Soft Charges, Soft Hair and Black Hole Entropy.

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Covariant Phase Space

To find soft charges, use the covariant phase space formalism: (Peierls, Ashtekar, Zuckerman, Witten and Crnkovic, Wald, Iyer, Lee, Zoupas.)

Action

$$I = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} \;\; R + \text{possible boundary terms}$$

Variation $g_{ab} \to g_{ab} + h_{ab}$

gives the equations of motion and a boundary term:

$$\delta_h I = \frac{1}{16\pi} \int_{\mathcal{M}} G^{ab} h_{ab} + \int_{\partial\mathcal{M}} \theta$$

G_{ab} = Einstein tensor. $\theta(g; h)$ = presymplectic potential

three-form. Think of θ as being $\sum p_i \delta q_i$ with p, q being the generalised coordinates and momenta.

Explicitly

$$(*\theta)_a = \frac{1}{16\pi}(\nabla_b h_a^b - \nabla_a h)$$

The presymplectic current is defined by

- $\omega(g, h, h') = \delta_h \theta(g, h') - \delta_{h'} \theta(g, h)$

  ω is the presymplectic current, a three-form in spacetime. The integral of ω on a spacelike (partial) Cauchy surface Σ in $\mathcal{M}$ defines the symplectic form $\Omega(g, h, h')$. Think of this as being $\sum dp_i \wedge dq_i$.

  $\Omega = \int_\Sigma \omega$

- The phase space of the theory is the classical solutions of the Einstein equations g_{ab}, together with the tangent vectors in the phase space h_{ab}, the solutions of the linearised Einstein equations.

  Ω is a two-form in the infinite-dimensional phase space of general relativity.

Soft Charges

- The Iyer-Lee-Wald charges are defined by

$$\Delta Q_\zeta(g;h) = \int_\Sigma \omega(g, h, L_\zeta g)$$

  As h' is now pure gauge, this can be reduced to a surface integral

$$\Delta Q_\zeta(g;h) = \int_S F(g, h, L_\zeta g)$$

  where $dF = \omega$ and S is a closed two-surface in Σ.

- ΔQ is the change in the charge conjugate to the vector field ζ between the spacetime g_{ab} and $g_{ab} + h_{ab}$ provided h_{ab} obeys the linearised Einstein equations.

An explicit formula for Q is

$$\Delta Q_\zeta(g;h) = \frac{1}{16\pi}\int_S dS_{ab}\ \ [-4\zeta^a\nabla^b h + 2\zeta^a\nabla_c h^{cb} - 2\zeta_c\nabla^b h^{ac} - h\nabla^a\zeta^b + 2h^{ac}\nabla_c\zeta^b].$$

or

$$\Delta Q_\zeta(g;h) = \int_S \delta_h Q_N - \iota_\zeta\theta$$

where the Noether current 2-form Q_N is

$$Q_N = *\frac{1}{16\pi}d\zeta$$

We have been a bit cavalier with our definition of ΔQ.

A physical charge must be an exact two-form in phase space in order for Q to be a function of state and not depend on the path taken from the metric g_{ab} to $g_{ab} + h_{ab}$.

$$\delta_{h_2} \Delta Q_\zeta(g; h_1) - \delta_{h_1} \Delta Q_\zeta(g; h_2) = 0.$$

In the derivation of Q from the action, there are various ambiguities which allow us to add extra terms to Q. These were cataloged by Wald and Zoupas. One may have to add a counterterm ΔQ_{ct} to make an exact charge, $\Delta \widehat{Q}$.

$$\Delta \widehat{Q}_\zeta(g; h) = \int_S \delta_h Q_N - \iota_\zeta \theta + \iota_\zeta \Theta.$$

- S might be the boundary of an asymptotically flat spacetime at infinity. Then if ζ were a time translation Killing vector, then Q would be the Komar mass. if ζ were a rotational Killing vector then Q would be the Komar angular momentum.
- S might be the horizon of a Kerr black hole. If ζ were the null generator of the Killing horizon, then Q would be the mass of the black hole. If ζ were the generator of axisymmetry that is a Killing vector, then Q would be the angular momentum of the hole.
- If S were some arbitrary closed 2-surface and ζ were a time translation, then Q would be a quasilocal mass enclosed by S.

BMS group

Asymptotically flat spacetime near $\mathcal{I}^+$

$$ds^2 = -Vdu^2 + 2e^{2\beta}dudr + g_{AB}(dz^A - U^A du)(dz^B - U^B du)$$

Bondi gauge:

$$g_{rr},\ g_{rA},\ \partial_r\left(\frac{\det g_{AB}}{r^4}\right) \quad \text{all vanish.}$$

u - retarded time
r - radial coordinate
z^A - coordinates on a sphere.
Similar expression on $\mathcal{I}^-$.

As $r \to \infty$

$$g_{AB} \to r^2\gamma_{AB} + rC_{AB} + \dots$$

$$V \to 1 - \frac{2M_B(z^A)}{r} + \dots$$

$$U^A \to -\frac{1}{2r^2}D_B C^{AB} + \frac{1}{3r^3}C^A_B(D_C C^{CB}) - \frac{2N^A}{3r^3} + \dots$$

$$\beta \to -\frac{1}{32r^2}C^{AB}C_{AB} + \dots$$

γ_{AB} metric on the unit sphere.
M_B Bondi Mass aspect
N^A Angular Momentum aspect
$C_{AB}\gamma^{AB} = 0$, C_{AB} contains data gravitational radiation.
D_A covariant with respect to γ^{AB}

Suppose

$$\zeta = f\partial_u + Y^A\partial_A + \dots$$

f any spherical harmonic - supertranslations

Y^A Killing or conformal Killing on the sphere at infinity - traditional super-rotations
Novel super-rotations use any other Y^A but these Y^A are singular somewhere. Controversial.
Then supertranslation charge is found from

$$\Delta Q_f = \frac{1}{16\pi}\int_{\text{sphere at infinity}} d\Omega \;\; 4f\,\delta M_B + \frac{1}{2}f\,N_{AB}\delta C^{AB}$$

$N_{AB} = \partial_u C_{AB}$ is the Bondi News function, representing the flux of graviational radiation. First term in the integrand is exact, the second is not. The physical charge is found simply by omitting the second term to give

$$\Delta\widehat{Q}_f = \frac{1}{16\pi}\int_{\text{sphere at infinity}} d\Omega \;\; 4f\,\delta M_B$$

Flux formula: Change in the charge along an interval on $\mathcal{I}$

$$\frac{1}{16\pi}\int du\ d\Omega\ f\,(\frac{1}{2}N_{AB}N^{AB}+8\pi r^2 T_{uu}+D_A D_B N^{AB})$$

First term - Gravitational Radiation
Second term - Matter
These form the hard charge contributions.
Third term - Soft charge. Similar formulae for super-rotation charges.

$$\Delta\widehat{Q}_Y=\frac{1}{8\pi}\int_{\text{sphere at infinity}} d\Omega\ \ Y^A N_A$$

Magnetic Charges

Magnetic charges were discovered by Pope, H. Godazgar and M. Godazgar. Can be derived from a topological action (H. Godazgar and M. Godazgar and MJP)

$$I_{\rm mag} = \int d(E^a \wedge T_a)$$

In the absence of torsion this is

$$I_{\rm mag} = \int R^{ab} \wedge E_a \wedge E_b$$

and is identically zero. Its variation is non-zero giving a charge

$$Q = \int \iota_\zeta \Gamma^{ab} E_a \wedge E_b$$

The supertranslated magnetic charge with $f = 1$ is the NUT charge.
Other charges from Euler, Pontryagin invariants. Also Lorentz transformations in the first order formalism.

Soft charges carry zero energy.
The vacuum is thereby infinitely degenerate.
The graviton is the Goldstone boson associated with breaking this degeneracy
Similar phenomena with other gauge fields.

Algebra

Diffeomorphisms ζ should form an algebra. So that

$$L_\zeta L_{\zeta'} - L_{\zeta'} L_\zeta = L_{[\zeta,\zeta']}$$

.

The same should be true of the charges:

$$[Q_\zeta, Q_{\zeta'}] = Q_{[\zeta,\zeta']}$$

as long as diffeomorphism invariance is a symmetry of nature.
For the BMS group, supertranslations are abelian and form a semi-direct product with six Y^A of traditional super-rotations that form the Lorentz algebra.

Conservation Law

- In an asymptotically flat spacetime, the charge Q_+ at the past endpoint of $\mathcal{I}^+$ is the same as the charge Q_- at the future end-point of $\mathcal{I}^-$ with the proviso that the fields in Q_- are taken to be antipodal to those at Q_+.
  Due to Christodoulou and Klainerman.

Black Holes

In a black hole spacetime formed by collapse, massless particles can leave the domain of outer communication through $\mathcal{I}^+$ or by passing through the horizon. Thus $\delta\widehat{Q}$ can change either because of effects on $\mathcal{I}^+$ or through $\mathcal{H}^+$.

Diffeomorphisms ζ should form an algebra. So

$$L_\zeta L_{\zeta'} - L_{\zeta'} L_\zeta = L_{[\zeta,\zeta']}$$

.

The same should be true of the charges

But explicit calculations reveal the possibility that

$$[Q_\zeta, Q'_\zeta] = Q_{[\zeta,\zeta']} + K_{\zeta,\zeta'}$$

and if $K \neq 0$ then diffeomorphism symmetry would be violated.

Do black holes have entropy?

- Classical No-hair theorems.
- Stationary black holes act as the source only of the $\ell = 0$ components of the electromagnetic field: electric charge.
- $\ell = 0, 1$ components of the gravitational field: mass, angular momentum, linear momentum.
- Black holes characterised by mass M, angular momentum J and electric charge Q.
- Could have been made in an infinite number of ways but are otherwise indistinguishable.
- Zero temperature. Infinite Entropy.

- First law of Black Hole Mechanics (Bardeen, Carter and Hawking):
  An infinitesimal change in black hole equilibrium states is described by

$$dM = \frac{\kappa dA}{8\pi} + \Phi dQ + \Omega dJ$$

A = Surface area of the event horizon.
κ = Surface gravity.
Φ = Electrostatic potential of the black hole.
Ω = Angular velocity of the black hole.
Need to include soft hair.

Boltzmann interpretation of entropy:

$$S = \ln W$$

W is density of states for black holes of fixed M, J and Q.

A central Question:

What are the quantum states of a black hole?

- Suppose now that we deal with observers external to a black hole.
  Their observations are made regarding the event horizon as the boundary of space.
- Q are then the charges of the black hole when we choose S to be the event horizon.

Black Hole Entropy

We can find vector fields that represent superrotations on the event horizon. (Haco, Hawking, MJP and Strominger). Suppose in the Kerr metric the inner and outer horizons are at $r = r_{\pm}$.
Let

$$w^{+} = \sqrt{\frac{r - r_{+}}{r - r_{-}}} e^{2\pi T_R \phi}, \quad w^{-} = \sqrt{\frac{r - r_{+}}{r - r_{-}}} e^{2\pi T_L \phi - t/2M},$$

$$y = \sqrt{\frac{r_{+} - r_{-}}{r - r_{-}}} e^{\pi (T_L + T_R)\phi - t/4M}$$

where

$$T_L = (r_{+} + r_{-})/4\pi a, \, T_R = (r_{+} - r_{-})/4\pi a$$

M is the mass of the Kerr black hole and a the rotation parameter.

The Kerr metric near the bifurcation surface, $w^+ = w^- = 0$,

$$ds^2 = 4\frac{\rho^2}{y^2}dw^+dw^- + \frac{16M^2a^2\sin^2\theta}{\rho^2y^2}dy^2 + \rho^2 d\theta^2 + \cdots$$

where

$$\rho^2 = r_+^2 + a^2\cos^2\theta.$$

Looks a bit like an S^1 bundle over AdS_3.

$$w^+ \to e^{4\pi^2 T_R}w^+,\ w^- \to e^{4\pi^2 T_L}w^-,\ y \to e^{2\pi^2(T_L+T_R)}y.$$

so more like $AdS_3/\Gamma \sim BTZ$.

- The vector field

$$\zeta^+ = \epsilon_n(w^+), \quad \zeta^y = \frac{1}{2} y \epsilon_n', \quad \zeta^\theta = \zeta^- = 0, \quad n \in Z.$$

with $\epsilon_n = 2\pi T_R (w^+)^{(1+\frac{in}{2\pi T_R})}$ has a Lie bracket that reproduces the Virasoro algebra.
- Replacing $+ \leftrightarrow -$, $T_R \leftrightarrow T_L$ produces a second vector field that obeys a second Virasoro algebra that commutes with the first.
- To define the exact charges associated to these vector fields, one needs to add a counterterm with Θ given by

$$\Theta = *(n^c q \cdot \nabla \ell_c)$$

where q_{ab} is the induced metric on the bifurcation surface and ℓ is normal to the horizon and n is a null vector orthogonal to q such that $\ell \cdot n = -1$.

One then finds that, for both families of vector fields, their charges obey algebras with a non-vanishing central term

$$[Q_n, Q_m] = i(n - m)Q_{n+m} + in^3 J\delta_{n+m,0}$$

This central term corresponds to a conventional central charge $12J$ with J being the angular momentum of the black hole.

Observers exterior to the black hole will observe a violation of diffeomorphism invariance unless they can find a way to cancel the central term.

Black hole scattering for Kerr black holes:

- For quanta of energy δE and angular momentum δJ the absorption probability P contains a factor of

$$P \sim |\Gamma(1 + \frac{i\omega_L}{2\pi T_L})|^2 |\Gamma(1 + \frac{i\omega_R}{2\pi T_R})|^2$$

where

$$\omega_L = \frac{2M^3}{J}\delta E, \qquad \omega_R = \frac{2M^3}{J}\delta E - \delta J.$$

This formula is well known from the ancient history of black hole scattering.

It is also part of the probability for absorption in a two-dimensional conformal field theory whose left movers are at a temperature T_L and right movers at a temperature T_R for quanta of energies ω_L and ω_R. We hypothesize that on a section of the horizon, an

observer outside the black hole would see such a conformal field theory with central charges c_L and c_R at temperatures T_L and T_R.

Such degrees of freedom on the horizon would cancel the anomaly.

The statistical entropy of such a conformal theory is given by a formula due to Cardy.

$$S = \frac{\pi^2}{3}(c_L T_L + c_R T_R).$$

Substituting in the results from the Kerr metric results in

$$S = \frac{1}{4}A.$$

We believe that the soft hair therefore accounts for the black hole entropy and the quantum states of the black hole are described by the states of such a theory.

- A similar calculation works in Kerr-Newman, (Haco, Hawking and MJP).
- A similar calculation works in AdS, (MJP and Rodriguez).
- The final results hold in Schwarzschild although there is something a bit degenerate since as $a \to 0$, $T_{L,R} \to \infty$ and $c_{L,R} \to 0$.
- Inherently holographic.

Some Issues

- Not a solution to the information paradox as it leaves unclear how to deal with the species problem or how the information in collapse gets encoded into this field theory.
- Freely falling observers do not see this conformal field theory as the horizon is not a boundary for their description of spacetime. An explanation of black hole complementarity?
- What happens at the singularity.
- No cloning?
- Can you "derive" Θ from something?
- What are these conformal field theories with $c = 0$ mod 6.
- What happens when you throw something into the hole. How does it affect the Hawking radiation?

- Related work:
  - Strominger and Vafa + ...: A CFT from string theory for states that are supersymmetric. Can be regarded as a special case of our treatment. Similarly, counting of supersymmetric brane configurations that have the same quantum numbers as black holes.
  - Kerr-CFT: A special case.


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