Taming method in modal logic and mosaic method in temporal logic

An outline of these methods application

Flávio Escada

Instituto Superior Técnico - Departamento de Matemática

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Objective

The scope of this presentation is to give an outline on how to apply the taming method in modal logic, stating the main procedure and illustrating with a specific branch of modal logic, the pair arrow logic, and also to synthetize the mosaic method used in temporal logic.

I want to give an idea on how to apply those methods and present an outline of the proofs on the key results.

Summary

Basic modal logic - Syntax and Semantics

Syntax - The alphabet:
* A set of atomic propositional variables $\phi_0 = \{p_1, p_2, ...\}$;
* Primitive logical symbols: $\bot$ (contradiction), $\neg$ (negation), $\wedge$ (conjunction), $\Diamond$ (possibility modality);
* Defined logical symbols: $\top$ (tautology), $\vee$ (disjunction), $\supset$ (material implication), $\equiv$ (equivalence), $\Box$ (necessity modality).

A set of well formed formulas (wff):
* Any atomic propositional variable;
* $\bot$;
* If φ and ψ are wff then $\varphi \wedge \psi$, $\neg\psi$, $\Diamond\psi$ are also wff.

Abbreviations:

* $\top = \neg \bot$;
* $\varphi \vee \psi = \neg (\neg\varphi \wedge \neg\psi)$;
* $\varphi \supset \psi = \neg\varphi \vee \psi$;
* $\varphi \equiv \psi = (\varphi \supset \psi) \wedge (\psi \supset \varphi)$;
* $\Box\varphi = \neg\Diamond\neg\varphi$.

Semantics:

* (Kripke) frame: F = (W,R), where W is the set of possible worlds (as points, states, ...) and R $\subset$ W $\times$ W is a binary relation;
* (Kripke) model: M = (W,R,π), where π: $\phi_0 \longrightarrow 2^W$ is a "valuation", a truth assignement;
* $\pi(p)$ represents the set of all worlds in which p is true;
* $w \in \pi(p)$ means that p is true in world w.

Semantics - Satisfaction:

Given a model M $=$ (W,R,π) and $w \in$ W:
(1) M,w $\nVdash \bot$;
(2) M,w $\Vdash$ p iff w $\in \pi(p)$ for any p $\in \phi_0$;
(3) M,w $\Vdash \neg\varphi$ iff M,w $\nVdash \varphi$;
(4) M,w $\Vdash \varphi \wedge \psi$ iff M,w $\Vdash \varphi$ and M,w $\Vdash \psi$;
(5) M,w $\Vdash \Diamond\varphi$ iff there exists u $\in$ W such that (w,u) $\in$ R and M,u $\Vdash \varphi$.

Taming method applied to pair arrow logic

Objective:
Finding "well-behaved" versions of certains logics.

Motivation:
*Some logics aren't "well behaved" in some desirable characteristics. As an example of failure in some characteristics we have undecidability of classic FOL (First Order Logic) or undecidability of several versions of AL (Arrow Logic) which is a branch of Modal Logic;
*Andréka ('95) proposed relativized versions of FOL as "modal fragments" of classical logic;
*AL is important because it is applied in computer science, being decidability a clearly desirable property;
*The most interesting connective is composition, which contributes to the dynamic character of the logic. However associative operations make AL non decidable.

Taming method applied to pair arrow logic

Process:

Some existential frames of logic condition may be dangerous causing undecidability and non-finite axiomatizability, so we need to get rid of those existential frames.

1st step:

Get rid of all the existential frame conditions and then add back those who cause no harm.

(1.1) We denote **L(K)** as being the modal logic, where K is a Kripke frame; **SubK** denotes al the substructures in the FOL model-theoretic sense - so **L(SubK)** is the core of **L(K)**;

L(SubK) is relatively close to **L(K)** because all the universal conditions are maintained;

Henkin ('85), using a operation called **relativization**, obtained the class **Alg(SubK)**.

Taming method applied to pair arrow logic

(1.2) As we got rid of all the existential conditions **L(SubK)** is remarkably weaker than **L(K)**. Now we have to find K' such that K $\subset$ K' $\subset$ SubK where **L(K')** has nice properties. Notice that in our specific case of analysis (pair arrow logic) **SubK** consists of all frames whose universe are binary arbitrary relations; if we add reflexive and simmetric relations it still has good properties.

2nd step: Although we might streghten the logics like we said before the power of these logics is usually strictly smaller than the original one. There may be connectives that aren't definable anymore. So this 2nd step relies in introducing new connectives to the logic without losing the desirable properties.
Some examples of new connectives introduced: **D** (difference operator) and $<$n$>$ (graded modalities).

Taming method applied to pair arrow logic

Illustrating the method - The Arrow Logic:

Definition

We define AL with the following connectives: the Booleans, the identity constant, id, a unary connective $\otimes$ *(called converse) and a binary connective* $\cdot$ *(called the composition).*
$<$W,F,C,I$>$ is called an arrow frame if $W \neq \emptyset$, *I is the unary relation, F the binary and C the ternary relation in W.*
An arrow model is an arrow frame together with a valuation v of the propositional variables.
Simbolically we write the truth of a formula φ *in a world w together with a model $<$W,F,C,I$>$ as* $w \Vdash_v \varphi$.

Taming method applied to pair arrow logic

Definition

We define PAL (Pair arrow logic) with the same syntax as AL and a frame $<W, C_W, F_W, I_W>$ *where* C_W *is the composition relation,* F_W *is the converse relation and* I_W *is the identity relation, all restricted to W. W is defined as* $W \subset U \times U$*, for some U that we call the base of the frame.*

For all $<x,x'>, <y,y'>, <z,z'> \in W$ we define:
$C_W <x,x'><y,y'><z,z'>$ iff $x = y$ and $x'=z'$ and $y'=z$;
$F_W <x,x'><y,y'>$ iff $x=y'$ and $x'=y$;
$I_W <x,x'>$ iff $x = x'$;
The class of all pair frames is denoted by PF.
Let $H \subset \{r, s, t\}$ (r stands for reflexivity, s for simmetry and t for transitivity). PF_H represents the subclass of PF where each element satisfies the properties in H.

Taming method applied to pair arrow logic

PF_{SQ} denotes the class of square pair frames, with universe of the form $U \times U$.
$PAL_H = L(PF_H)$.
PAL_H's universe is any binary relation satisfying the conditions in H, and it's the relativized version of PAL. $PAL_\emptyset$ is the completely relativized version of PAL.
In PAL_{SQ} transitivity of the universe ensures that composition is associative and that causes Hilbert incompleteness and undecidability (Andréka '91 and '95).

Applying the method:

Some notation needed:
$[|\varphi|] := \{w \in W : w \Vdash_v \varphi\}$;
For X,Y $\subset$ W define X $\circ_W$ Y := { <w,u> $\in$ W : $\exists$ v (<w,v> $\in$ X and (<v,u> $\in$ Y)}.

Taming method applied to pair arrow logic

1st step:
(1.1.) Core(PAL_{SQ}) = $PAL_{\emptyset}$
Get rid of existential conditions by relativization.
In a model for PAL_H with universe $W \subset U \times U$,
$[|\varphi \cdot \psi|] = [|\varphi|] \circ_W [|\psi|] = ([|\varphi|] \circ_{U \times U} [|\psi|]) \bigcap$ W.
There are non-transitive relations W such that <w,w'> $\in$ W, and for some x, <w,x> $\in [|\varphi|] \circ_W [|\psi|]$ and <x,w'> $\in [|\chi|]$, while $[|\psi|] \circ_W [|\chi|] = \emptyset$.
That is $[|(\varphi \cdot \psi) \cdot \chi|] \neq \emptyset = [|\varphi \cdot (\psi \cdot \chi)|]$.

(1.2.) $PAL_{\emptyset}$ is weaker than PAL_{SQ}. For example the formula $\varphi \cdot id \equiv \varphi$ is not valid on $PAL_{\emptyset}$ but it is valid on PAL_{SQ} (because we now admit irreflexive universes).
We may consider pair frames with reflexive and/or symmetric universes getting back some of the existential frame conditions on PF_{SQ}, getting PAL_H (H $\subset \{r, s\}$).

Taming method applied to pair arrow logic

Theorem (1)

Let $H \subset \{r, s, t\}$ be arbitrary. Then:
(1) PAL_H has a strongly sound and complete Hilbert calculus iff $t \notin H$;
(2) PAL_H is decidable iff $t \notin H$;
(3) PAL_H has the Craig interpolation iff $t \notin H$;
(4) PAL_H has the Beth definability property iff $t \notin H$.

Craig interpolation:

If a formula $\varphi \implies \psi$ and the two have at least one atomic variable symbol in common, then there is a 3rd formula ρ, called an interpolant such that every nonlogical symbol in ρ occurs both in φ and ψ; $\varphi \implies \rho$ and $\rho \implies \psi$.

Taming method applied to pair arrow logic

Beth definability property:
Not formally, a property is implicitly definable in a theory with language L (via introduction of a new symbol φ of an extended language L') only if that property is explicitly definable in that theory (by a formula ψ in the original language L).

2nd step: In PAL_{SQ} there are connectives that are not definable in PAL_H ($H \subset \{r, s\}$).
$[|\Diamond\varphi|] = \{w \in W : (\exists u \in W)u \in [|\varphi|]\}$.
To stregthen the logic we define the D operator as:
$[|D\varphi|] = \{w \in W : (\exists u \in W)w \neq u \text{ and } u \in [|\varphi|]\}$.

Taming method applied to pair arrow logic

Theorem (2)

For $H = \{r, s\}$, PAL^D_H has a strongly sound and complete Hilbert-style calculus.

Theorem (3)

PAL^D_H is decidable iff $t \notin H$.

Definition (Sahlqvist's formulas)

Every Sahlqvist formula φ corresponds to an effectively obtained first-order formula φ^* in the FOL of the frames of the logic such that for every frame F, $F \models \varphi$ iff $F \models \varphi^*$;
For every set Γ of Sahlqvist's formulas the derivation system consisting of the axioms, Γ, and the rules modus ponens, generalization and substitution, is strongly sound and complete for $\models_K$, where $K := \{F : F \models \Gamma\}$.

Taming method applied to pair arrow logic

Next we state Sahlqvist's theorem in a very synthetic and simplified way:

Theorem (Sahlqvist)

If we have a set of Sahlqvist's formulas then the modal logic with respect to the class of frames satisfying those formulas is sound and complete.

Proof of theorem 3 - Outline:

1st step: Define the following class of pair frames:
$PFD_{r,s}$:= { F = $<$V,C_V,F_V,I_V,$\neq$$>$: $<$V,C_V,F_V,$I_V$$>$ $\in PF_{r,s}$} where $\neq$ denotes inequality and is the accessibility relation of the D-operator.
By definition $PAL^D_{r,s}$ is the logic of the class $PFD_{r,s}$.

Taming method applied to pair arrow logic

Note that an acessibility relation it's a relationship between 2 possible worlds and transmits the idea that a statement may not take the same truth in all possible worlds.
2nd step Define a class of arrow frames KD using Sahlqvist formulas such that for every set of $PAL^D_{r,s}$ formulas $\Gamma \bigcup \{\varphi\}$, we have $\Gamma \models_{KD} \varphi$ iff $\Gamma \models_{PFD_{r,s}} \varphi$.
Due to Sahlqvist's theorem we have a completeness result with respect to the class KD, which implies completeness in $PFD_{r,s}$ (and in consequence in PAL^D_H). $\square$

We can add a new operator called "graded modality <n>" such that:

$$| < n > | = \begin{cases} w & \text{if } [|\varphi|] \geq \text{n} \\ \emptyset & \text{otherwise} \end{cases}$$

Taming method applied to pair arrow logic

Theorem

The graded logic PAL_H^{grad} is decidable iff $t \notin H$.

So, by this theorem we can add all the <n> to PAL_H without losing decidability (having in mind that $t \notin H$).
Marx, Németi and Sain showed that PAL_H^D and PAL_H^{grad} don't preserve the Craig interpolation and Beth definability properties.
There are relativized versions of FOL that behave nicely, however FOL_3^2, the 3 variable fragment of FOL with binary predicates is equivalente to PAL_{SQ} (Henkin, Tarski). So whenever we obtain results about PAL_{SQ} these apply to FOL_3^2.

Temporal logic - Syntax and Semantics

Temporal logic is a branch of modal logic. Language contained in temporal logic consists of countably infinite propositional variables and connectives: $\neg$, $\wedge$, $\vee$, $\implies$, $\equiv$, with the same meanings as in the modal logic but also temporal connectives: F (future) and P (past).

A frame F is defined as F := (T,<), where $T \neq \emptyset$.

A model M is a frame F, together with a valuation v.

We say that:
$t \Vdash F\varphi$ iff $t' \Vdash \varphi$ for some t'>t;
$t \Vdash P\varphi$ iff $t' \Vdash \varphi$ for some t'<t;

Temporal logic - Syntax and Semantics

Besides F and P we can have defined temporal connectives G ("always in the future") and H ("always in the past") with the following relations:

$G\varphi = \neg F \neg \varphi$;
$H\varphi = \neg P \neg \varphi$;

We assume that linear order as irreflexive ($t \nless t$ for every t $\in$ T).

Mosaic method for temporal logic

We want to apply the mosaic method for proving decidability and Hilbert-style completeness of temporal logics over linear flows of time.
The mosaic approach serves as a general method to prove decidability of certain frames of logic.
The main key is to show that the existence of a model is equivalent to the existence of a finite set of partial models, called mosaics. This gives us a procedure to check the theoremhood of a formula.
First we present the definition of mosaics and saturated sets of mosaics (SSM), and then we state the key lemma (giving an outline of the proof), following, as consequence of that lemma, decidability and completeness of temporal logics.

Fact: The existence of an SSM for a formula φ is equivalent to the existence of a model for φ.

Mosaic method for temporal logic

Definition (Mosaic)

Let X be a fixed set of formulas closed under subformulas and single negation. A mosaic μ is a structure $(<m_0, m_1>, l)$ such that the function l associates a subset of X to each element of the base $\{m_0, m_1\}$ of μ such that, for every formula in X and index $i < 2$,

(1) $\varphi \in l(m_i)$ *iff* $\neg\varphi \notin l(m_i)$;

(2) $\varphi \wedge \psi \in l(m_i)$ *iff* $\{\varphi, \psi\} \in l(m_i)$;

(3) $G\varphi \in l(m_0) \implies G\varphi \in l(m_1)$;

(4) $G\varphi \in l(m_0) \implies \varphi \in l(m_1)$;

(5) $H\varphi \in l(m_1) \implies H\varphi \in l(m_0)$;

(6) $H\varphi \in l(m_1) \implies \varphi \in l(m_0)$.

We can allow mosaics with a singleton basis (where $m_0 = m_1$) requiring conditions one and two only.

We call conditions 5 and 6 the mirror images of 3 and 4 respectively.

Mosaic method for temporal logic

Definition (SSM)

A set M of mosaics is an SSM if satisfies the following saturation conditions: For every $\mu = (<m_0, m_1>, l) \in M$ *such that:*

(1) If $F\varphi \in l(m_1)$ *then there is a mosaic* $\mu' = (<m'_0, m'_1>, l') \in M$ *such that* $l(m_1) = l'(m'_0)$ *and* $\varphi \in l'(m'_1)$*;*

(2) If $F\varphi \in l(m_0)$*, then either* $F\varphi \in l(m_1)$ *or there are mosaics* $\mu' = (<m'_0, m'_1>, l')$ *and* $\mu'' = (<m''_0, m''_1>, l'') \in M$ *such that* $l(m_0) = l'(m'_0)$*,* $l(m_1) = l''(m''_1)$ *and* $\varphi \in l'(m'_1) = l''(m''_0)$*;*

(3) If $P\varphi \in l(m_1)$ *then there is a mosaic* $\mu' = (<m'_0, m'_1>, l') \in M$ *such that* $l(m_0) = l'(m'_1)$ *and* $\varphi \in l'(m'_0)$*;*

(4) If $P\varphi \in l(m_1)$*, then either* $P\varphi \in l(m_0)$ *or there are mosaics* $\mu' = (<m'_0, m'_1>, l')$ *and* $\mu'' = (<m''_0, m''_1>, l'') \in M$ *such that* $l(m_0) = l'(m'_0)$*,* $l(m_1) = l''(m''_1)$ *and* $\varphi \in l'(m'_1) = l''(m''_0)$*.*

Mosaic method for temporal logic

Given a set Γ of formulas we say that there exists an SSM for Γ if there is an SSM such that Γ is contained in the label of one of the points of a mosaic. If Γ is a singleton $\{\gamma\}$ we talk about γ-SSM.

Lemma (1)

For any set Γ of formulas, Γ is satisfiable iff there exists an SSM for Γ.

Proof - Outline:

($\Longrightarrow$) Let (T,<,v) be a model and $t \in T$ such that $t \Vdash \Gamma$.
Let $X \supset \Gamma$ closed under subformulas and simple negation. For any $s \in T$, l(s) $= \{\varphi : s \Vdash \varphi\} \bigcap X$.
Now we define a mosaic $\mu(u, v) =$ (<u,v>,l(u),l(v)), u<v $\in$ T. This is an SSM for Γ since it contains all the mosaics from the same model for Γ.

Mosaic method for temporal logic

($\Longleftarrow$) We want to construct a model satisfying Γ, using the elements of the SSM, M, as bulding blocks. The objective is to cure the "defects" of the participating mosaics "gluing" them together.
Let X be the labelling set of M. We define the possbile defects as $D = \{ <q,F\varphi>,<q,P\varphi> : q \in \mathbb{Q}$ and $F\varphi$, $P\varphi \in X \}$.
We know that $|D| \leq |\mathbb{Q}| = \omega$.
Define **J** = (J,<,l) as a labelled structure such that $J \subset \mathbb{Q}$, $<$ is the ordering of rational numbers and l: J $\longrightarrow \mathbb{P}(X)$. We define a "future defect" as a pair <q,Fφ> if:
*q $\in$ J; $F\varphi \in$ l(q) and for every p $\in$ J such that p<q, $\varphi \notin l(p)$.
We say that **J** is coherent if satisfies the coherency conditions of the mosaic definition. Let's consider $J_n = (J_n, <, l_n)$ coherent labelled substructures and $\sigma : \omega \longrightarrow \omega$ such that for every $j \in \omega$ there are infinitely many $k \in \omega$ such that $\sigma(k) = j$. This σ will behave like a scheduling function which selects the defect $D(\sigma(n))$ to cure.

Mosaic method for temporal logic

step 0: Let μ be a mosaic with $\Gamma \subset l(m_i)$, i<2. We have 2 cases: $m_0 \neq m_1$ and we define $J_0 = (\{0,1\},<0,1>,l_0)$, where $l_0(j) = l(m_j)$, j<2; the base of μ is a singleton, hence $J_0 = (\{0\}, \emptyset, l(m_0))$, so J_0 is coherent.
n+1st step: Assume we have a finite linear order of rational numbers $<i_0, i_1, ..., i_k>$ with a coherent labelling l_n. Let $\sigma(n+1) = l$. Consider the list D of possible defects and take the $l-th$ element. We assume that $D(l)$ has the form $<i, F\varphi>$. Now check if $D(l)$ is an actual defect of J_n. If is not we can define $J_{n+1} = J_n$. If $D(l)$ is in fact a defect of J_n take i_j as the greatest element that causes the defect.

Mosaic method for temporal logic

We have two cases:
(1) i_j is the last point in the order defined at the beggining ($j = k$), and then there exists a mosaic in M (M is SSM) which we can glue to the others adding a new point to our order, creating J_{n+1} as this order: $<i_0, i_1, ..., i_k, i_{k+1}>$;
(2) $i_j < i_k$, by definition (2) of SSM we can find two mosaics such that we can add a new point i' gluing those mosaics to the others existent defining J_{n+1} as the order: $<i_0, i_1, ..., i_j, i', i_{j+1}, ..., i_k>$.
ω step: Take union **J** of the labelled structures defined before, and notice that they don't have any defect.
Now to define our model we take the valuation v defined as i $\in$ v(p) iff p $\in$ l(i) where $i \in J$ is a point and p is an atom. Considering the model (J,<,v) we can prove, by induction on the complexity of φ that $i \Vdash \varphi$ iff $\varphi \in l(i)$.

Mosaic method for temporal logic

Using previous lemma (1) we can prove:

Lemma (2)

For any set Γ of formulas, Γ is consistent iff there exists an SSM for Γ.

and also:

Lemma (3)

For any formula ϵ, it is decidable whether there is an SSM for ϵ.

Some questions

*Can we generalize the two studied methods to only one?
Since they have similar objectives we could think in uniformize the algorithm to answer this question.

*Find some more types of logics where this method can be applied.

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