Holographic entanglement entropy for perturbative higher-curvature gravities

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[2012.14033 - P. Bueno, J. Camps, AVL]

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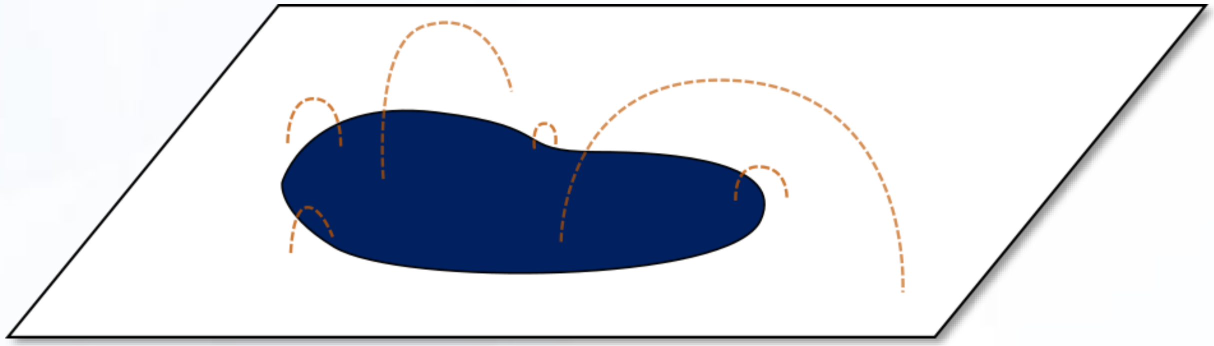
Why entanglement entropy?

Entanglement entropy allows us to gain significant physical insight into quantum field theories.

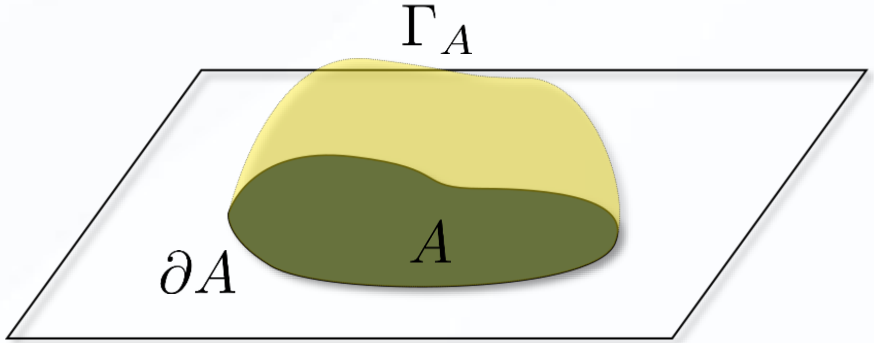


But it is usually difficult to calculate!

[0603001, Ryu, Takayanagi]

In a holographic theory dual to GR, it is possible to compute entanglement entropy via a geometric problem (minimization of the area of a bulk surface).

$$S^{\mathrm{E}}_{\mathrm{HEE}}(A) = \frac{\mathcal{A}(\Gamma_A)}{4G_N}$$



Is this a **deep statement about quantum gravity?**

Why higher-curvature gravity?

Higher-curvature gravity theories appear naturally within the context of AdS/CFT:

- They are expected as **quantum or stringy corrections** to the classical bulk action, corresponding to corrections to the strong coupling and large-N limits of the CFT.
- Their holographic duals are inequivalent to the one defined by Einstein gravity *(e.g.*, $a \neq c$ in 4d). This has been very fruitful in discovering universal properties of CFTs, or as toy models to probe interesting CFT physics.

We aim to understand **how entanglement entropy is to be calculated** in the bulk when we include **higher-curvature corrections**.

Why perturbative?

The general form of the entanglement entropy functional is known:

[1310.5713, X. Dong]
[1310.6659, J. Camps]

$$S_{\text{HEE}}^{\mathcal{L}(\text{Riem})}(A) = S_{\text{Wald}} + S_{\text{Anomaly}}$$

$$S_{\text{Wald}} = -2\pi \int_{\Gamma_A} d^{d-1}y\sqrt{h}\, \epsilon_{\mu\nu}\epsilon_{\rho\sigma} \frac{\partial \mathcal{L}_E}{\partial R_{\mu\nu\rho\sigma}}$$

$$S_{\text{Anomaly}} = 2\pi \int_{\Gamma_A} d^{d-1}y\sqrt{h} \sum_\alpha \left(\frac{8\partial^2 \mathcal{L}_E}{\partial \text{Riem}^2} K^2 \right)_\alpha \frac{1}{1+q_\alpha}$$

$$\left(\frac{8\partial^2 \mathcal{L}_E}{\partial \text{Riem}^2} K^2 \right) \equiv 2\left[\perp^{\lambda_1\lambda_2}\left(\perp_{\mu_1\mu_2}\perp_{\nu_1\nu_2} - \epsilon_{\mu_1\mu_2}\epsilon_{\nu_1\nu_2}\right) + \epsilon^{\lambda_1\lambda_2}\left(\perp_{\mu_1\mu_2}\epsilon_{\nu_1\nu_2} + \epsilon_{\mu_1\mu_2}\perp_{\nu_1\nu_2}\right)\right] \times \frac{\partial^2 \mathcal{L}_E}{\partial R_{\mu_1\rho_1\nu_1\sigma_1}\partial R_{\mu_2\rho_2\nu_2\sigma_2}} K_{\lambda_1\rho_1\sigma_1} K_{\lambda_2\rho_2\sigma_2}$$

Just a projected second derivative contracted with K's!

$$\epsilon_{\mu\nu} \equiv \epsilon_{ab} n^a{}_\mu n^b{}_\nu\,, \quad \perp_{\mu\nu} \equiv \delta_{ab} n^a{}_\mu n^b{}_\nu\,, \quad g_{\mu\nu} \equiv h_{\mu\nu} + \perp_{\mu\nu}$$

The holographic surface should be obatained by extremizing the new functional. If we work **perturbatively, it can be taken to be the RT surface**.

Why perturbative?

$$S_{\text{HEE}}^{\mathcal{L}(\text{Riem})}(A) = S_{\text{Wald}} + S_{\text{Anomaly}}$$

$$S_{\text{Anomaly}} = -2\pi \int_{\Gamma_A} d^{d-1}y\sqrt{h} \sum_{\alpha} \left(\frac{8\partial^2 \mathcal{L}_E}{\partial \text{Riem}^2} K^2 \right)_{\alpha} \frac{1}{1+q_\alpha}$$

The functional comes from applying a replica trick-like technique in the bulk which involves **regularizing a conical singularity** at the entanglement surface:

[1304.4926, Lewkowycz, Maldacena]

- Different Riemann tensor components contribute differently at the conical singularity, thus they must be expanded as:

$$R_{z\bar{z}z\bar{z}} = \tilde{R}_{z\bar{z}z\bar{z}} - \frac{1}{8}\underbrace{K^{aij}K_{aij}}_{q_\alpha = 1}\,, \quad \underbrace{R_{z\bar{z}zi}}_{q_\alpha = 1/2}\,, \quad R_{z\bar{z}ij} = \tilde{R}_{z\bar{z}ij} - 2\underbrace{K_{z[i|}{}^{k}K_{\bar{z}|j]k}}_{q_\alpha = 1}\,, \quad \ldots$$

$\epsilon_{z\bar{z}} = i/2$

$\perp_{z\bar{z}} = 1/2$

- The **regularization is not unique** *(splitting problem)*, and it must be fixed by the EoM of the theory. This has only been done for GR, and perturbative corrections to it.

[Exceptions: Lovelock, f(R), quadratic gravity] [1503.05538, R.-X. Miao] [1605.08588, J. Camps]

The anomaly expansion

$$S_{\text{Anomaly}} = -2\pi \int_{\Gamma_A} d^{d-1}y\sqrt{h} \sum_\alpha \left(\frac{8\partial^2 \mathcal{L}_E}{\partial \text{Riem}^2} K^2 \right)_\alpha \frac{1}{1+q_\alpha}$$

This is a mess (and difficult to understand what it is doing):

$$\mathcal{L}_E = R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\delta\gamma} R_{\delta\gamma}{}^{\mu\nu} \qquad \left(\frac{8\partial^2 \mathcal{L}_E}{\partial \text{Riem}^2} K^2 \right) = 48 R^{zi\bar{z}j} K_{zi}{}^k K_{\bar{z}jk} \qquad [K_a = 0]$$

$$\sum_\alpha (R^{zi\bar{z}j})_\alpha = \tilde{R}^{zi\bar{z}j} - \underbrace{K^{zik} K^{\bar{z}j}{}_k}_{q_\alpha = 1}$$

$$\sum_\alpha (R^{zi\bar{z}j})_\alpha \frac{1}{1+q_\alpha} = \tilde{R}^{zi\bar{z}j} - \frac{1}{2} K^{zik} K^{\bar{z}j}{}_k = R^{zi\bar{z}j} + \frac{1}{2} K^{zik} K^{\bar{z}j}{}_k$$

Generates higher-order terms in K!

$$\sum_\alpha \left(\frac{8\partial^2 \mathcal{L}_E}{\partial \text{Riem}^2} K^2 \right)_\alpha \frac{1}{1+q_\alpha} = 48 R^{zi\bar{z}j} K_{zi}{}^k K_{\bar{z}jk} + 24 K^{zik} K^{\bar{z}j}{}_k K_{zi}{}^l K_{\bar{z}jl}$$

The anomaly expansion revisited

It is possible to understand this anomaly term in a simpler way:

$$S_{\text{Anomaly}} = 2\pi \int_{\Gamma_A} d^{d-1}y\sqrt{h} \sum_{S=0}^{\infty} \frac{1}{S!} \int_0^1 du\, 2u : \left(-(1-u^2)\hat{\partial}_A - (1-u)\hat{\partial}_B\right)^S : \left(\frac{8\partial^2 \mathcal{L}_E}{\partial \text{Riem}^2} K^2\right)$$

$$\hat{\partial}_A = \Big[\frac{1}{2} \perp^{\lambda_1\lambda_2} h^{\tau_1\tau_2} h^{\omega_1\omega_2} \epsilon_{\mu\nu}\epsilon_{\rho\sigma} - 2\epsilon^{\lambda_1\lambda_2} h^{\tau_1}_{\rho} h^{\tau_2}_{\sigma} h^{\omega_1\omega_2}\epsilon_{\mu\nu} - 2 \perp^{\lambda_1\lambda_2} h^{\tau_1}_{\mu} h^{\omega_1}_{\rho} h^{\tau_2}_{\nu} h^{\omega_2}_{\sigma}$$
$$- 2(\perp^{\lambda_1\lambda_2}\perp_{\mu\rho} + \epsilon^{\lambda_1\lambda_2}\epsilon_{\mu\rho}) h^{\tau_1}_{\nu} h^{\tau_2}_{\sigma} h^{\omega_1\omega_2}\Big] K_{\lambda_1\tau_1\omega_1} K_{\lambda_2\tau_2\omega_2} \frac{\partial}{\partial R_{\mu\nu\rho\sigma}}$$
$$+ 2\left(\perp_{\mu_2}{}^{\mu_1}\perp_{\rho_2}{}^{\rho_1} - \epsilon_{\mu_2}{}^{\mu_1}\epsilon_{\rho_2}{}^{\rho_1}\right) h^{\nu_1}_{\nu_2} h^{\sigma_1}_{\sigma_2} R_{\mu_1\nu_1\rho_1\sigma_1} \frac{\partial}{\partial R_{\mu_2\nu_2\rho_2\sigma_2}}$$

$$\hat{\partial}_B = 4\left[\perp^{\mu_1}_{\mu_2} h^{\nu_1}_{\nu_2} h^{\rho_1}_{\rho_2} h^{\sigma_1}_{\sigma_2} + \perp^{\mu_1}_{\mu_2}\perp^{\nu_1}_{\nu_2}\perp^{\rho_1}_{\rho_2} h^{\sigma_1}_{\sigma_2}\right] R_{\mu_1\nu_1\rho_1\sigma_1} \frac{\partial}{\partial R_{\mu_2\nu_2\rho_2\sigma_2}}$$

Generates higher order terms in K (certain Riemann tensor components)

Only terms up to S=n-2 in an n-th order Lagrangian

This is both algorithmically and conceptually clearer!

The anomaly expansion: general results

1. Theories of the form f(Ricci) do not have anomaly term (perturbatively):

$$S_{\text{HEE}}^{\mathcal{L}(\text{Ricci})} = \frac{\mathcal{A}(\Gamma_A)}{4G} + \frac{\lambda}{8G_N}\int_{\Gamma_A} d^{d-1}y\sqrt{h}\,\frac{\partial\mathcal{L}}{\partial R_{\mu\nu}}\perp_{\mu\nu} + \mathcal{O}(\lambda^2)$$

$$I_E^{\mathcal{L}(\text{Ricci})} = -\frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{|g|}\left[\frac{d(d-1)}{L^2} + R + \lambda\mathcal{L}(g_{\mu\nu}, R_{\rho\sigma})\right]$$

2. Riccis do not generate K's. An n-th order Lagrangian containing n_R Riemann tensors and n-n_R Riccis (or scalars) has extrinsic curvatures up to the power 2n_R-2

$$S_{\text{Anomaly}} \sim \int_{\Gamma_A}\sum \text{Ricci}^{n-n_R}\left(\text{Riem}^{n_R-2}K^2 + \text{Riem}^{n_R-3}K^4 + \cdots + \text{Riem}\,K^{2n_R-4} + K^{2n_R-2}\right)$$

$$\mathcal{L} \sim \text{Riem}^{n_R}\text{Ricci}^{n-n_R}$$

In particular, densities with 0 or 1 Riemann tensors have no anomaly (perturbatively).

The anomaly expansion: general results

3 The anomaly term for Lovelock theories has a nice expression:

$$S^{\mathcal{X}_{2n}}_{\text{Anomaly}} = 2\pi \int_{\Gamma_A} d^{d-1}y\sqrt{h}\left[2K_{aik}K^{a}{}_{lj}\frac{\partial}{\partial R_{ijkl}}\right]^{-1}\left[\exp\left(2K_{aik}K^{a}{}_{jl}\frac{\partial}{\partial R_{ijkl}}\right)-1\right]\left(\frac{8\partial^2\mathcal{X}_{2n}}{\partial \text{Riem}^2}K^2\right)$$

$$\mathcal{X}_{2n}(R) \equiv \frac{1}{2^n}\delta^{\mu_1\mu_2\cdots\mu_{2n-1}\mu_{2n}}_{\nu_1\nu_2\cdots\nu_{2n-1}\nu_{2n}} R^{\nu_1\nu_2}_{\mu_1\mu_2}\cdots R^{\nu_{2n-1}\nu_{2n}}_{\mu_{2n-1}\mu_{2n}}$$

Combined with the Wald term, this becomes totally intrinsic, and it coincides with the Jacobson-Myers functional:

$$S^{\mathcal{X}_{2n}}_{\text{HEE}} = -4\pi n\int_{\Gamma_A}\mathrm{d}^{d-1}y\,\sqrt{h}\,\mathcal{X}_{2(n-1)}(\mathcal{R})$$

Cubic functionals

With this general form of the functional, **we have obtained the contribution of cubic (and quartic) theories**:

$$I_E^{\text{Riem}^3} = -\frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{|g|}\left[\frac{d(d-1)}{L^2} + R + L^4\sum_{i=1}^{8}\beta_i\mathcal{L}_i^{(3)}\right]$$

$$\mathcal{L}_1^{(3)} \equiv R_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \delta\ \gamma}R_{\delta\ \gamma}^{\ \mu\ \nu} \qquad \mathcal{L}_2^{(3)} \equiv R_{\mu\nu}^{\ \ \rho\sigma}R_{\rho\sigma}^{\ \ \delta\gamma}R_{\delta\gamma}^{\ \ \mu\nu}$$

$$\mathcal{L}_3^{(3)} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\delta}R^{\sigma\delta} \qquad \mathcal{L}_4^{(3)} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R$$

$$\mathcal{L}_5^{(3)} \equiv R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} \qquad \mathcal{L}_6^{(3)} \equiv R_\mu^\nu R_\nu^\rho R_\rho^\mu$$

$$\mathcal{L}_7^{(3)} \equiv R_{\mu\nu}R^{\mu\nu}R \qquad \mathcal{L}_8^{(3)} \equiv R^3$$

$$\boxed{S_{\text{HEE}}^{\text{Riem}^3} = \frac{\mathcal{A}(\Gamma_A)}{4G_N} + \frac{L^4}{4G_N}\int_{\Gamma_A} d^{d-1}y\sqrt{h}\sum_{i=1}^{8}\beta_i\Delta_i^{(3)} + \mathcal{O}(\beta_i^2)}$$

...

$$\Delta_2^{(3)} = +3R^{ab\rho\sigma}R_{ab\rho\sigma} - 6K_{ai}{}^{k}K_{bjk}\left(R^{aibj} - R^{biaj}\right) - 6K_{aik}K^{ajk}R^{bi}{}_{bj} + 3K_{ai}{}^{j}K_{bj}{}^{k}K^{a}{}_{k}{}^{l}K^{b}{}_{l}{}^{i} - 6K_{ai}{}^{j}K^{a}{}_{j}{}^{k}K_{bk}{}^{l}K^{b}{}_{l}{}^{i}$$

...

$$\Delta_6^{(3)} = +\frac{3}{2}R^{a\mu}R_{a\mu}$$

Recap

$$S_{\text{HEE}}^{\mathcal{L}(\text{Riem})}(A) = S_{\text{Wald}} + S_{\text{Anomaly}}$$

$$S_{\text{Wald}} = -2\pi \int_{\Gamma_A} d^{d-1}y\sqrt{h}\, \epsilon_{\mu\nu}\epsilon_{\rho\sigma} \frac{\partial \mathcal{L}_E}{\partial R_{\mu\nu\rho\sigma}}$$

$$S_{\text{Anomaly}} = 4\pi \int_{\Gamma_A} d^{d-1}y\sqrt{h} \int_0^1 du\, u : e^{-F(u)} : \left(\frac{8\partial^2 \mathcal{L}_E}{\partial \text{Riem}^2} K^2 \right) , \quad F(u) \equiv (1-u^2)\hat{\partial}_A + (1-u)\hat{\partial}_B$$

Projected second derivative

Operators made out of
projected derivatives and K's

Universal terms of entanglemet entropy

Entanglement entropy of a (smooth) region A in a d-dimensional CFT:

$$S_{\rm EE}(A) = p_1\left(\frac{H}{\delta}\right)^{d-2} + p_2\left(\frac{H}{\delta}\right)^{d-4} \cdots + \begin{cases} p_{(d-2)/2}\,(H/\delta)^2 + c\log(H/\delta) & [d\,\text{even}] \\ p_{(d-1)/2}\,(H/\delta) + \tilde{c} & [d\,\text{odd}] \end{cases}$$

Typical lengthscale of A or IR regulator

UV regulator

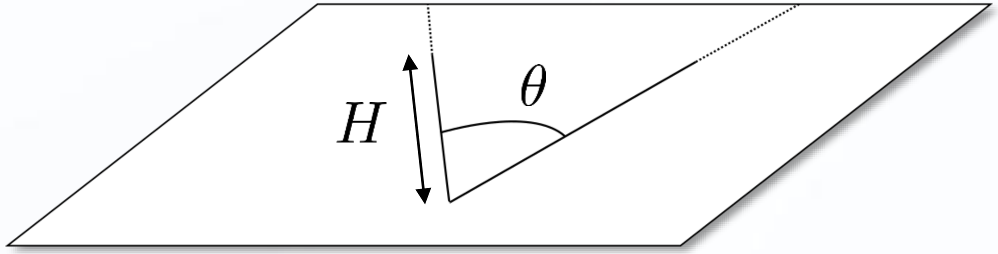
Non-universal terms

Universal terms

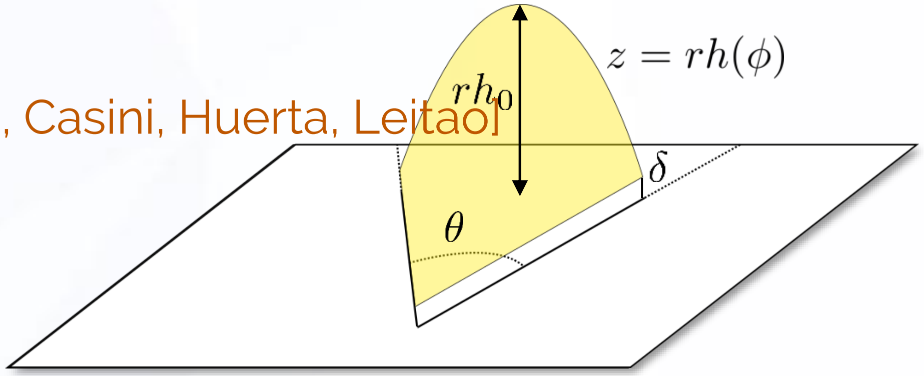
Geometric singularities introduce new terms. For a corner in d=3 in vacuum:

$$S_{\rm EE}(A) = b_1\frac{H}{\delta} - a(\theta)\log(H/\delta) + b_0$$

Universal piece (corner function)



Holographic corner functions



Some well-known properties: [0811.1968, Casini, Huerta, Leitao]

$$a(\theta) \geq 0\,, \quad \partial_\theta a(\theta) \leq 0\,, \quad \partial_\theta^2 a(\theta) \geq -\frac{\partial_\theta a(\theta)}{\sin\theta} \qquad (\theta \in [0,\pi])$$

Sharp limit: $a(\theta) \underset{\theta\to 0}{\sim} \frac{\kappa}{\theta} + \mathcal{O}(\theta)$

Smooth limit: $a(\theta) \underset{\theta\to \pi}{\sim} \sigma\,(\theta-\pi)^2 + \mathcal{O}\,(\theta-\pi)^4$

$$\mathrm{d}s^2 = \frac{L_\star^2}{z^2}\left(\mathrm{d}\tau^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2\right)$$

Only explicitly known for simple models (free bosons/fermions, EMI model, ...) and, holographically, for Einstein gravity:

$$S^{\mathrm{E}}_{\mathrm{HEE}} = \frac{L_\star^2}{2G_N}\int_{\delta/h_0}^{H}\frac{dr}{r}\int_0^{\theta/2-\epsilon} d\phi\,\frac{\sqrt{1+h^2+\dot{h}^2}}{h^2}$$

$$a_{\mathrm{E}}(\theta) = \frac{L_\star^2}{2G_N}\int_0^{+\infty} dy\left[1-\sqrt{\frac{1+h_0^2(1+y^2)}{2+h_0^2(1+y^2)}}\right]$$

$$\theta = \int_0^{h_0} dh\,\frac{2\sqrt{1+h_0^2}\,h^2}{\sqrt{1+h^2}\sqrt{(h_0^2-h^2)(h_0^2+(1+h_0^2)h^2)}}$$

Holographic corner functions

Quadratic theories do not add anything new: $\mathcal{L}_1^{(2)} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,, \quad \mathcal{L}_2^{(2)} = R_{\mu\nu}R^{\mu\nu}\,, \quad \mathcal{L}_3^{(2)} = R^2$

$$a_{\text{Riem}^2}(\theta) = [1 - 6\alpha_2 - 24\alpha_3]\; a_{\text{E}}(\theta)$$

But (some) cubic theories do! $\mathcal{L}_1^{(3)} \equiv R_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \delta\ \gamma}R_{\delta\ \gamma}^{\ \mu\ \nu}\,, \quad \mathcal{L}_2^{(3)} \equiv R_{\mu\nu}^{\ \ \rho\sigma}R_{\rho\sigma}^{\ \ \delta\gamma}R_{\delta\gamma}^{\ \ \mu\nu}\,, \quad \ldots$

$$a_{\text{Riem}^3}(\theta) = [1+6\beta_1+12\beta_2+6\beta_3+24\beta_4+27\beta_5+27\beta_6+108\beta_7+432\beta_8]a_{\text{E}}(\theta)+\sum_{i=1}^{2}\beta_i g_i(\theta)$$

$$g_1(\theta) \equiv +\frac{L_\star^2}{2G_N}\int_0^{+\infty}\frac{3(1+h_0^2)\left[3+h_0^2(5+4y^2)+2h_0^4(1+y^2)^2\right]}{\left[1+h_0^2(1+y^2)\right]^{7/2}\sqrt{2+h_0^2(1+y^2)}}dy$$

$$g_2(\theta) \equiv -\frac{L_\star^2}{2G_N}\int_0^{+\infty}\frac{6(1+h_0^2)\left[3+h_0^2(7+8y^2)+4h_0^4(1+y^2)^2\right]}{\left[1+h_0^2(1+y^2)\right]^{7/2}\sqrt{2+h_0^2(1+y^2)}}dy$$

This is the **first holographic corner function** with a functional dependence **different from the Einstein gravity one**!

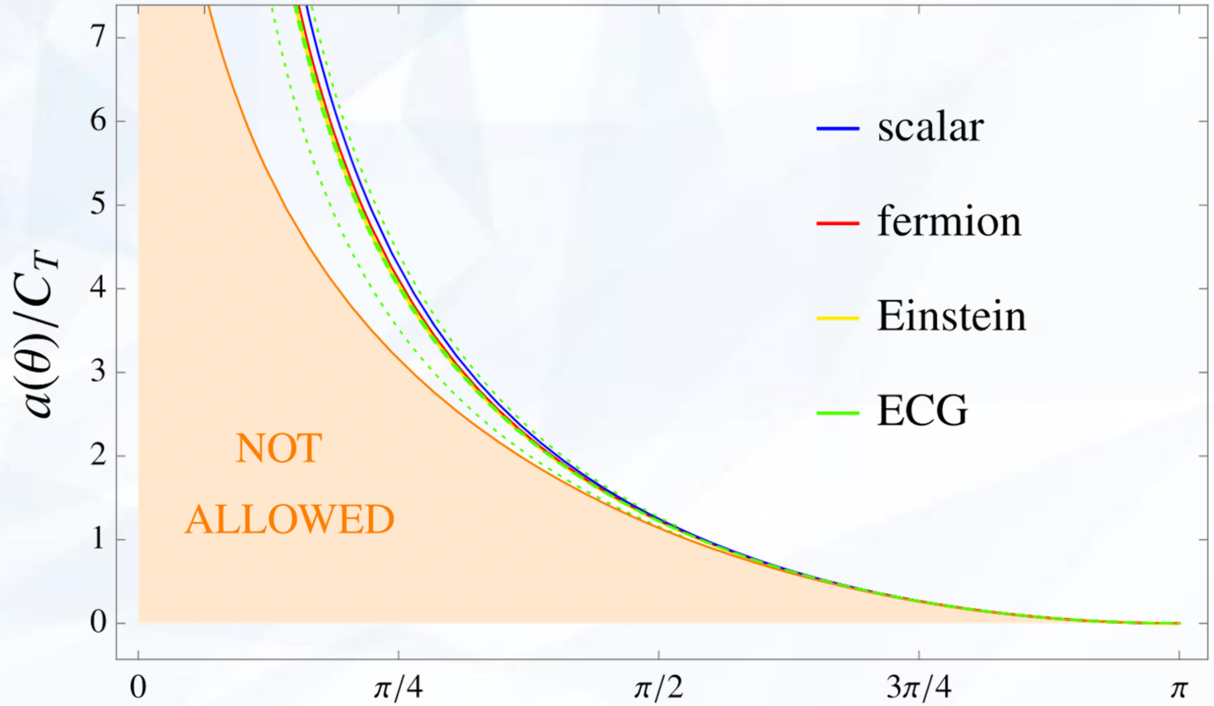
Holographic corner functions

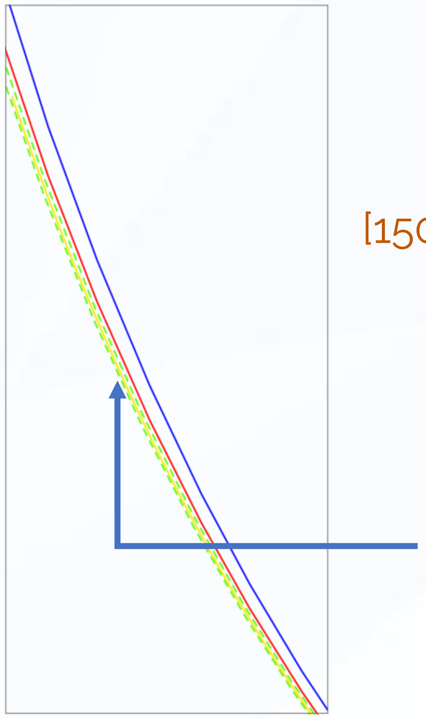
Take Einsteinian Cubic Gravity as an example:

[1607,06463, P. Bueno, P. Cano]

$$I_E^{\rm ECG} = -\frac{1}{16\pi G_N}\int d^4x\sqrt{|g|}\left[\frac{6}{L^2} + R - \frac{\mu_{\rm ECG} L^4}{8}\mathcal{P}\right]$$

$$\left[\mathcal{P} \equiv 12\mathcal{L}_1^{(3)} + \mathcal{L}_2^{(3)} - 12\mathcal{L}_5^{(3)} + 8\mathcal{L}_6^{(3)}\right]$$





All extremely close when normalized by C_T!

[1505,04804, Bueno, Myers, Witczak-Krempa]

Some are below the Einstein gravity result, which was conjectured to be a lower bound

Conclusions and future work

- ✓ We have presented a **rewriting of the entanglement entropy functional** for perturbative HCG, which is both **computationally and conceptually simpler**.
- ✓ This was used to compute the functional corresponding to cubic corrections – and quartic, see paper!
- ✓ It also gave us a better understanding of the structure of the functional: n_R Riemann tensors produce $2n_R$-2 extrinsic curvatures.
- ✓ We used the cubic functionals to compute universal contributions to the EE. For a corner in d=3, we have obtained the **first holographic corner function which is not proportional to the Einstein gravity one** (for other shapes, see paper!).

Future work:

- Can this be generalized for non-perturbative corrections?
- Is the new formula useful in other contexts, *e.g.*, to prove second laws for BHs?

Obrigado!

FONDO EUROPEO DE DESENVOLVEMENTO REXIONAL
“Unha maneria de facer Europa"