Information geometry and quantum field theory

Johanna Erdmenger

Julius-Maximilians-Universität Würzburg

AdS/CFT correspondence:

Duality mapping QFT (without gravity) to gravity theory

Original motivation from string theory (Maldacena '97):

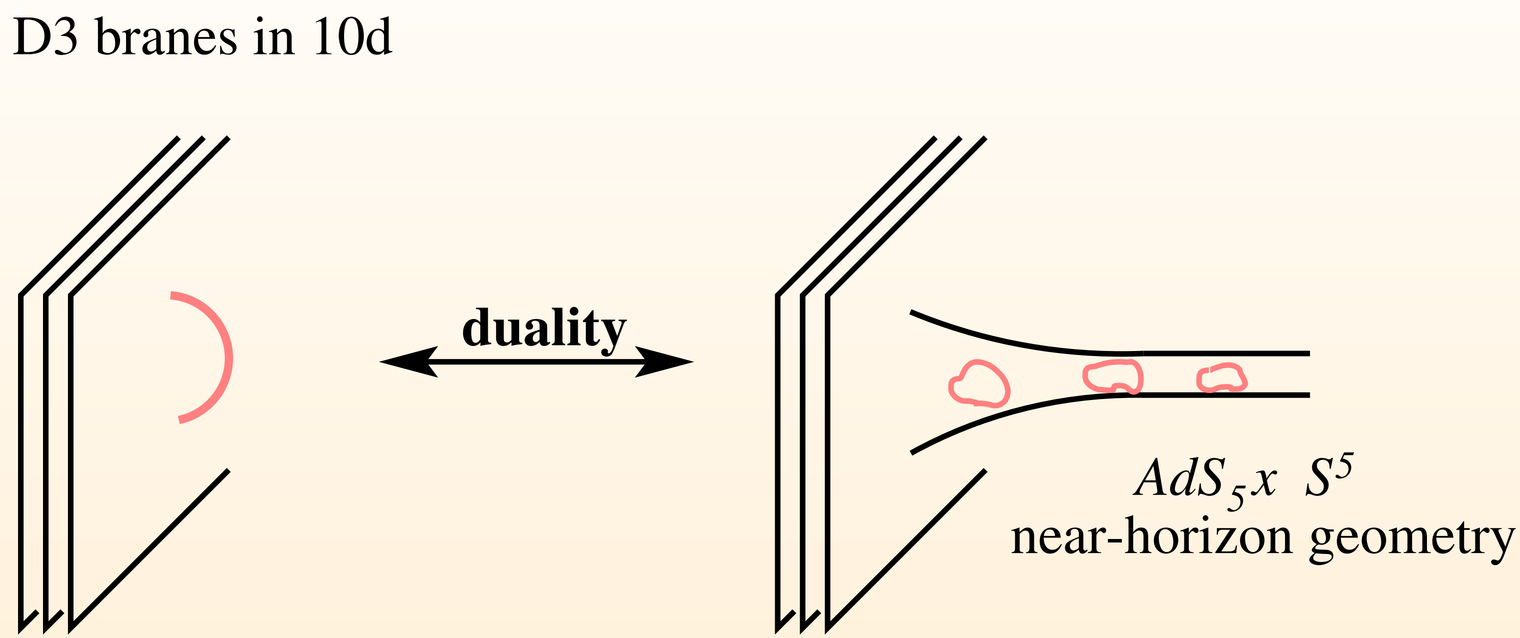
Near-horizon limit of D-branes

New approach to quantum gravity

Essential ingredients:

Conformal symmetry, large N limit (saddle point approximation)

String theory origin of the AdS/CFT correspondence



$\Downarrow$ Low-energy limit: $g_s = \lambda/N \to 0$, $\lambda = L^4/\alpha'^2$ large

$\mathcal{N} = 4$ $SU(N)$ gauge theory in four dimensions $\Leftrightarrow$ Classical supergravity on $AdS_5 \times S^5$

AdS/CFT correspondence

- Field-operator correspondence:

$$\langle e^{\int d^d x\, \phi_0(\vec{x})\mathcal{O}(\vec{x})}\rangle_{CFT} = e^{-S_{\rm sugra}}\Big|_{\phi(0,\vec{x})=\phi_0(\vec{x})}$$

Saddle point approximation

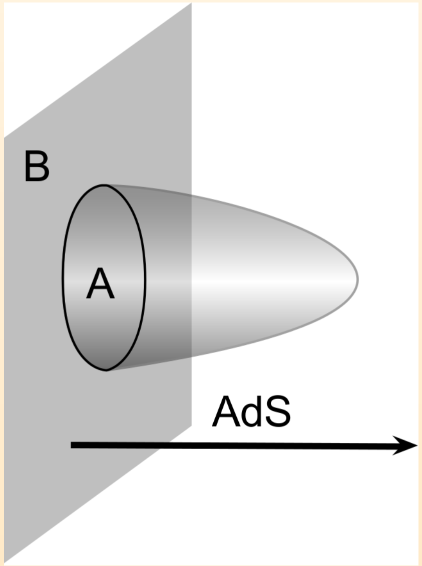
Is the duality valid more generally?

Is the duality valid more generally?

Information and AdS/CFT

Ryu+Takayanagi 2006: Holographic Entanglement entropy

Leading term in entanglement entropy given by
area of minimal surface in holographic dimension



Motivation and Overview

Concepts from information theory in quantum field theory related to geometry:

1. Information geometry;   Fisher information metric
2. Modular Hamiltonian/ Modular flow
3. Computational Complexity

Concepts from information theory in quantum field theory related to geometry:

1. Information geometry; Fisher information metric
2. Modular Hamiltonian/ Modular flow
3. Computational Complexity

Open question: Dynamics?

Thanks to my collaborators:

- Information Geometry:

  J.E., Kevin T. Grosvenor and Ro Jefferson, SciPost Phys. 8 (2020) 5, 073; arXiv: 2001.02683.

- Fisher metric in AdS for mixed states:

  Souvik Banerjee, J.E., Debayjoti Sarkar, JHEP 08 (2018) 001; arXiv:1701.02319.

- Modular flow:

  J. E., Pascal Fries, Ignacio A. Reyes, Christian P. Simon, JHEP 12 (2020) 126; arXiv: 2008.07532.

- Complexity in CFT:

  J.E., Marius Gerbershagen, Anna-Lena Weigel, JHEP 11 (2020) 003, arXiv: 2004.03619.

Branch of mathematics: Uses differential geometry to study probability theory

Shun-ichi Amari

Statistical manifold:

Riemannian manifold whose points correspond to probability distributions

Probability distributions $p(x,\vec{\theta})$; $\int dx\, p(x,\vec{\theta}) = 1$

x stochastic variable, parameters $\vec{\theta}$

Smooth manifold spanned by the $\vec{\theta}$ coordinates

Fisher information metric

Fisher metric in information theory: Metric on space of probability distributions

Probability distribution $p(x, \vec{\theta})$, x a stochastic variable, $\vec{\theta}$ a set of n external parameters

Spectrum $\gamma(x, \vec{\theta}) \equiv - \ln p(x, \theta)$

Fisher metric

$$g_{\mu\nu}(\vec{\theta}) \; = \; \int dx \, p(x, \vec{\theta}) \frac{\partial \gamma(x, \theta)}{\partial \theta^{\mu}} \frac{\partial \gamma(x, \theta)}{\partial \theta^{\nu}} \; = \; \langle \partial_{\mu} \gamma \partial_{\nu} \gamma \rangle$$

For Gaussian distribution (saddle point approximation)

$$p(x_1, \ldots, x_n) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp \left(- \sum_{i=1}^{n} \frac{(x_i - \bar{x}_i)^2}{2\sigma^2} \right)$$

Fisher metric gives Anti-de Sitter space:

$$ds^2 = \frac{1}{\sigma^2} \left(d\bar{x}_i d\bar{x}^i + 2n d\sigma^2 \right)$$

Fisher metric: Discussion

Fisher metric for Gaussian probability distribution is AdS metric!

Fisher metric for Gaussian probability distribution is AdS metric!

Caveats: Clingman, Murugan, Shock 1504.03184

Non-uniqueness: Infinitely many distributions give the same geometry

(eg. 1d Gaussian and Cauchy-Lorentz give 2d hyperbolic space)

Many statistical models lead to the same information geometry

Fisher metric inherits symmetries of statistical model but can have more

(Example in CMS'15 of a probability distribution that gives 2d hyperbolic space but has none of its symmetries)

Blau, Narain, Thompson hep-th/0108122:

Yang-Mills instantons in 3+1 dimensions

Uses prescription of Hitchin (1990): Probability distribution is taken to be the Lagrangian density on the state

$$p_{(\Lambda,a)}(x) = \mathrm{tr} F^{\mu\nu} F_{\mu\nu}\,, \qquad F^a_{\mu\nu} = -4\eta^a{}_{\mu\nu}\frac{\Lambda^2}{(x-a)^2+\Lambda^2}$$

FIsher metric gives AdS metric (Radial coordinate given by instanton radius)

Works also for scalar instanton

Fisher metric for theory space and state space

J.E., Grosvenor, Jefferson 2020

Theory space spanned by (renormalized) couplings (Ising model, RG flows)

State space (Examples: Instantons, coherent states)

Fisher metric for theory space and state space

J.E., Grosvenor, Jefferson 2020

Theory space spanned by (renormalized) couplings (Ising model, RG flows)

State space (Examples: Instantons, coherent states)

String theory: Heckman 1305.3621

Worldsheet action for Fisher metric

Conformal invariance of 2d worldsheet action implies Einstein equations for target space

Fisher metric for 2d Ising model

J.E., Grosvenor, Jefferson 2020

Square lattice

$$H = -J \sum_{i,j=1}^{N} \sigma_{i,j}\sigma_{i+1,j} - K \sum_{i,j=1}^{N} \sigma_{i,j}\sigma_{i,j+1}$$

Partition function: $Z = \prod_i \sum_{\sigma_i=\pm 1} e^{-\beta H(\sigma)}$

Probability distribution: $p_{\beta J,\beta K}(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)}$

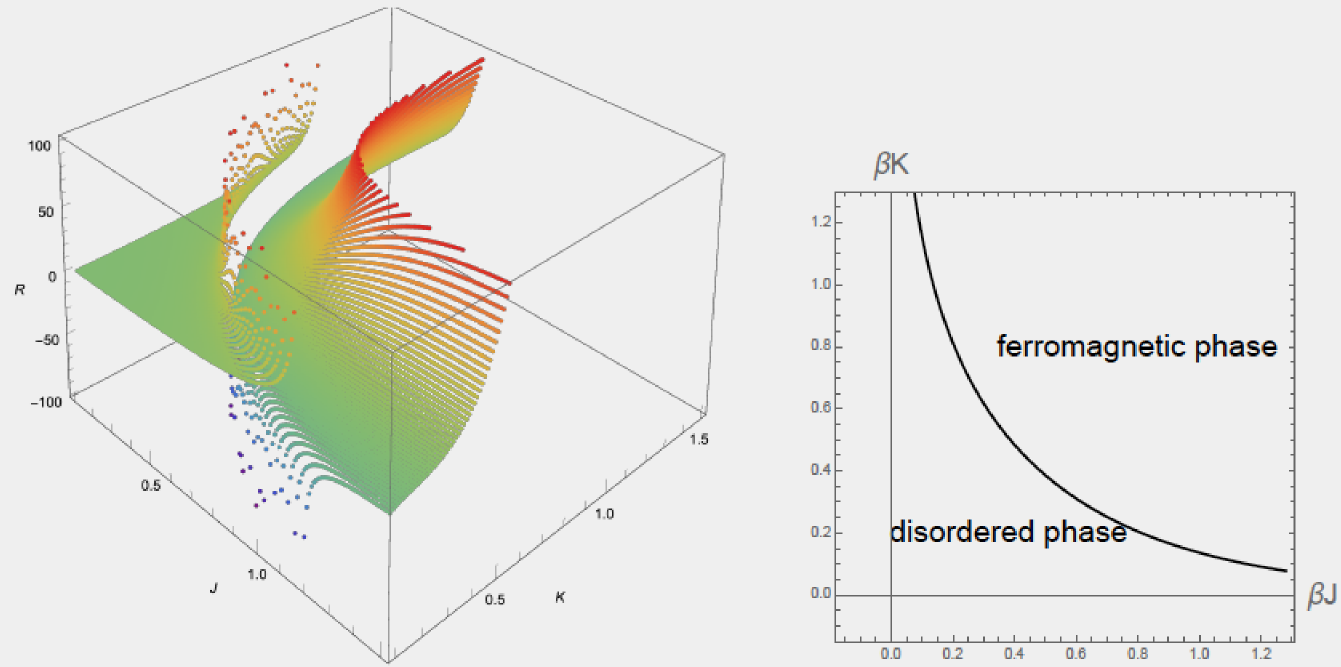
$g_{ab} = \partial_a \partial_b f$ with f reduced free energy per site

Previous work in condensed matter/ quantum field theory:

Janke et al, Brian Dolan et al

See also the recent Di Giulio, Tonni 2006.00921

Fisher metric for 2d Ising model



(a) Ricci curvature as function of couplings J, K ($\beta = 1$)

(b) 2d Ising Model phase diagram for $\beta J, \beta K \geq 0$

The scalar curvature diverges at the critical curve given by

$$\sinh(2\beta J)\,\sinh(2\beta K) = 1$$

Fisher metric for 2d Ising model

Set $J = K = 1$ and the critical temp. is $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right) \approx 0.44$.

$g_{\beta\beta}$ is the specific heat:

$$g_{\beta\beta} = \frac{d^2 f}{d\beta^2} \simeq \ln\frac{1}{|\beta-\beta_c|} \simeq \ln\frac{1}{|m|},$$

This may be reproduced in field theory:
The critical 2d Ising model corresponds to a field theory of free fermions

$$S = \int \frac{d^2 z}{2\pi}\left(\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi} + im\overline{\psi}\psi\right)$$

$$g_{ij} = \frac{1}{\text{spacetime vol}}\Big(\langle\partial_i S\,\partial_j S\rangle - \langle\partial_i S\rangle\langle\partial_j S\rangle\Big)$$

$$g_{mm} \simeq \int_0^{\Lambda} \frac{p\,dp}{p^2+m^2} \simeq \ln\frac{\Lambda}{|m|} \simeq \ln\frac{1}{|m|}$$

'Divergence': Bifunctional $D(p||q)$of two probability distributions p and q measuring their difference

Example: Relative entropy $D(p||q) = \int dx p \ln \frac{p}{q}$

$$g_{ij}(\xi) = \frac{\partial^2}{\partial \xi'^i \, \partial \xi'^j} D(p_\xi || p_{\xi'}) \Big|_{\xi'=\xi}$$

Bures metric for quantum states given by density matrix ρ:

Bures distance $D_B(\rho_1, \rho_2) = 2\big(1 - \big|\mathrm{tr}\sqrt{\rho_1^{1/2} \rho_2 \rho_1^{1/2}}\big|\big)$

For pure states $D_B(|\psi_1\rangle, |\psi_2\rangle) = 2\big(1 - \big|\langle\psi_1|\psi_2\rangle\big|\big)$.

The *Bures metric* is the leading (quadratic) term in the expansion of D_B around $\rho_2 \approx \rho_1$.

Fisher metric: Coherent States

Nozaki, Ryu, Takayanagi 2012

Two free Majorana fermions $\{a, a^\dagger\} = \{b, b^\dagger\} = 1$.

Fermionic coherent state: $|\psi_\lambda\rangle = \sqrt{\frac{1}{1+|\lambda|^2}}\, e^{-\lambda a^\dagger b^\dagger}|\Omega\rangle$ with $\lambda \in \mathbb{C}$.

One free complex scalar $[a, a^\dagger] = [b, b^\dagger] = 1$.

Bosonic coherent state: $|\psi_\lambda\rangle = \sqrt{1-|\lambda|^2}\, e^{-\lambda a^\dagger b^\dagger}|\Omega\rangle$ with $\lambda \in \mathbb{C}$.

Fermions: $ds^2 = \frac{d\lambda\, d\bar{\lambda}}{(1+|\lambda|^2)^2}$ and Bosons: $ds^2 = \frac{d\lambda\, d\bar{\lambda}}{(1-|\lambda|^2)^2}$

Fermions: 2-sphere and Bosons: 2d hyperbolic space

Consistent with the density matrix symmetries $\rho \to \rho' = U\rho U^\dagger$

Fisher information on the gravity side of AdS/CFT

Banerjee, J.E., Sarkar 1701.02319

Fidelity susceptibility F –Related to Bures distance by $D_B = 2(1 - F)$

Fisher metric

$$G_{mn} = \frac{\partial^2 F}{\partial \lambda^m \partial \lambda^n}$$

Couplings λ_m dual to to deformations of the AdS metric

Takayanagi et al 1507.07555: For pure states and marginal couplings:

$$G^{\text{pure}} = \frac{n_d}{G} \frac{\text{vol}_{d-1}}{\epsilon^{d-1}}$$

Proposal for mixed states: $F = C_d \left(\text{vol}(m^2) - \text{vol}(0) \right)$ (finite expression)

For m a metric or relevant scalar deformation, the result has the expected scaling behaviour with the size of the entangling region, $R^{2\Delta}$

II. Modular Hamiltonian and modular flow

Starting point: State given by density matrix ρ

Entangling region V

Entropy generalizes to entanglement entropy $S_V = -\mathrm{tr}(\rho_V \ln \rho_V)$

Hamiltonian generalizes to modular Hamiltonian K_V, defined implicitly via

$$\rho_V := \frac{e^{-K_V}}{\mathrm{tr}\left(e^{-K_V}\right)}$$

Generalized time evolution

Entanglement spectrum has many applications in many body physics and QFT

Topological order; relative entropy

AdS/CFT: Essential for gravity bulk reconstruction from QFT boundary data

Modular Hamiltonian and modular flow

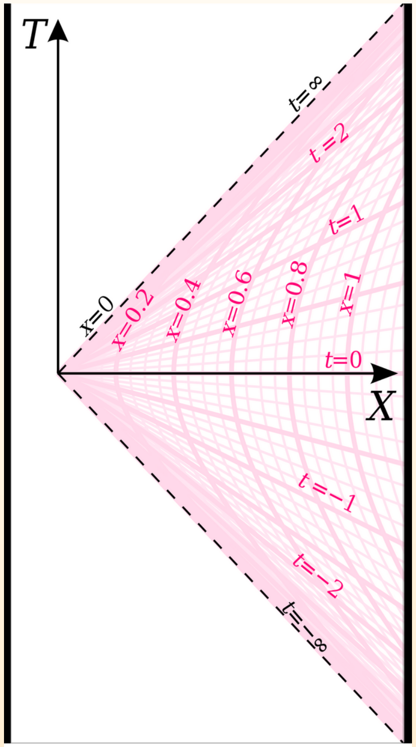
Modular Hamiltonian known explicitly only in a small number of cases

Universal and local result for QFT on Rindler spacetime:
(accelerated reference frame in Minkowski spacetime)

$$K - K_{\mathrm{vac}} = \frac{2\pi}{\hbar} \int_0^\infty dx\, x T_{tt}$$

(Bisognano-Wichmann theorem)

Further examples: CFT vacuum on a ball,
CFT_2 for single interval, vacuum on the cylinder or
thermal state on real line



(WIkipedia)

Modular flow generated by modular Hamiltonian:

Generalised time evolution with the density matrix:

$$\sigma_t(\mathcal{O}) := \rho^{it} \mathcal{O} \rho^{-it}$$

In general, modular flow is non-local

Modular Hamiltonian and modular flow

J.E., Fries, Reyes, Simon 2008.07532 for free fermions in 1+1 dimensions:

For disjoint intervals $V = \bigcup_n [a_n, b_n]$:

$$\sigma_t \left(\psi^\dagger(y)\right) = \int_V \mathrm{d}x \, \psi^\dagger(x) \Sigma_t(x, y) \,,$$

$$\Sigma_t = \left(\frac{1 - G|_V}{G|_V}\right)^{it} .$$

Modular flow expressed in terms of reduced propagator $G|_V$

A few facts from Tomita-Takesaki modular theory: (see S. Hollands, 1904.08201)

Tomita conjugation:

$$S\mathcal{O}|\Omega\rangle := \mathcal{O}^\dagger|\Omega\rangle$$

for operator $\mathcal{O}$ in von Neumann algebra $\mathcal{R}$

S may be decomposed into $J\Delta^{1/2}$, J antiunitary and Δ positive

Tomita theorem: $J\mathcal{R}J^\dagger = \mathcal{R}'$, $\Delta^{it}\mathcal{R}\Delta^{-it} = \mathcal{R}$

Modular flow: $\sigma_t(\mathcal{O}) = \Delta^{it}\mathcal{O}\Delta^{-it}$

Modular Hamiltonian: $e^{-itK} := \Delta^{it}$

Two operators satisfy the KMS (Kubo-Martin-Schwinger) condition

$$\langle\Omega|\mathcal{O}_1\sigma_t(\mathcal{O}_2)|\Omega\rangle = \langle\Omega|\sigma_{t+i}(\mathcal{O}_2)\mathcal{O}_1|\Omega\rangle$$

by analogy to time evolution at finite temperature

Modular two-point function for free fermions

Modular two-point function

$$G_{\text{mod}}(x,y;t) := \begin{cases} -\langle\Omega|\sigma_t(\psi^\dagger(y))\psi(x)|\Omega\rangle & \text{for } 0 < \mathrm{Im}(t) < 1 \\ +\langle\Omega|\psi(x)\sigma_t(\psi^\dagger(y))|\Omega\rangle & \text{for } -1 < \mathrm{Im}(t) < 0. \end{cases}$$

Introduce Σ_t as test or smearing function

$$\sigma_t\big(\psi^\dagger(y)\big) = \int_V \mathrm{d}x\, \psi^\dagger(x)\Sigma_t(x,y)$$

From fermion anticommutator it follows that

$$G_{\text{mod}}(x,y;t-i0^+) - G_{\text{mod}}(x,y;t+i0^+) = \Sigma_t(x,y)$$

Locality properties

We compute the free fermion modular flow for a number of examples:

plane, cylinder (Ramond and Neveu-Schwarz sectors), torus

Locality:

Non-local: Kernel $\Sigma_t(x,y)$ is a smooth function on all of the region V

Bi-local: $\Sigma_t(x,y) \sim \delta(f(x,y))$. Discrete set of contributions. Couples pairs of distinct points since $x \neq y$ at $t=0$.

Local: As bi-local but with $x=y$ at $t=0$

Locality properties depend on boundary conditions and temperature

Reflected in structure of poles and cuts in modular two-point function

Degree of locality as function of temperature

High temperatures: Thermal and local behaviour of modular flow

Lowering the temperature, modular flow adds bilocal couplings between an infinite discrete set of points

Can be traced back to anti-periodicity of thermal propagator that is required by KMS condition

For $T \to 0$, the discrete couplings condense to a continuum (branch cut)

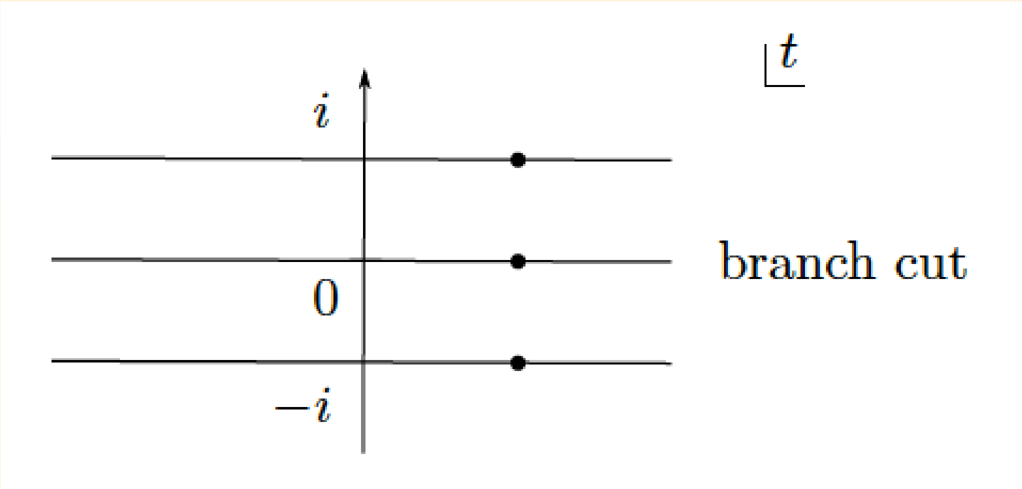
Modular two-point function: Pole structure

Presence of poles and cuts in G_{mod} gives information about non-locality of modular flow

Causes couplings between isolated points or entire regions

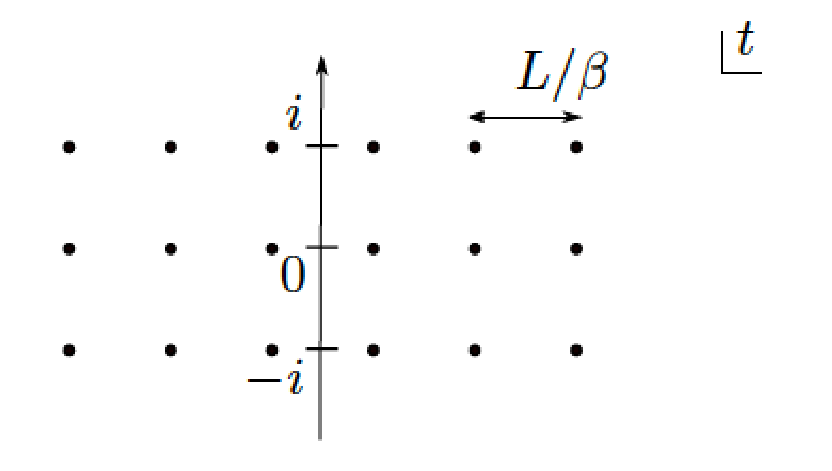
Plane: Only simple poles, modular flow local

Cylinder (P): Additional branch cuts



Modular two-point function: Pole structure

Torus (AP):



Reproduces result for cylinder when $T \Rightarrow 0$

Complexity:

Consider set of predefined unitary transformations in a Hilbert space

How many of these need to be applied to reach any given state
from a reference state?

Complexity

Consider a reference state $|r\rangle$ and a set of unitary operators U_1, U_2, ... (gates)

The complexity $\mathcal{C}(|\psi\rangle)$ of a state $|\psi\rangle$ is given by the minimal number of gates required to map $|r\rangle$ to $|\psi\rangle$ up to a given tolerance

$$C(|\psi\rangle) = \min\{n \in \mathbb{N} |\, U_{i_1} \dots U_{i_n} |r\rangle = |\psi\rangle, \text{up to tolerance}\}$$

- Well-defined for pure states in finite-dimensional Hilbert spaces

- No standard definition for quantum field theories

  (Recent progress for free field theories (Myers et al, Heller et al)

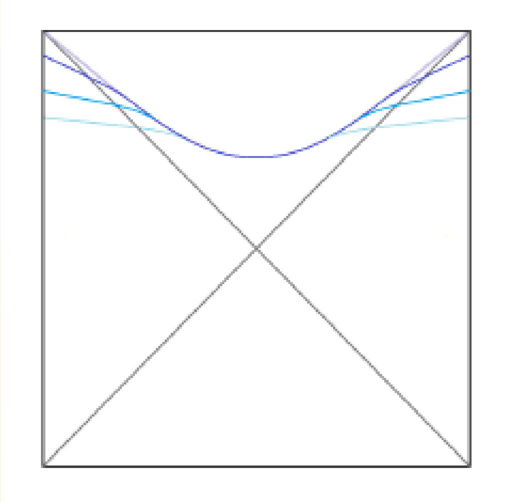
  also Headrick et al, 1804.01561; Policastro and Ge 1904.03003)

Holographic complexity proposals

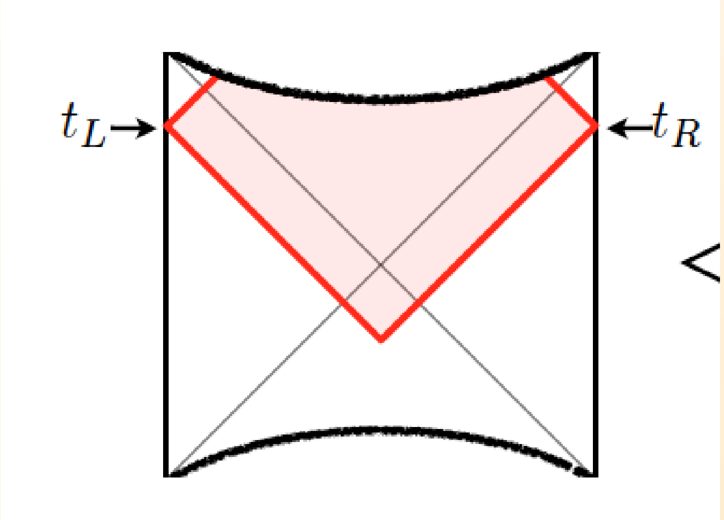
Susskind et al: Fortsch.Phys. 64 (2016) 24-43, Phys.Rev.Lett. 116 (2016) no.19, 191301

Consider evolution of two copies of a CFT initially entangled in a thermofield double state

'Complexity = Volume':
Volume of Einstein-Rosen bridge



'Complexity = Action':
Action on Wheeler-de Witt patch



Both proposals evolve linearly with time.

Continuous set of gates to be determined

Examples for reference states:

Spatially unentangled states

Highest weight state in a symmetry multiplet

Cost function determines expense of applying a particular gate

Complexity for conformal field theories

Caputa, Magan 1807.04422

Gates from conformal transformations: $Q(t) = \frac{1}{2\pi}\int d\sigma \epsilon(\sigma,t)T(\sigma)$

Reference state: Primary state

$Q(t)$ generates unitary transformations

Cost function: Expectation value of gate w.r.t state, $\mathcal{F} = |\mathrm{t}r[\rho(t)Q(t)]|$

J.E., Gerbershagen, Weigel 2004.03619

Complexity functional equivalent to geometric group action on coadjoint orbits of the Virasoro group

Extension to Kac-Moody algebras

Flory, Heller 2005.02415, 2007.11555

Assign cost to circuits based on the
Fubini-Study metric (fidelity susceptibility)

Optimal circuits are determined by Euler-Arnold equations

Equations of classical mechanics viewed as geodesic flow on Riemannian manifold

Conclusion and outlook

- Inspired by gauge/gravity duality, many beautiful new connections between quantum field theory and geometry have recently emerged
- Important ingredient: Information theory
- Challenge: To understand working mechanisms of gauge/gravity duality in general
- Dynamics ?