Electromagnetic Quasitopological Gravities

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Introduction

- **Higher-derivative theories** of gravity (also called **higher-curvature** or **higher-order** theories) = GR + terms with any number of Riemann tensors and covariant derivatives. For instance:

$$\mathcal{L} = \frac{1}{16\pi G}\left(R + \ell^2 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) \, .$$

- Relevance: effective action of UV-complete theory of gravity expected to have infinite tower higher-derivative terms (String Theory).
- However, recently interest in studying higher-order gravities by themselves and take an **EFT approach**:

$$\mathcal{L} = \frac{1}{16\pi G}\left(R + \sum_{n=2}^{\infty} \ell^{2n}\mathcal{R}_n\right) \, ,$$

$\mathcal{R}_n$: terms with p Riemanns and $2q$ covariant derivatives, $2q + p = n$. E.g. for $n = 2$:

$$\mathcal{R}_2 = \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \, .$$

Examples of higher-curvature gravities

- **Lovelock gravities** [Lovelock]:

$$\mathcal{L} = \frac{1}{16\pi G}\left(R + \sum_{n=2}^{[(d-1)/2]} \lambda_n \, \ell^{2n-2} \, \chi_{2n}\right) ,$$

where $$\chi_{2n} = \frac{(2n)!}{2^n}\delta^{[\mu_1}_{\nu_1} \dots \delta^{\mu_{2n}]}_{\nu_{2n}} R_{\mu_1\mu_2}{}^{\nu_1\nu_2} \dots R_{\mu_{2n-1}\mu_{2n}}{}^{\nu_{2n-1}\nu_{2n}} .$$

  - Most general theory with second-order Equations of Motion (EoM).
  - The term χ_{2n} only dynamical when $d > 2n$.

- $f(R)$ **theories** [Buchdahl]:

$$\mathcal{L} = \frac{1}{16\pi G} f(R) .$$

  - Easier than generic higher-order theory but encapsulate higher-curvature phenomena.
  - In general, EoMs are not second-order.

Generalized Quasitopological Gravities (GQs).

- Apart from Lovelock gravities, no other theory will have 2nd order EoMs $\rightarrow$ Try to find theories whose **EoMs are second-order under certain circumstances**.
- With this idea $\rightarrow$ **Generalized Quasitopological Gravities** (GQs)[1] [Bueno, Cano, Hennigar, Kubizňak, Mann, Oliva, Ray...].
- GQs are defined by admitting static, spherically-symmetric (SSS) solutions satisfying $g_{tt}g_{rr} = -1$:

$$\mathrm{d}s_f^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{d-2}^2\,.$$

- Equivalently, a theory is a GQ iff its Lagrangian $\mathcal{L}$ satisfies that

$$\frac{\partial L_f}{\partial f} - \frac{d}{dr}\frac{\partial L_f}{\partial f'} + \frac{d^2}{dr^2}\frac{\partial L_f}{\partial f''} + \cdots = 0\,, \quad L_f = r^2\mathcal{L}|_{ds_f^2}\,.$$

[1] Lovelock gravities are a subclass of GQs.

Properties of GQs.

- The **EoM for** $f(r)$ is at most[2] of **order 2**.
- **Linearized EoM** on max. symmetric backgrounds are **second-order**.
- **BH thermodynamics** can be computed **analytically**.
- Any higher-derivative theory can be mapped[3] via **field redefinitions to a GQ** [Bueno, Cano, Moreno, ÁM].
- There exist non-trivial GQs in 4 dimensions: **Einsteinian Cubic Gravity** [Bueno, Cano]:

$$\mathcal{L}_{\mathsf{ECG}} = \frac{1}{16\pi G}\Big(R + 12 R_{a\ b}^{\ c\ d} R_{c\ d}^{\ e\ f} R_{e\ f}^{\ a\ b} + R_{ab}^{cd} R_{cd}^{ef} R_{ef}^{ab} - 12 R_{abcd} R^{ac} R^{bd} + 8 R_a^b R_b^c R_c^a\Big) \, .$$

[2] We assume no explicit covariant derivatives of the curvature appear in the action.

[3] Proven for theories without covariant derivatives; strong evidence for theories with covariant derivatives.

Motivation for Electromagnetic Quasitopological Gravities

- **GQs** are **purely gravitational theories**, no coupling to matter.
- Desirable to extend definition of GQs to include matter. Simple and relevant example: an Abelian gauge field.
- GQs with minimally-coupled vector field have been considered [Bueno, Cano, Frassino, Hennigar, Rocha...]. But this is very restrictive. In general, higher-derivative actions may contain all possible couplings.
- Question: is it possible theories analogous to GQs with non-minimal couplings between curvature and gauge field?
- Answer: Yes! **Electromagnetic Quasitopological Gravities (EQs)**.

Electromagnetic Quasitopological Gravities (EQs)

- We will fix from now on the space-time dimension to 4.
- We search for higher-order theories of gravity with a non-minimally coupled vector field satisfying (**EQ conditions**):
  1. Diffeomorphism- and gauge-invariance.
  2. Assuming electric or magnetic ansatz for vector field, existence of SSS solutions characterised by single function $f(r)$:

$$\mathrm{d}s_f^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{d-2}^2 \quad \text{and} \quad \begin{cases} F^{\mathrm{e}} = -\Phi'(r)\mathrm{d}t \wedge \mathrm{d}r\,. \\ F^{\mathrm{m}} = \chi'(\theta)\mathrm{d}\theta \wedge \mathrm{d}\varphi\,. \end{cases}$$

  3. The equation of motion for $f(r)$ is at most second-order[4].

[4] We shall not consider any explicit covariant derivative on the curvature or the field strength in the action.

Electromagnetic Quasitopological Gravities (EQs)

- A clever way to implement EQ conditions:
  1. Set $F = -\Phi'(r)\mathrm{d}t \wedge \mathrm{d}r$ or $F = \chi'(\theta)\mathrm{d}\theta \wedge \mathrm{d}\varphi$.
  2. Search for some choice of $\Phi_{\mathsf{sol}}(r)$ or $\chi_{\mathsf{sol}}(\theta)$ solving Maxwell equation for any higher-order theory (under general SSS ansatz).
  3. Determine which theories become GQs after imposing F_{sol}.
- For **magnetic** vector fields, the previous programme **can be carried out**, because

$$F^{\mathrm{m}} = P \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi$$

  always solves[5] the Maxwell equation for an SSS metric.
- For **electric** vector fields, the programme **does not work**: if $F^{\mathrm{e}} = -\Phi'(r)\mathrm{d}t \wedge \mathrm{d}r$, then $\Phi(r)$ depends on the theory.
- How to define theories canonically admitting electric solutions? $\rightarrow$ **Dualizing theory** with magnetic solutions!

[5] For theories constructed out of monomials of Riemanns and field strengths.

Electromagnetic Quasitopological Gravities (EQs)

- **Dualization**: A map between two theories:

$$\begin{pmatrix} g_{\mu\nu} \\ \\ F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} \\ \\ \mathcal{L}(R,F) \end{pmatrix} \longrightarrow \begin{pmatrix} g_{\mu\nu} \\ \\ G_{\mu\nu} = 2\partial_{[\mu}B_{\nu]} \\ \\ \mathcal{L}'(R,G) = \mathcal{L}(R,F(G)) - 2F(G)^{\mu\nu}(\star G)_{\mu\nu} \end{pmatrix}$$

where $F_{\mu\nu}(G_{\rho\sigma})$ is obtained by inverting

$$\frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} = 2(\star G)_{\mu\nu} \tag{1}$$

- Imposing (1) on the EoMs and Bianchi identity of dual theory, one recovers the set of EoMs and Bianchi of original theory[6].
- If we have a theory with an SSS magnetic solution, the **dual theory** will have **SSS electric solutions**!

[6] Net effect is exchange of Bianchi identities and Maxwell equations.

Electromagnetic Quasitopological Gravities (EQs)

Definition 1 (Electromagnetic Quasitopological Gravities)

A given theory $\mathcal{L}(R, F)$ *is an* **Electromagnetic Quasitopological Gravity** *iff its Lagrangian or the Lagrangian of its dual theory admits SSS magnetic solutions characterised by a single metric function* $f(r)$.

Equivalently, a theory is an EQ iff its Lagrangian or the Lagrangian of its dual theory $\mathcal{L}$, after **evaluation on the magnetic SSS ansatz** with $g_{tt}g_{rr} = -1$:

$$L_f = r^2 \mathcal{L}|_{ds^2_f, F^{\mathrm{m}}} ,$$

the **Euler-Lagrange equation for** $f(r)$ **vanishes** identically:

$$\frac{\partial L_f}{\partial f} - \frac{d}{dr}\frac{\partial L_f}{\partial f'} + \frac{d^2}{dr^2}\frac{\partial L_f}{\partial f''} + \cdots = 0 \, .$$

Examples of EQs

- Any GQ with a minimally-coupled Abelian vector field is an EQ.
- Non-trivial examples:

$$\mathcal{L}^{(a)}_{n,m} = \left(2nR_{\mu}{}^{\alpha}\delta_{\nu}{}^{\beta} - (3n-3+4m)R^{\alpha\beta}{}_{\mu\nu}\right)\left(R^{n-1}\right)^{\mu\nu}{}_{\rho\sigma}F^{\rho\sigma}F_{\alpha\beta}\left(F^2\right)^{m-1},$$

$$\mathcal{L}^{(b)}_{n,m} = \left(F^2\right)^{m-1}F_{\mu\nu}F^{\rho\sigma}\left(\frac{n}{2}R\left(R^{n-1}\right)^{\mu\nu}{}_{\rho\sigma} + \right.$$
$$\left. + \frac{(n+4-4m)}{4}(3n-3+4m)\left(R^{n}\right)^{\mu\nu}{}_{\rho\sigma}\right) - n\left(F^2\right)^{m-1}F_{\alpha\nu}F^{\rho\sigma}R_{\mu}{}^{\alpha}\left\{\right.$$
$$\left.(1+2n)\left(R^{n-1}\right)^{\mu\nu}{}_{\rho\sigma} - (n-1)R^{\beta}{}_{\rho}\left(R^{n-2}\right)^{\mu\nu}{}_{\beta\sigma}\right\},$$

$$\mathcal{L}^{(c)}_{n,m} = \left(R^{n-1}\right)^{\mu\nu}\left[nRg^{\alpha\beta} - (4n+4m-3)R^{\alpha\beta}\right]F_{\mu\alpha}F_{\nu\beta}\left(F^2\right)^{m-1}.$$

- Both $\mathcal{L}^{(a)}_{n,m}$ and $\mathcal{L}^{(b)}_{n,m}$ have algebraic EoM for $f(r)$, while $\mathcal{L}^{(c)}_{n,m}$ has a 2nd-order EoM for $f(r)$.
- This proves that **EQs exist at all orders**, both with algebraic and 2nd-order EoMs.

Properties of Electromagnetic Quasitopological Gravities and its solutions

Properties of EQs

Most relevant properties of EQs:

1. They admit **electrically/magnetically charged solutions** (by def.).
2. The **EoM** for $f(r)$ is at most **second-order**.
3. The only gravitational mode propagated on maximally-symmetric backgrounds is a **spin 2-massless graviton**.
4. **BH thermodynamics** can be computed **analytically**. In addition, the following first law of BH thermodynamics holds:

$$\mathrm{d}M = T\mathrm{d}S + \Psi_h \mathrm{d}P \,,$$

   where M mass, T temperature, S entropy, Ψ_h the **electric potential** and P the (electric or magnetic) charge. Also, perfect match between Noether charge and Euclidean action approaches for free energy.
5. There is a subset of EQs which **admit regular solutions**.

Regular solutions

- Let us consider:

$$\mathcal{L} = R + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \ell^{2(n+m-1)}\left(\lambda_{n,m}\mathcal{L}_{n,m}^{(a)} + \gamma_{n,m}\mathcal{L}_{n,m}^{(b)}\right).$$

- After **setting magnetic SSS ansatz** with $g_{tt}g_{rr} = -1$, we find:

$$1 - f - \frac{2M}{r} + \sum_{n=0}^{\infty}(1-f)^{n-1}[\alpha_n(r) + \beta_n(r) f] = 0\,,$$

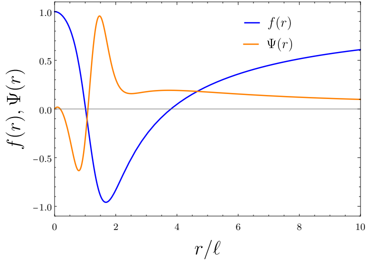
  with $\alpha_n(r) = \sum_{m=1}^{\infty} \frac{\alpha_{n,m}}{r^{4m+2n-2}}$ and $\beta_n(r) = \sum_{m=1}^{\infty} \frac{\beta_{n,m}}{r^{4m+2n-2}}$, being $\alpha_{n,m}$ and $\beta_{n,m}$ linear combinations of $\lambda_{n,m}$ and $\gamma_{n,m}$.

- Polynomial equation for $f(r)$! It generically[7] admits a well-behaved $f(r)$ which produces a **globally regular** metric!

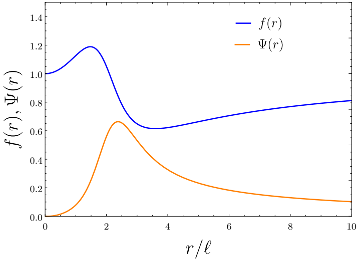
[7] Up to possible bounds on $\lambda_{n,m}$ and $\gamma_{n,m}$. If only $n = 1$ and $m = 0$ terms are included, couplings must be related though.

Regular solutions

- When it comes to analyse the **electric potential**[8], it does not need to be always regular...
- However, for **subspace of moduli space** of couplings, the electric potential $\Psi(r)$ is **also regular** everywhere.



(a) $2P = M = 2\ell$,
$2\gamma_{1,1} = 2 = -2\gamma_{2,1} = \lambda_{1,1}$ and $\lambda_{2,1} = \frac{25}{16}$.



(b) $P = M = \ell$,
$3\gamma_{1,1} = 6 = -6\gamma_{2,1} = \lambda_{1,1}$ and $\lambda_{2,1} = \frac{53}{4}$.

[8] Obtained from dual field strength.

Regular solutions

- How to obtain electric regular solutions? $\rightarrow$ By **dualization** of magnetic ones with regular electric potential!
- Proceeding this way $\rightarrow$ **first explicit theory regularizing** gravitational and EM fields **for any M and Q** [Cano, ÁM]:

$$\mathcal{L}(R,F) = R - F_{\mu\nu}F_{\rho\sigma}\chi^{\mu\nu\rho\sigma}\,, \quad \chi^{\mu\nu}{}_{\rho\sigma} = 6\delta^{[\mu\nu}{}_{\rho\sigma}\left(\mathcal{Q}^{-1}\right)^{\alpha\beta]}{}_{\alpha\beta}\,,$$

where $\left(\mathcal{Q}^{-1}\right)^{\alpha\beta}{}_{\mu\nu}\mathcal{Q}^{\mu\nu}{}_{\rho\sigma} = \delta^{\alpha\beta}{}_{\rho\sigma}$ and

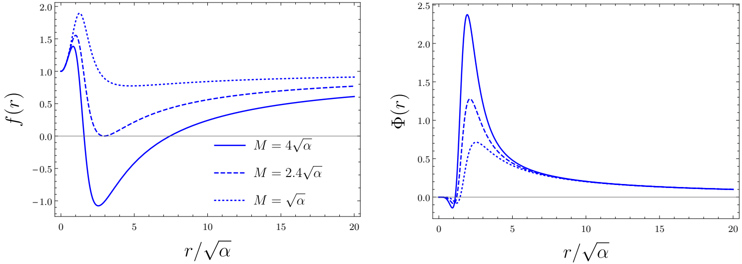
$$\begin{aligned}\mathcal{Q}^{\mu\nu}{}_{\rho\sigma} = \delta^{\mu\nu}{}_{\rho\sigma} &+ \alpha\left(6R^{[\mu}{}_{[\sigma}\delta^{\nu]}{}_{\rho]} + 7R^{\mu\nu}{}_{\rho\sigma} + \frac{1}{2}R\delta^{\mu\nu}{}_{\rho\sigma}\right)\\ &+ \alpha^2\left(\frac{9}{4}R_{\alpha}{}^{[\mu}R^{\nu]\alpha}{}_{\rho\sigma} + \frac{9}{4}R^{\alpha}{}_{[\rho}R^{\mu\nu}{}_{\sigma]\alpha} + \frac{1}{4}RR^{\mu\nu}{}_{\rho\sigma}\right.\\ &\left.+\frac{35}{8}R^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma} + \frac{1}{2}R_{\lambda}{}^{[\mu}\delta^{\nu]\lambda}{}_{\beta[\rho}R_{\sigma]}{}^{\beta}\right)\,,\end{aligned}$$

Regular solutions

- No need to know $\left(\mathcal{Q}^{-1}\right)^{\alpha\beta}{}_{\rho\sigma}$ to solve the EoMs!

$$2\mathcal{E}^E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 12\hat{F}_\mu{}^\alpha\hat{F}_{[\nu\alpha}\,\mathcal{Q}_{\rho\sigma]}{}^{\rho\sigma} + 3\,g_{\mu\nu}\hat{F}^{\alpha\beta}\hat{F}_{[\alpha\beta}\,\mathcal{Q}_{\rho\sigma]}{}^{\rho\sigma}$$
$$+ 6\hat{F}^{\alpha\beta}\hat{F}_{[\alpha\beta}\frac{\partial\mathcal{Q}_{\rho\sigma]}{}^{\rho\sigma}}{\partial R^{\mu\lambda\tau\gamma}}R_\nu{}^{\lambda\tau\gamma} + 12\nabla^\lambda\nabla^\gamma\left(\hat{F}^{\alpha\beta}\hat{F}_{[\alpha\beta}\frac{\partial\mathcal{Q}_{\rho\sigma]}{}^{\rho\sigma}}{\partial R^{\mu\lambda\nu\gamma}}\right) + (\mu\leftrightarrow\nu)\,,$$
$$\mathcal{E}^M_\nu = \nabla_\mu\hat{F}^\mu{}_\nu\,, \quad \text{where } F_{\mu\nu} = 6\hat{F}_{[\rho\sigma}\,\mathcal{Q}_{\mu\nu]}{}^{\rho\sigma}\,.$$

- Setting $F = -\Phi'(r)\mathrm{d}t\wedge\mathrm{d}r$ and single-function SSS ansatz for metric:



Extremal Black Holes

- We want to study properties of **extremal BHs** in EQs.
- In general, inaccessible problem: we'd better focus on subfamilies of EQs to try to grasp general features of extremal BHs in EQs.
- First: among theories with algebraic EoM for $f(r)$ and magnetic solutions, we restrict to those which are at most quadratic in F:

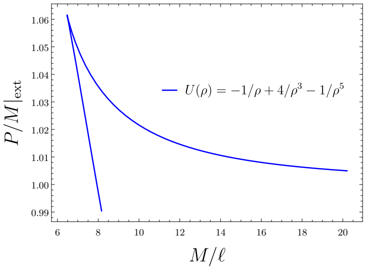
$$\frac{2M}{\ell} = \rho - U(\rho)\frac{P^2}{\ell^2}\,, \quad U(\rho) = -\sum_{n=0}^{\infty} \frac{2^n}{\rho^{2n+1}}\lambda_{n,1}\,, \quad \rho = \frac{r_h}{\ell}\,.$$

- **Extremal charge-to-mass** ratio:

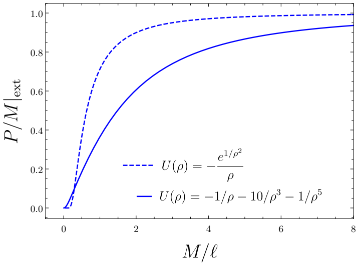
$$\left.\frac{P}{M}\right|_{\text{ext}} = \frac{2\sqrt{U'(\rho)}}{\rho\, U'(\rho) - U(\rho)}\,.$$

- Pick different choices of $U(\rho)$ to understand behaviour of extremal charge-to-mass ratio.

Extremal Black Holes



(a) WGC ✓

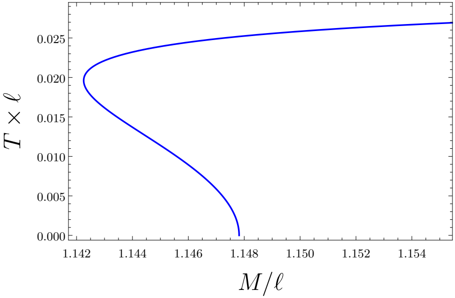


(b) WGC ✗

- WGC: $P/M|_{\mathrm{ext}}$ must not increase as mass increases.
- Extremal BH solutions do not exist below a minimal mass.

Extremal Black Holes

- Now we study extremal BHs in EQs with 2nd-order EoM for $f(r)$.
- On top of previously commented phenomena, we find EQ theories whose **extremal solutions do not represent the minimal mass** state for a given charge!



- It happens when $\left.\left(\frac{\partial^2 M}{\partial T^2}\right)_P\right|_{\rm ext} < 0.$

Conclusions and Future Directions

Conclusions and Future Directions

- Message to take Home: A new type of gravitational theories with a non-minimally coupled Maxwell field has been identified. These theories are defined by admitting electrically/magnetically charged SSS solutions with nice and reasonable properties:
  1. Amenability to computations.
  2. Physically meaningful solutions.

Future directions:

- How general are EQs? Can every theory be mapped via field redefinitions to an EQ? (this happens for GQs...)
- Holographic dual of these theories?
- Higher-dimensional generalizations of EQs?

Obrigado pela sua atenção

“O que não tem solução, solucionado está.”