Parter vertices and Parter Sets

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27 de maio 2016

Notation

- Let T be a tree on n vertices $1, 2, \ldots, n$.

- $\mathcal{S}(T)$ is the set of all $n \times n$ real symmetric matrices $A = (a_{ij})$ whose graph is T,

  That is $a_{ij} \neq 0$, with $i \neq j$, if and only if there is an edge between i and j.

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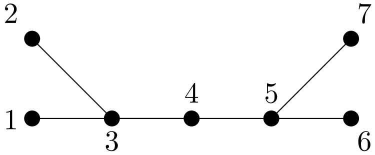
Consider the following tree with $n = 7$ vertices.



$$A = \begin{pmatrix} -1 & & & & & & \\ & -2 & & & & & \\ & & 3 & & & & \\ & & & \frac{1}{2} & & & \\ & & & & 0 & & \\ & & & & & \frac{1}{5} & \\ & & & & & & -4 \end{pmatrix}.$$

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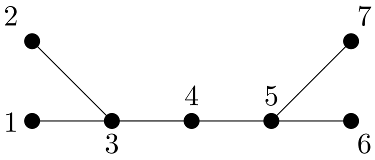

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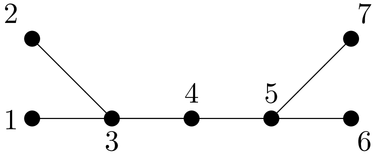

$$A = \begin{pmatrix} -1 & 0 & 5 & & & & \\ 0 & -2 & -1 & & & & \\ 5 & -1 & 3 & & & & \\ & & & \frac{1}{2} & & & \\ & & & & 0 & & \\ & & & & & \frac{1}{5} & \\ & & & & & & -4 \end{pmatrix}.$$

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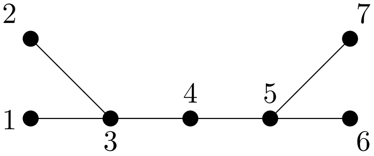

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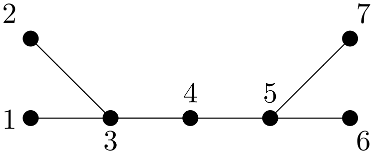

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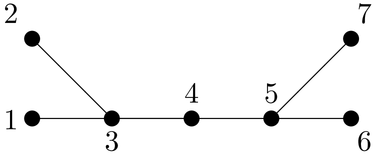

$$A = \begin{pmatrix} -1 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 & 0 \\ 5 & -1 & 3 & -\frac{1}{4} & & & \\ 0 & 0 & -\frac{1}{4} & \frac{1}{2} & & & \\ 0 & 0 & & & 0 & & \\ 0 & 0 & & & & \frac{1}{5} & \\ 0 & 0 & & & & & -4 \end{pmatrix}.$$

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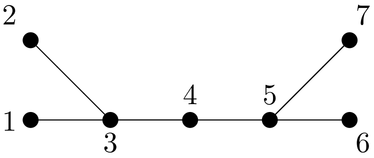

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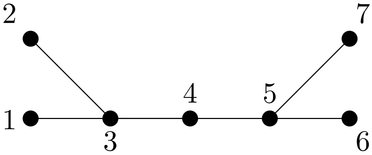

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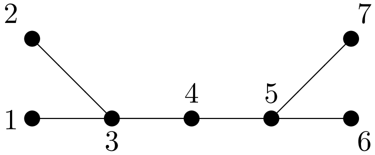

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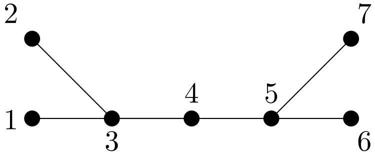

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- Let $\alpha \subseteq \{1, 2, \ldots, n\}$ be an index set.

- We denote the principal matrix of $A \in \mathcal{S}(T)$ resulting from deletion of rows and columns α by $A(\alpha)$.

- We denote the principal matrix of $A \in \mathcal{S}(T)$ resulting from retention of rows and columns α by $A[\alpha]$.

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Notation

- When α consists of a single index i, instead of $A(\{i\})$ we simply write $A(i)$.

- $A(i)$ is a direct sum whose summands we call blocks and correspond to components of $T - i$ (which we call branches of T at i or of $T - i$).

- We denote the multiplicity of $\lambda \in \mathbb{R}$ as an eigenvalue of $A \in \mathcal{S}(T)$ by $m_A(\lambda)$.

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There is a simple relationship between $m_{A(i)}(\lambda)$ and $m_A(\lambda)$ when A is Hermitian:

$$m_{A(i)}(\lambda) = m_A(\lambda) - 1 \quad \text{or} \quad m_{A(i)}(\lambda) = m_A(\lambda) \quad \text{or}$$

$$m_{A(i)}(\lambda) = m_A(\lambda) + 1.$$

PW-theorem

Theorem

Let T be a tree on n vertices, let $A \in \mathcal{S}(T)$ and $\lambda \in \mathbb{R}$ is such that $m_A(\lambda) \geq 2$. Then, there is a vertex i of T such that $m_{A(i)}(\lambda) = m_A(\lambda) + 1$ and λ occurs as an eigenvalue in direct summands of A that corresponds to at least three branches of T at i.

In 2003, PW-theorem was generalized to the case $m_A(\lambda) = 1$.

Theorem

Let A be a real symmetric matrix whose graph is a tree T, and suppose that there exists a vertex v of T and a real number λ such that λ is eigenvalue of A and of $A(v)$. Then

(a) *there is a vertex u of T such that $m_{A(u)}(\lambda) = m_A(\lambda) + 1$.*

(b) *if $m_A(\lambda) \geq 2$, then u may be chosen so that $deg\ u \geq 3$ and so that there are at least three branches $T_1,\ T_2,\ T_3$ of T at u such that $m_{A[T_i]}(\lambda) \geq 1$, $i = 1, 2, 3$.*

(c) *if $m_A(\lambda) = 1$, then u may be chosen so that $deg\ u \geq 2$ and so that there are two branches $T_1,\ T_2$ of T at u such that $m_{A[T_i]}(\lambda) = 1$, $i = 1, 2$.*

- We say that i is a Parter vertex of T, for λ relative to A, if i satisfies (a) and ((b) or (c)) of the previous Theorem.

- We say that i is a weak Parter vertex of T, for λ relative to A, if i satisfies (a) of the previous Theorem.

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- We say that i is a neutral vertex of T, for λ relative to A, if i satisfies

$$m_{A(i)}(\lambda) = m_A(\lambda).$$

- We say that i is a downer vertex of T, for λ relative to A, if i satisfies

$$m_{A(i)}(\lambda) = m_A(\lambda) - 1.$$

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Parter set

Definition

Let T *be a tree on* n *vertices,* $A \in \mathcal{S}(T)$*,* $i_1, i_2, \ldots, i_k$ *be vertices of* T *and* λ *be an eigenvalue of* A*.*
We say that $R = \{i_1, i_2, \ldots, i_k\}$ *is a Parter set of* T *for* λ *relative to* A *if for each* $j = 1, \ldots, k$*,* i_j *is a Parter vertex of* T *for* λ *relative to* A *and* $m_{A(R)}(\lambda) = m_A(\lambda) + k$*.*

Auxiliary results

Proposition: (Jonhson and Sutton, 2004)

Let T be a tree, $\lambda \in \mathbb{R}$, and $A \in \mathcal{S}(T)$ with $m_A(\lambda) \geq 1$. Let x and y be distinct Parter vertices of T, for λ relative to A, then

$$m_A(\lambda) - m_{A(x,y)}(\lambda) \in \{-2, 0\}.$$

□

Proposition

Let T be a tree, $\lambda \in \mathbb{R}$, and $A \in \mathcal{S}(T)$ with $m_A(\lambda) \geq 2$. Let v and t be distinct Parter vertices of T, for λ relative to A. Let S be the branch of $T - v$ that contains t. Then, $m_{A[S]}(\lambda) \geq 1$.

Proof: Since t is a Parter vertex and $m_A(\lambda) \geq 2$, then λ is eigenvalue of, at least, 2 blocks of $A[S](t)$. Using the interlacing theorem, $m_{A[S]}(\lambda) \geq m_{A[S](t)}(\lambda) - 1 \geq 2 - 1 = 1$. □

$m_A(\lambda) = 1$

Remark

If x is a Parter vertex of T, for λ relative to A, when $m_A(\lambda) = 1$, then, by definition, there are two branches T_1 and T_2 of $T - x$ such that $m_{A[T_1]}(\lambda) = m_{A[T_2]}(\lambda) = 1$, and, for all other branches S of $T - x$, $m_{A[S]}(\lambda) = 0$. $\square$

$m_A(\lambda) = 1$

Proposition

Let T be a tree, $\lambda \in \mathbb{R}$, and $A \in \mathcal{S}(T)$ with $m_A(\lambda) = 1$. Let x and y be distinct Parter vertices of T, for λ relative to A. Then $\{x, y\}$ is a Parter set of T, for λ relative to A.

Proof: Let R_1 and R_2 be the branches of $T - x$ such that $m_{A[R_1]}(\lambda) = m_{A[R_2]}(\lambda) = 1$ and let T_1 and T_2 be the branches of $T - y$ such that $m_{A[T_1]}(\lambda) = m_{A[T_2]}(\lambda) = 1$. Suppose , w.l.g., that $y \notin R_2$ and $x \notin T_2$. Thus,

$$m_{A(x,y)}(\lambda) \geq m_{A[R_2]}(\lambda) + m_{A[T_2]}(\lambda) = 2 = m_A(\lambda) + 1.$$

Then, by proposition 2.1, $m_{A(x,y)}(\lambda) = m_A(\lambda) + 2$ and $\{x, y\}$ is a Parter set of T, for λ relative to A.

$m_A(\lambda) = 1$

Proposition

Let T be a tree, $\lambda \in \mathbb{R}$, and $A \in \mathcal{S}(T)$ with $m_A(\lambda) = 1$. Let x and y be distinct Parter vertices of T, for λ relative to A. If U_1 is the branch of $T - x$ where y belongs and V_1 is the branch of $T - y$ where x belongs, then x is a Parter vertex of V_1, for λ relative to $A[V_1]$, and y is a Parter vertex of U_1, for λ relative to $A[U_1]$.

$m_A(\lambda) = 1$

Theorem

Let T be a tree, $\lambda \in \mathbb{R}$, and $A \in \mathcal{S}(T)$ with $m_A(\lambda) = 1$. Let $x_1, \ldots, x_p$, with $p \geq 2$, be distinct Parter vertices of T, for λ relative to A. Then $\{x_1, \ldots, x_p\}$ is a Parter set of T, for λ relative to A.

$m_A(\lambda) = 2$

Remark

If x is a Parter vertex of T, for λ relative to A, when $m_A(\lambda) = 2$, then, by Definition there are three branches T_1, T_2, and T_3 of $T - x$ such that $m_{A[T_1]}(\lambda) = m_{A[T_2]}(\lambda) = m_{A[T_3]}(\lambda) = 1$, and, for all other branches S of $T - x$, $m_{A[S]}(\lambda) = 0$. □

$m_A(\lambda) = 2$

Theorem

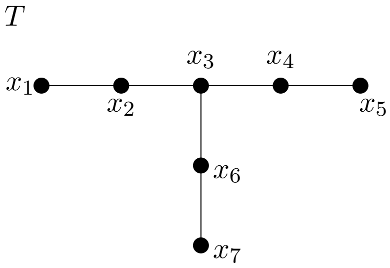
Let T be a tree, $\lambda \in \mathbb{R}$, and $A \in \mathcal{S}(T)$ with $m_A(\lambda) = 2$. Let $v_1, \ldots, v_p$, with $p \geq 2$, be distinct Parter vertices of T, for λ relative to A. Then

1) *$\{v_1, \ldots, v_p\}$ is a Parter set of T, for λ relative to A.*
2) *there is a path of T where $v_1, \ldots, v_p$ belong.*

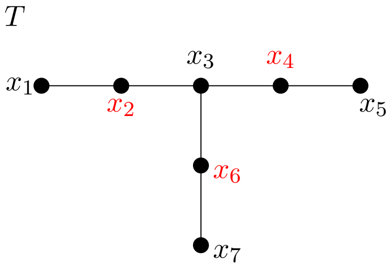
Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The eigenvalues of A are -1.5616, -0.6180, -0.6180, 1.0000, 1.6180, 1.6180, 2.5616.



The vertex x_i of T corresponds to row i of A, $i = 1, \ldots, 7$.



T
x1
x2
x3
x4
x5
x6
x7

$m_A(\lambda) = 3$

Remark

If x is a Parter vertex of T, for λ relative to A, when $m_A(\lambda) = 3$, then,

1. there are four branches T_1, T_2, T_3 and T_4 of $T - x$ such that $m_{A[T_1]}(\lambda) = m_{A[T_2]}(\lambda) = m_{A[T_3]}(\lambda) = m_{A[T_4]}(\lambda) = 1$, and, for all other branches S of $T - x$, $m_{A[S]}(\lambda) = 0$.

2. there are three branches T_1, T_2, and T_3 of $T - x$ such that $m_{A[T_1]}(\lambda) = m_{A[T_2]}(\lambda) = 1$ and $m_{A[T_3]}(\lambda) = 2$, and, for all other branches S of $T - x$, $m_{A[S]}(\lambda) = 0$.

Proposition

Let T be a tree and $A \in \mathcal{S}(T)$ with $m_A(\lambda) = 3$. Let $v_1, v_2, \ldots, v_p$ be Parter vertices of T for λ relative to A. Then $\{v_1, v_2, \ldots, v_p\}$ is a Parter set of T for λ relative to A.

$M(T)$ denotes the maximum possible multiplicity for an eigenvalue among the matrices $A \in \mathcal{S}(T)$.

In 1999, Johnson and Leal Duarte associated to a tree T a positive integer $P(T)$ defined as the minimum number of vertex disjoint paths, occurring as induced subgraphs of T, that cover all vertices of T. They also proved the following important theorem:

Theorem(Jonhson, Leal Duarte, 1999)

If T is a tree then

$$M(T) = P(T).$$

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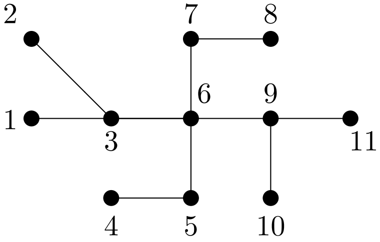
If T is a tree then

$$M(T) = P(T).$$

Consider the following tree T with $n = 11$ vertices.



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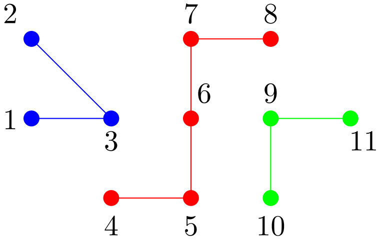


Theorem(Jonhson, Leal Duarte, Saiago, 2008)

Suppose that T is a tree with n vertices, $A \in \mathcal{S}(T)$ and λ is an eigenvalue of A such that $m_A(\lambda) = M(T)$. Then, no vertex of T is a neutral vertex of A for λ.

Theorem(Jonhson, Leal Duarte, Saiago, 2008)

Suppose that T is a tree with n vertices, $A \in \mathcal{S}(T)$ and λ is an eigenvalue of A such that $m_A(\lambda) = M(T)$. The removal of a P-vertex of A in T does not change the status of any other vertex.

Paths

If T is a path with vertices $1, \ldots, n$ whose terminal vertices are 1 and n, and $A \in \mathcal{S}(T)$, then A is an n-by-n irreducible tridiagonal symmetric matrix.

Corollary

Let T be a path, $A \in \mathcal{S}(T)$ and λ be an eigenvalue of A. Then $m_A(\lambda) = 1$.

Proposition

If A is an n-by-n irreducible tridiagonal symmetric matrix, then the eigenvalues of $A(1)$ and $A(n)$ each strictly interlace those of A.

When T is a tree, as usual, we denote by $L(T)$ the Laplacian matrix, i.e., $L(T) = A(T) - D(T)$ where $A(T)$ is the adjacency matrix of T and $D(T)$ is a diagonal matrix with the degree of vertices of T. It is well known that zero is an eigenvalue of $L(T)$.

Theorem

Let T be a path with n vertices and terminal vertices 1 and n (1 and n may be the same vertex). Let $L(T)$ be the Laplacian matrix of T . Then

$$m_{L(T)}(0) = 1 = P(T)$$

and there are no P-vertices of $L(T)$.

Lemma

Let T be a path with n vertices and let $A \in \mathcal{S}(T)$ such that $m_A(0) = 1$. If x and y are two P-vertices of A then x isn't adjacent to y.

The number of P-vertices of $A \in \mathcal{S}(T)$ is denoted by $P_v(A)$.

Proposition

Let T be a path with n vertices and let $A \in \mathcal{S}(T)$ such that $m_A(0) = 1$. Then

$$0 \leq P_v(A) \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof: Using Proposition 6.2 we know that a P-vertex isn't a terminal vertex of T. By Lemma 6.4 we know that there aren't adjacent P-vertices. So, $P_v(A) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. □

Definition

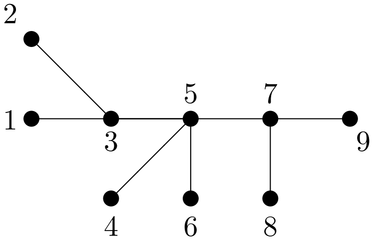
Let $T = (X, \mathcal{E})$ *be a tree. Let*

$$\mathcal{A} = \{R : R \subseteq X,\ T - R \text{ is a union of paths }, P(T-R) = P(T) + |R|, \\ T - (R - x) \text{ is not a union of paths}, \forall x \in R\}.$$

Let r_T be the number

$$r_T = \min\{|R| : R \in \mathcal{A}\}.$$

Consider the following tree T with $n = 9$ vertices.



The set $\mathcal{A}$ is the set

$$\{\{3, 7\},\ \{5\}\}.$$

So, $r_T = 1$

Proposition

Let T be a tree and let $A \in \mathcal{S}(T)$ with $m_A(0) = M(T)$. Then

$$P_v(A) \geq r_T.$$

For each $R \in \mathcal{A}$ let

$$X_R = \max\{\, |Y| : Y \subseteq T - R, d_{T-R}(x) = 2, \text{ for all } x \in Y, \text{ and the subgraph spanned by the vertices of } Y \text{ is the null graph}\}.$$

We denote by c_T the integer

$$c_T = \max\{\, |R| + X_R : R \in \mathcal{A}\}.$$

Proposition

Let T be a tree that is not a path and let $A \in \mathcal{S}(T)$ with $m_A(0) = M(T)$. Then

$$P_v(A) \leq c_T.$$