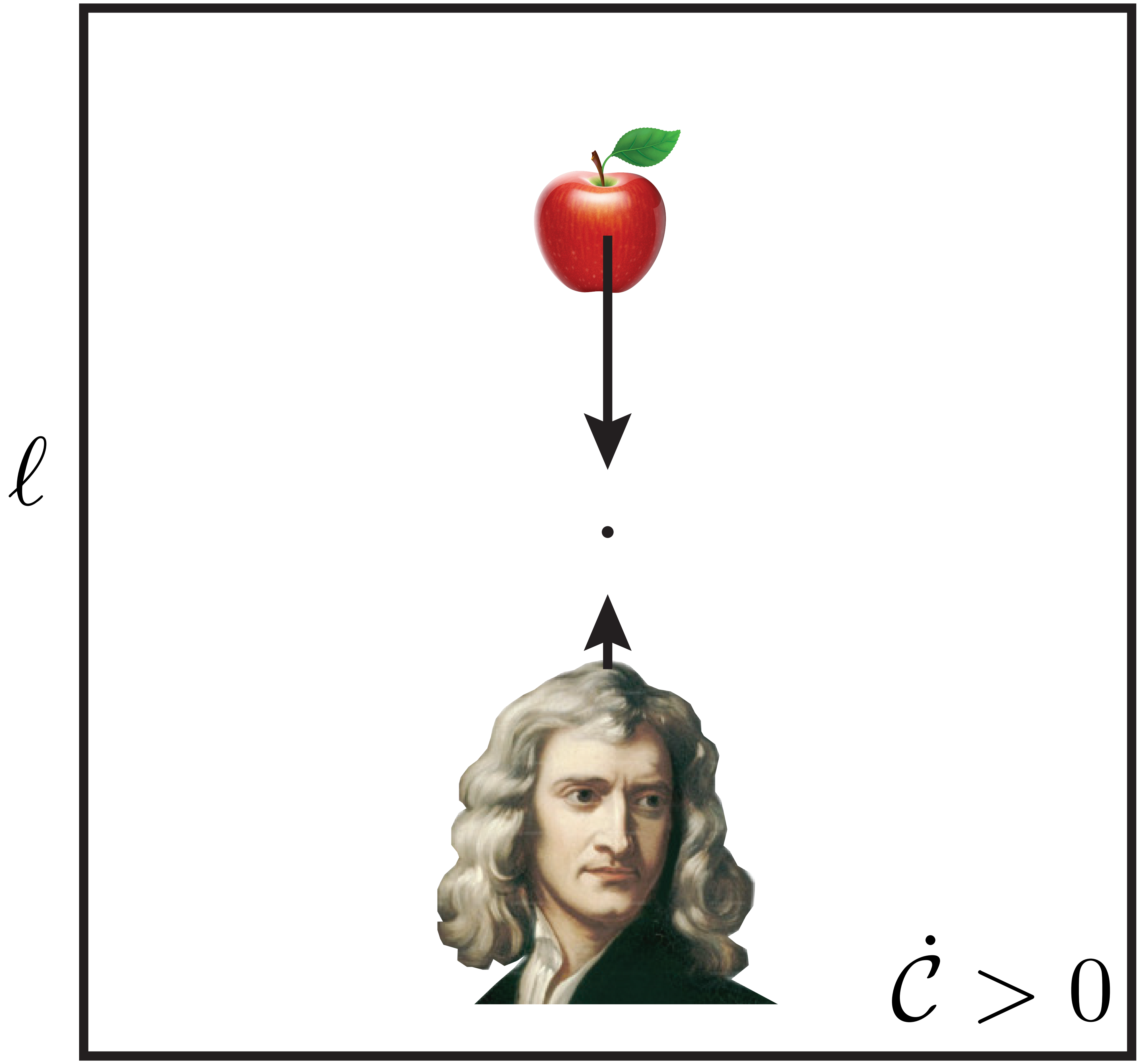

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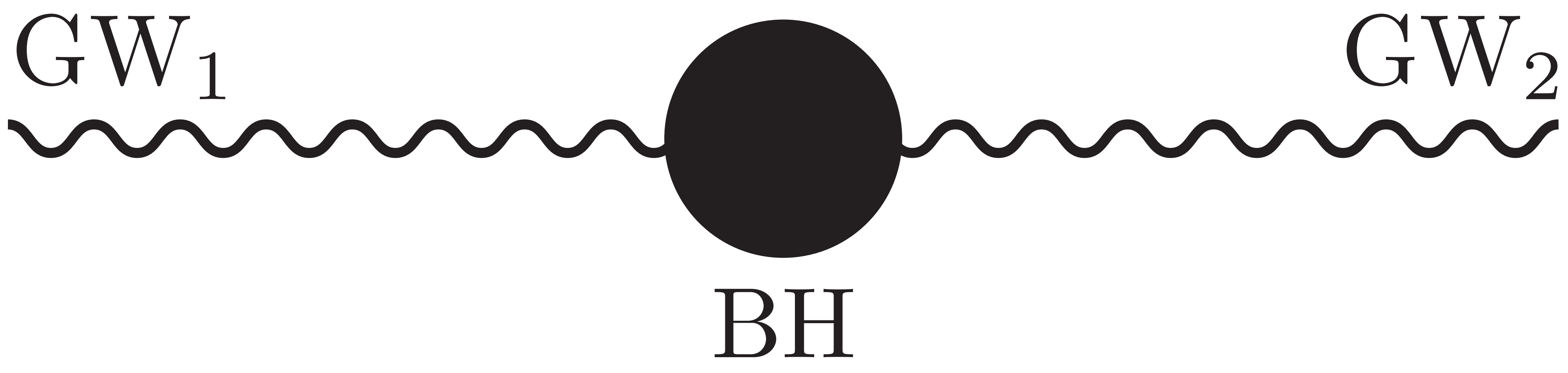
- There is a center of infall set by the box.
- Complexity is a radial moment of inertia.
- For an unbound gravitational system

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{C} < 0$$

Obstruction for exact PVC: gravitational radiation

- For vacuum one-sided geometries, the whole complexity rate is given by the remainder.



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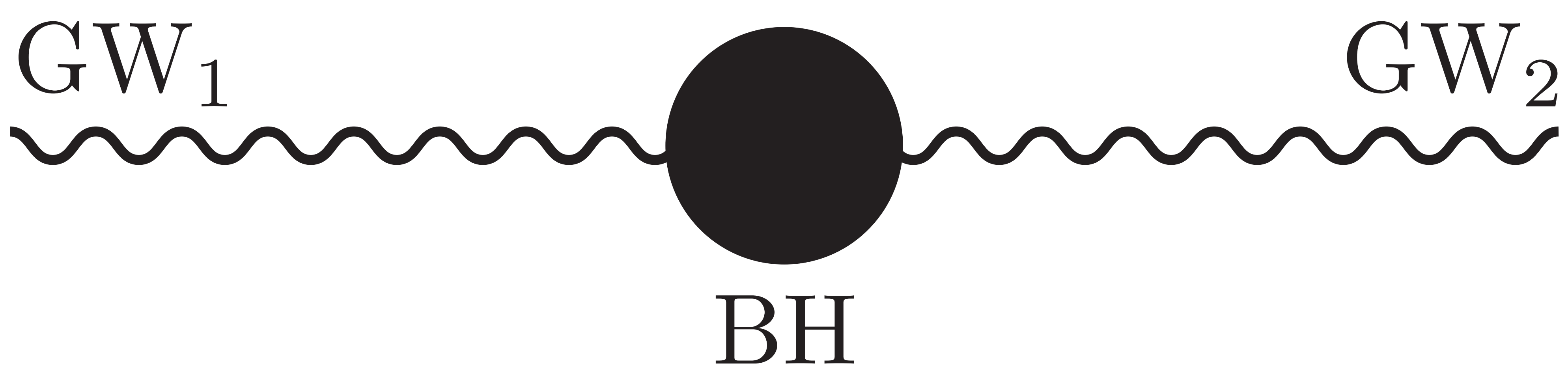

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- No candidate for exact momentum of gravity. Perturbatively, we expect that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C} \approx \int_{\Sigma_0} \boldsymbol{\mathcal{P}}_{\mathrm{grav}} \qquad\qquad \boldsymbol{\mathcal{P}}_{\mathrm{grav}} = -N^{\mu}_{\Sigma_0}\, t_{\mu\nu}\, C^{\nu}_0$$

for some notion of conserved energy-momentum pseudotensor (Landau-Lifshitz or others).

Generalization

- **Theorem**: Given a tangent tensor field $M^{\mu\nu\rho}$ which asymptotically is

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then the **generalized PVC correspondence** holds

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C} = \int_{\Sigma} \boldsymbol{\mathcal{P}}_C + \int_{\Sigma} \boldsymbol{\mathcal{W}}_M + \int_{\Sigma} \boldsymbol{R}_M}$$

where $C^{\mu} = h_{\nu\rho} M^{\nu\mu\rho}$ and

$$\boldsymbol{\mathcal{W}}_M = -\frac{1}{16\pi G} N^{\mu}_{\Sigma} W_{\mu\nu\rho\sigma} M^{\nu\rho\sigma}$$

- The idea is that now $\boldsymbol{R}_M = 0$ is always possible in one-sided situations because of the extra components of the infall tensor.

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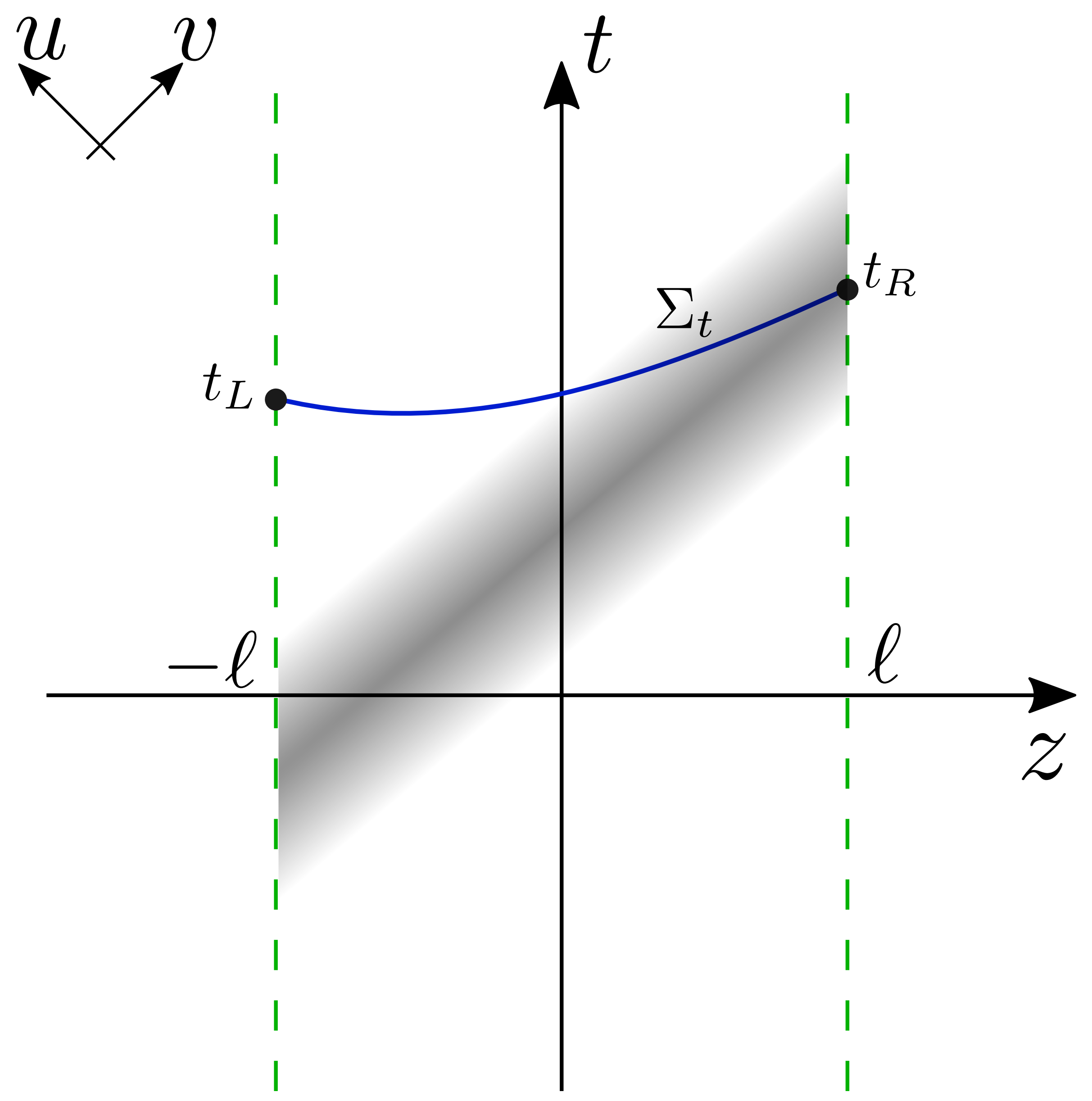
- We checked that the generalized PVC correspondence works for a **pp-wave spacetime**.

$$\mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v + L^2(u)\left(e^{2\beta(u)}\,\mathrm{d}x^2 + e^{-2\beta(u)}\,\mathrm{d}y^2\right)$$



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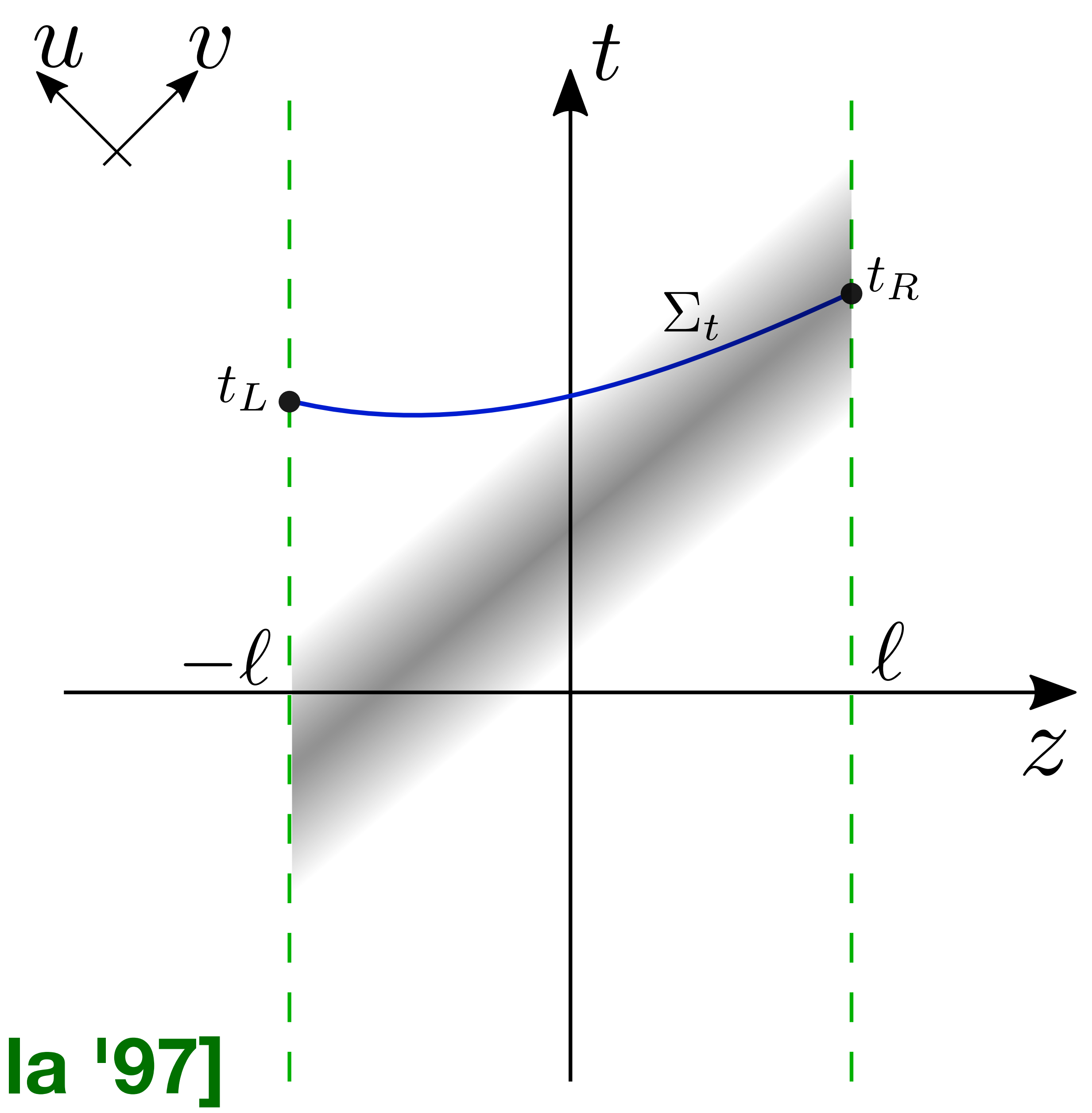

and the "Weyl-momentum"

$$\boldsymbol{\mathcal{W}}_M = -N^\mu_\Sigma\, t_{\mu\nu}\, \tilde{C}^\nu$$

for $t_{\mu\nu}$ the square root of the Bel-Robinson tensor. **[Bonilla, Senovilla '97]**

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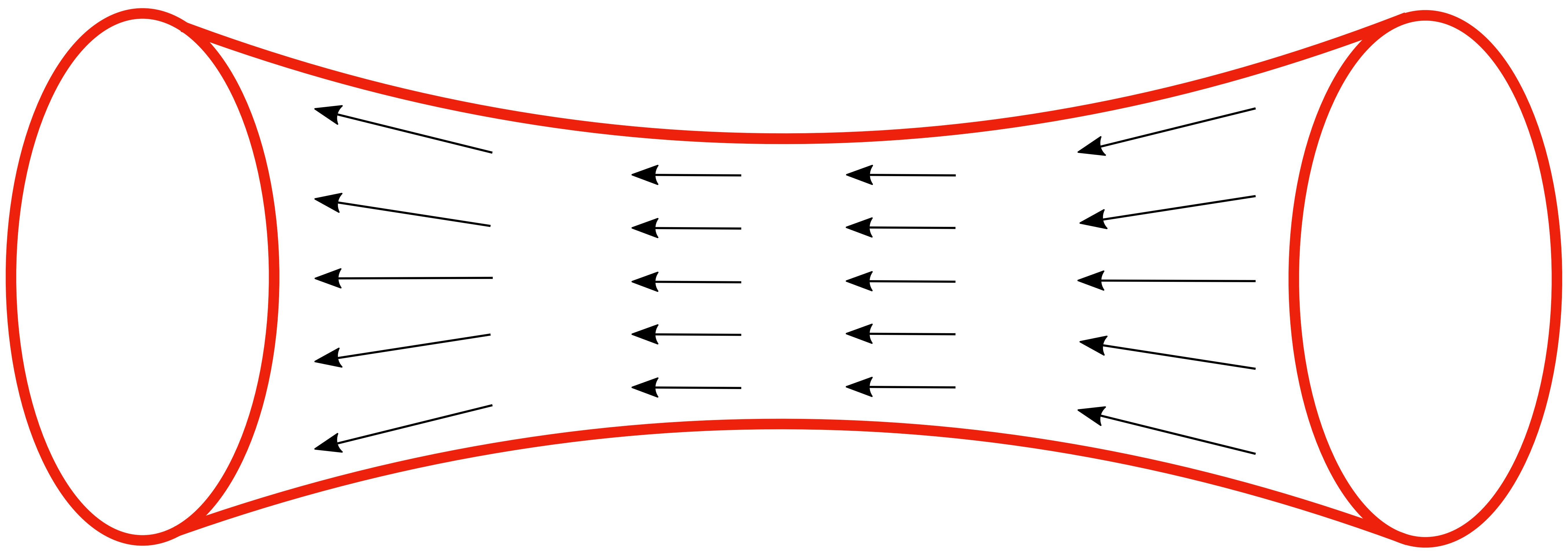
- Another obstruction is topology. There is an intrisic volume growth of ERB that cannot be attributed to matter infall nor to the gravitational radiation.
- It seems reasonable to expect that there are inequalities for the second derivative of VC if we assume energy conditions for matter.
- More speculatively, the second law of complexity is the statistical tendency for matter to gravitationally clump together. For that we need to understand what VC really is.

Obrigado !

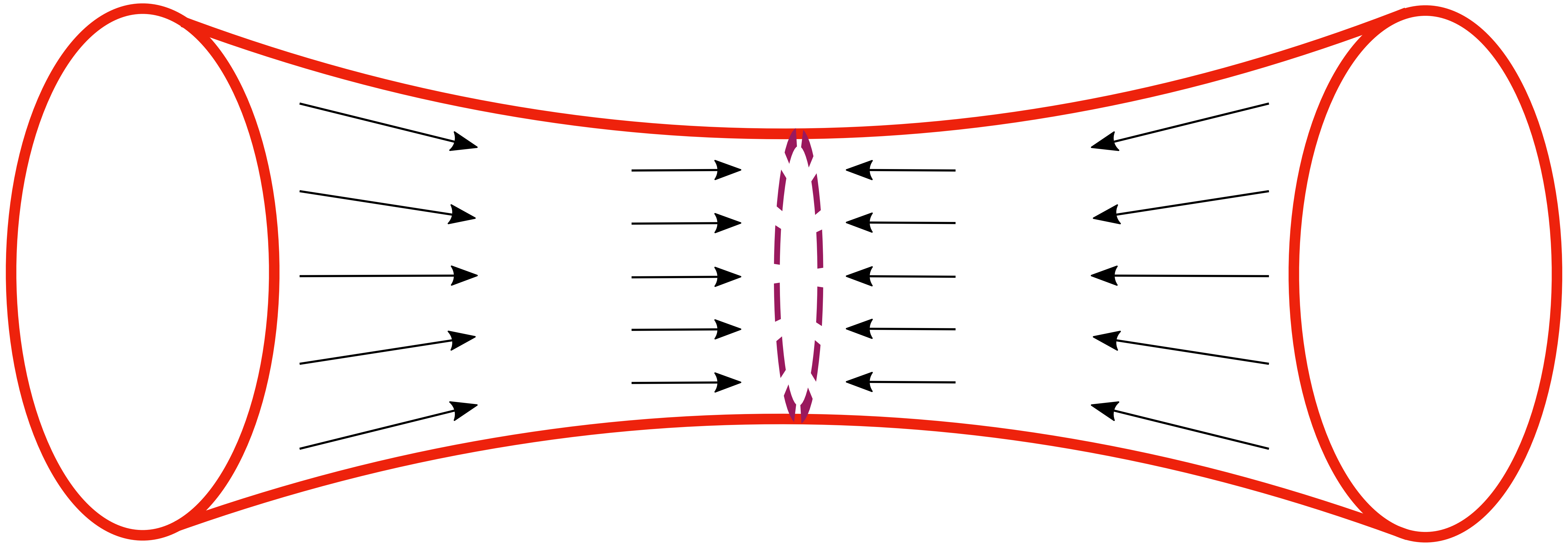
Topological Obstruction

- Consider two holographic CFTs with Hamiltonian $H_L + H_R$. For product states, everything works the same way since $\mathcal{C} = \mathcal{C}_L + \mathcal{C}_R$.
- For entangled states representing classical wormholes, left and right infall are incompatible for the conformal Killing vector.



- Exact PVC holds for the Killing Hamiltonian $H_R - H_L$.

- For $H_L + H_R$, one can introduce a topological defect that swaps the direction of infall. The only contribution to the rate of complexity comes from the remainder on the defect.



$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C} = \int_{\mathrm{defect}} \boldsymbol{R}_C \sim 2M$$