Counting curves and the Stabilized Embedding Conjecture

Dusa McDuff

Department of Mathematics, Barnard College, Columbia University

dusa@math.columbia.edu

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Plan of lecture:

I will first introduce the stabilized embedding conjecture, that concerns the question of when one ellipsoid embeds symplectically into another.

To solve it, we need to show that certain obstructions exist.

The most geometrically direct way to do this is to show that certain J-holomorphic curves exist in four dimensions. I will then describe four or five different ways to construct such curves.

Introduction:

We work in $\mathbb{R}^{2n} = \mathbb{C}^n$ (and mostly with $n = 2$) with the standard symplectic form

$$\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n, \quad x_j + iy_j = z_j \in \mathbb{C}.$$

A symplectomorphism is a diffeomorphism that preserves the symplectic form. The boundary ∂E of an ellipsoid E with center 0 is given by the level set of a positive definite quadratic form. After a symplectic linear transformation and appropriate scaling, they can be put in the following normal form

$$E(a_1, \ldots, a_n) := \Big\{ z \in \mathbb{C}^n \Big| \sum_i \frac{\pi |z_i|^2}{a_i} \le 1 \Big\}, \quad 0 < a_1 \le a_2 \le \cdots \le a_n.$$

Question: When does $E(a_1, \ldots, a_n)$ embed symplectically in $E(b_1, \ldots, b_n)$? If such exists, we write $E(a_1, \ldots, a_n) \overset{s}{\hookrightarrow} E(b_1, \ldots, b_n)$.

The answer is known in dimension four, but unknown in general, (and there are no good conjectures about what happens in the general case.)

Embedding 4-dim'l Ellipsoids

Let $E(a,b)$ be the ellipsoid
$\{(z_1,z_2) : \pi\big(\frac{|z_1|^2}{a} + \frac{|z_2|^2}{b}\big) \le 1\}$.

Hofer conjectured around 2010 that
$\exists\ \mathrm{int}E(a,b) \overset{s}{\hookrightarrow} \mathrm{int}E(c,d)$ *iff* $\mathcal{N}(a,b) \le \mathcal{N}(c,d)$.

Here $\mathcal{N}(a,b)$ is the set of all numbers $ka+\ell b,\ \ k,\ell \ge 0,$ arranged with multiplicities in increasing order. So

$\mathcal{N}(2,2) = (0, \underbrace{2,2}, \underbrace{4,4,4}, \underbrace{6,6,6,6}, \underbrace{8,8,8,8,8}, \dots)$, and

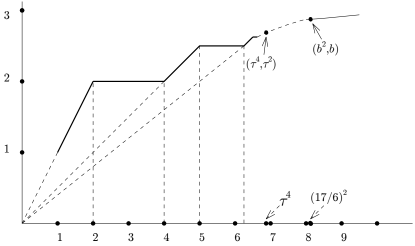
$\mathcal{N}(1,4) = (0, \underbrace{1,2}, \underbrace{3,4,4}, \underbrace{5,5,6,6}, \underbrace{7,7,8,8,8}, \dots$ Thus $\mathcal{N}(1,4) \le \mathcal{N}(2,2)$

because the first sequence is termwise no larger than the second.

- An embedding $\mathcal{N}(1,4) \overset{s}{\hookrightarrow} \mathcal{N}(2,2)$ can be constructed either indirectly using symplectic inflation, or via an almost toric fibration.
- These numbers are the actions of the ECH generators: ECH = embedded contact homology — a 4-dimensional Floer-type homology theory related to gauge theory; whose generators are unions of closed orbits of the boundary Hamiltonian flow.
- This conjecture proved by McDuff (2012).

The "ellipsoid into ball" capacity function

For $a \geq 1$ define $c(a) := \inf\{\mu : E(1,a) \text{ embeds sympl. in } B^4(\mu) = E(\mu,\mu)\}$.
This function was calculated by McDuff–Schlenk (2012).



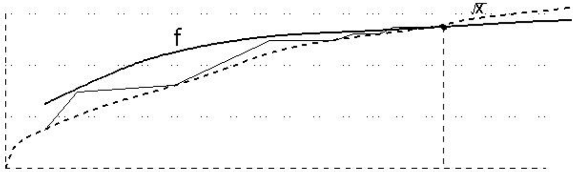
- For $a < \tau^4 \approx 6.7$ (where $\tau = \frac{1+\sqrt{5}}{2}$) there is an infinite staircase (with numerics based on the Fibonacci numbers),
- for $a \geq 8\frac{1}{36} = \left(\frac{17}{6}\right)^2$, $c(a) = \sqrt{a}$ – no obstruction except for volume
- $\tau^4 < a < 8\frac{1}{36}$ is a transitional region;
- there are rather few results or plausible guesses as to behavior in dim > 4.
  The obvious analog of Hofer's conjecture is false (Guth) since
  $E(1,S,S) \overset{s}{\hookrightarrow} E(3+\varepsilon, 3+\varepsilon, S^3)$ for all S

The stabilized embedding problem:

Define $c_k(a) = \inf\{\mu : E(1,a) \times \mathbb{R}^{2k} \overset{s}{\hookrightarrow} B^4(\mu) \times \mathbb{R}^{2k}\}$, $k \geq 0$.

- (Cristofaro-Gardiner – Hind The Fibonacci stairs stabilize!
  i.e. $c_k(a) = c_0(a)$ for $1 \leq a \leq \tau^4$.
- (Hind) If $a > \tau^4$ and $k > 0$ then $c_k(a) \leq \frac{3a}{1+a}$ by an explicit folding construction

The volume obstruction $c_0(a) \geq \sqrt{a}$ disappears when $k > 0$ – and $x = \tau^4$ is exactly where the graphs of $\sqrt{x}$ and $f(x) = \frac{3x}{x+1}$ cross!

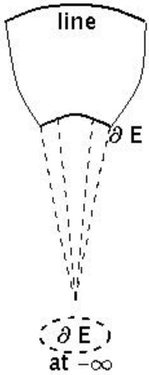


Here f is the graph of $\frac{3x}{x+1}$; the dotted graph is that of $g(x) = \sqrt{x}$ and the zigzag is the Fibonacci stairs, which goes between the two.

Conjecture: $c_k(a) = \frac{3a}{a+1}$ for all k and all $a > \tau^4$.
True for all integers $3m-1, m \geq 1$ (McD, 2018), for the 'ghost stairs' $\frac{F_{4m+2}}{F_{4m-2}}, m \geq 1$ (CGHM,2017) and finitely many other values (Siegel (2019)).

Embedding obstructions from curves:

- Note that $E(1,x) \overset{s}{\hookrightarrow} B^4(\mu)$ iff $E(1,x) \overset{s}{\hookrightarrow} \mathbb{C}P^2(\mu)$; so we can compactify the target.
- given an embedding $\iota : E(1,x) \overset{s}{\hookrightarrow} \mathbb{C}P^2(\mu)$ (with $x > 1$, large μ);
  - remove its image $\iota(E(1,x))$;
  - complete the manifold $\mathbb{C}P^2(\mu) \setminus \iota(E(1,x))$ by attaching a negative end of the form $\partial E(1,x) \times (-\infty, 0]$ to obtain $\mathbb{C}P^2_x$;
- Hofer analyzed the properties of finite energy J-hol curves in noncompact spaces such as $\mathbb{C}P^2_x$ and showed that they have negative ends asymptotic to the periodic orbits on $\partial E(1,x)$.



- if x is irrational, there are just two such orbits, the short one β of action 1 and the long one $\widetilde{\beta}$ of action $x > 1$.
- every curve in $\mathbb{C}P^2_x$ has positive action (or energy): so if it has degree d, and ends of total multiplicities $m, \widetilde{m}$ on $\beta, \widetilde{\beta}$ we must have $d\mu \geq m + \widetilde{m}x$, i.e.$\mu \geq \frac{m+\widetilde{m}x}{d}$.

Criterion for persistence under stabilization : 7.

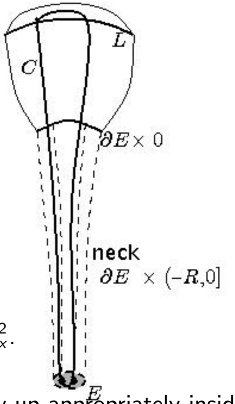
Theorem: [CG-H-M] *If C is a genus zero curve in $\mathbb{C}P^2_x$ with just one negative end asymptotic to some multiple of the short orbit β and of Fredholm index 0, then C persists under perturbations (eg of J and μ and $\iota : E(1,x) \overset{s}{\hookrightarrow} B^4(\mu)$) and stabilization.*

- If C has degree d and one end on β_m (the m-fold cover of the short orbit of $\partial E(1,x)$ its index is $2(3d-1-m-\lfloor \frac{m}{x} \rfloor)$.
- If $x = p/q + \varepsilon$ and C has degree d with one negative end of multiplicity p on the β, $p + \lfloor \frac{pq}{p+\varepsilon'} \rfloor = p+q-1$, so that C has index 0 if $3d = p+q$.
- The corresponding obstruction is $d\mu \geq p$ so that (with $\varepsilon \to 0$) the obstruction at $x = \frac{p}{q}$ is $\mu > \frac{p}{\frac{p+q}{3}} = \frac{3x}{1+x}$ — just what we want!
- The genus zero and single end conditions on C are essential: for higher genus curves or many ends the Fredholm index becomes negative as k (the stabilization dim) increases so curves disappear for generic J.
- But C need not be embedded. So in general we are not dealing with the curves of ECH (which are embedded and can have any genus).
- However, for the Fibonacci corners, the relevant curves C are embedded, and so can be found by using ECH (as in [CG-H])

Constructing suitable curves I:

given $\iota : E(1,x) \to \mathbb{C}P^2(\mu)$, we can think of $\mathbb{C}P^2_x$ as the top part of the space got from $\mathbb{C}P^2(\mu)$ by stretching the neck along $\partial E(1,x)$.

Hence we can find curves in $\mathbb{C}P^2_x$ by starting with a curve C in $\mathbb{C}P^2(\mu)$ (going through constraints in $\mathrm{int}E(1,x)$), and seeing what happens to it in the limit as we stretch the neck along $\partial E(1,x)$. Since C must also intersect any line in $\mathbb{C}P^2(\mu)\backslash E(1,x)$, C must split into (at least) two pieces, with top part in $\mathbb{C}P^2_x$.



- for the Fibonacci corners at $\frac{F_{2k+5}}{F_{2k+1}}$, we can blow up appropriately inside $E(1,x)$ and start with the exceptional obstructing curve that gives the staircase obstruction. Using ECH methods, it is relatively easy to show that, when we stretch the neck, the top has just one negative end.

[CG-H] did not compactify the target, instead looking at curves in the completed cobordism $X^{1^+}_x$, with top the distorted ball $\partial B(\mu, \mu+\varepsilon')$ and bottom $\partial E(1,x)$; they also stabilize under conditions analogous to those given above.

Constructing suitable curves II :

- for the ghost stairs, at $x = \frac{F_{4k+6}}{F_{4k+2}}$, we started in [CG-HM] with a family of 12 genus zero curves C with one double point, and use ECH methods to show that when we stretch the neck at least three must have just one negative end. This was hard. (When you stretch the neck the double point might go into the neck or even be in the lower part of the curves.)
- in the 'next case', i.e. numbers of the form $x = 11, 76/11, \ldots$, one would have to start with a family of 620 genus zero curves, each with three double points. – this does not seem promising.
- Hind–Kerman construct new curves C by attaching cylinders to known genus zero curves with one end. This works for some integers of the form $x = 3m - 1$.
- I constructed curves for all integers $x = 3m - 1$ by using the obstruction bundle gluing developed by Hutchings–Taubes.
- The last two approaches are limited because it is hard to prove that J-holomorphic cylinders with appropriate ends do exist, even in 4-dim. Homological arguments show that you must have a broken cylinder, but to make use of those you would need a much better understanding of how to glue multiply covered and nonregular curves. (Note: often multiple covers of index zero curves have negative index... and such negative curves cannot be avoided simply by perturbing J, you need a proper regularization scheme such as would be provided by SFT...)

Curves in $\mathbb{C}P^2$ satisfying local tangency constraints I:

Here is a completely different approach suggested by Kyler Siegel.

- Consider the set $\mathcal{J}_0$ of J on $\mathbb{C}P^2$ that are integrable in a small nbhd $\mathrm{Op}(p)$ of p, fix a smooth complex curve $D \subset \mathrm{Op}(p)$ through p, and choose a holomorphic map $g : (\mathrm{Op}(p), p) \to (\mathbb{C}, 0)$ s.t. $D = g^{-1}(0)$ and $dg(p) \neq 0$.
- As in Cieliebak–Mohnke [CM], we say that u has tangency order $m-1$ to (D, p) at the marked point z (and say u satisfies $\langle\!\langle\langle \mathcal{T}^{m-1} p \rangle\!\rangle\rangle$ if

$$\frac{d^j(g \circ u \circ f)}{d\zeta^j}\Big|_{\zeta=0} = 0 \qquad \text{for } j = 1, ..., m-1, \tag{1}$$

- As shown in [CM], $\langle\!\langle\langle \mathcal{T}^{m-1} p \rangle\!\rangle\rangle$ imposes a condition on curves of codimension $2m$. e.g., there is a unique degree 2 curve in $\mathbb{C}P^2$ that satisfies $\langle\!\langle\langle \mathcal{T}^4 p \rangle\!\rangle\rangle$: — take the unique conic though 5 points, and move them together (along D).
- We define $N_d \langle\!\langle\langle \mathcal{T}^{3d-2} p \rangle\!\rangle\rangle$ to be the number of genus 0, degree d curves u in $\mathbb{C}P^2$ that satisfy $\langle\!\langle\langle \mathcal{T}^{3d-2} p \rangle\!\rangle\rangle$. We prove in [MSie] that this is well defined, i.e. independent of choices of D, generic $J \in \mathcal{J}_0$ etc.
- As J varies in a generic 1-parameter family this count is constant — by a version of automatic transversality all curves count positively.

Local tangency constraints (II):

- Given a partition $\mathcal{P} = (m_1, m_2, \ldots, m_b)$ we also show there is a well defined invariant $N_d\langle\langle\langle \mathcal{T}^{\mathcal{P}} p\rangle\rangle\rangle$ that counts degree d curves with b branches tangent to D at p to orders $m_1 - 1, m_2 - 1, \ldots, m_b - 1$.
- We establish combination rules showing how counts behave when the constraints at two points p, p' are combined as p moves to p': e.g.

$$N_d\langle\langle\langle \mathcal{T}^{(m)} p, \mathcal{T}^{(m')} p'\rangle\rangle\rangle = N_d\langle\langle\langle \mathcal{T}^{(m+m')} p\rangle\rangle\rangle + N_d\langle\langle\langle \mathcal{T}^{(m,m')} p\rangle\rangle\rangle, \quad m \neq m'.$$

- Using that we derive an algorithm to compute $N_d\langle\langle\langle \mathcal{T}^{\mathcal{P}} p\rangle\rangle\rangle$ for all partitions $\mathcal{P}$ of $3d-1$ and in particular $N_d\langle\langle\langle \mathcal{T}^{(3d-1)} p\rangle\rangle\rangle = N_d\langle\langle\langle \mathcal{T}^{3d-2} p\rangle\rangle\rangle$. (Note that $(3d-1)$ is the partition of $3d-1$ with one entry...)
- This algorithm has $\pm$ coeffs, so does NOT imply $N_d\langle\langle\langle \mathcal{T}^{(3d-1)} p\rangle\rangle\rangle > 0\ \forall d$. BUT it is easy to see this: if you start with C, you can choose the constraints so they are satisfied:Take any rational curve C of degree d, pick a regular point $p \in C$ and J integrable near p and such that C is J-hol; finally pick J-hol local D so that C is tangent to D to order exactly $3d-2$. Thus there is a curve satisfying $\langle\langle\langle \mathcal{T}^{(3d-1)} p\rangle\rangle\rangle$, and it is automatically regular

Connection to stabilized embedding problem:

- Given an embedding $\iota : E(1,x) \to \mathbb{C}P^2(\mu)$, where $x = 3d - 1 + \varepsilon$, we must show there is a genus zero, degree d curve in $\mathbb{C}P^2_x$ with a single negative end on $\beta_{3d-1}(x)$, i.e. it goes $3d-1$ times round the short orbit.
- We saw above that the index contribution of m-times the short orbit on $\partial E(1,x)$ is $2(m + \lfloor \frac{m}{x} \rfloor)$ Thus this index contribution is constant for all $x > m = 3d - 1$, and we show in [MSie] that in this case the number of degree d curves with this single negative end does not change as $x > 3d - 1$ varies. (We call such $E(1,x)$ a skinny ellipsoid.)
- Denote by $N^E_d \langle\langle\langle (3d-1) \rangle\rangle\rangle$ the number of degree d curves in $\mathbb{C}P^2_x$ with one negative end on $\partial E(1,x)$ (that we assume to be skinny). We show that this is well defined. We also show $N^E_d \langle\langle\langle (3d-1) \rangle\rangle\rangle = N_d \langle\langle\langle \mathcal{T}^{(3d-1)} p \rangle\rangle\rangle$ — but the argument is quite complicated, — and all we need is that $N^E_d \langle\langle\langle (3d-1) \rangle\rangle\rangle \neq 0$.
- However, it is not hard to see that $0 \neq N_d \langle\langle\langle \mathcal{T}^{(3d-1)}_D p \rangle\rangle\rangle \leq N^E_d \langle\langle\langle (3d-1) \rangle\rangle\rangle$: Simply take a curve C satisfying $\mathcal{T}^{(3d-1)}_D p$ and stretch the neck around a small skinny ellipsoid containing p. Then show that the top of the limiting building can only have one negative end. The proof involves using writhe estimates to understand the structure of the curve just before breaking — the kind of argument used by Hutchings–Nelson in their geometric construction of cylindrical contact homology.

A more general framework:

- To solve the problem in general, I think you need to put it in a larger context and use algebraic structures to help you. Siegel develops a very promising approach in *Computing Higher Symplectic Capacities*. Given an ellipsoid $E(1, x)$ (x irrat, $\beta, \widetilde{\beta}$ the short, long orbits) he defines a filtered $\mathcal{L}_\infty$-algebra $\mathcal{L}(x)$ with generators (roughly) given by elements $\beta_{(m_1,\dots,m_b)} \otimes \widetilde{\beta}_{(\widetilde{m}_1,\dots,\widetilde{m}_{\widetilde{b}})}$ — representing an orbit set consisting of b (unordered) ends on β of multiplicities $(m_1, \dots, m_b)$ and similarly on $\widetilde{\beta}$. The filtration is by the action.
- Given an embedding $\iota : E(1, x) \to E(\mu, \mu y)$, let X_x^y be the completion of $\mu E(1, y) \backslash \iota(E(1, x))$. Then (assuming $\exists$ suitable version of SFT) he defines a filtered $\mathcal{L}_\infty$-homom $\Phi_{y,x} : \mathcal{L}(y) \to \mathcal{L}(x)$ by 'counting' disconnected rational curves in X_x^y where each component has one negative end (and many top ends).
- We want to show that if $y = 1^+$ (so top is the distorted ball) the entry $\Phi_{1^+,x}(\widetilde{\beta}_{(1,\dots,1)})$ is nonzero. Siegel constructs an algorithm showing this in all cases he has calculated (again it has $\pm$ coeffs).
- BUT so far, $\nexists$ suitable regularization procedure.
- How far can you get with traditional methods?

Well-defined counts of curves:

Let $\mathcal{M}_{y,x}(\mathcal{P},\widetilde{\mathcal{P}})$ be the mod space of somewhere injective, connected, index 0, J-hol rational curves in X_x^y with one negative end and top on $\beta_{\mathcal{P},\widetilde{\mathcal{P}}} = \beta_{(m_1,\dots,m_b)} \otimes \widetilde{\beta}_{(\widetilde{m}_1,\dots,\widetilde{m}_{\widetilde{b}})}$. The count of such curves is well-defined if

- The number of elements in this mod space does not change as J etc vary in a generic 1-parameter family;
- There is no index zero building with these ends.

- For this to hold there cannot be any (disconn) index zero rational curve in $\mathbb{R} \times \partial E(1,y)$ with top partition $\mathcal{P},\widetilde{\mathcal{P}}$ (and each comp with one neg end). (These are branched covers of trivial cylinders that change $\mathcal{P},\widetilde{\mathcal{P}}$.)
- Example: Let β be the short orbit on $\partial E(1,1+)$. Then the two-fold cover of $\mathbb{R} \times \beta$ with two top ends and one neg end has index zero.
- We say $\mathcal{P},\widetilde{\mathcal{P}}$ are minimal wrt y if such covers do NOT exist. Then:

Proposition [MSie2] (i) If $\mathcal{P},\widetilde{\mathcal{P}}$ are minimal wrt y, $x >> 0$ is 'skinny' (we say $x = sk$), and $\nexists$ mult covers, then the count of curves in $\mathcal{M}_{y,x}(\mathcal{P},\widetilde{\mathcal{P}})$ is well defined.
(ii) If $y = 1^+$, then $\forall x$ the count of curves in X_x^{1+} with top on $\widetilde{\beta}_{(1,\dots,1)}$ is well defined. (and this is precisely the count we want to be $\neq 0$.)

The capacity $\mathfrak{g}_k$:

We saw that $\#\mathcal{M}_{x,sk}(\mathcal{P},\widetilde{\mathcal{P}})$ is well defined for $\mathcal{P},\widetilde{\mathcal{P}}$ minimal.
Define the action of $C \in \mathcal{M}_{x,sk}(\mathcal{P},\widetilde{\mathcal{P}})$ to be $|\mathcal{P}| + |\widetilde{\mathcal{P}}|x$ (where $|\mathcal{P}| = \sum m_i, |\widetilde{\mathcal{P}}| = \sum \widetilde{m}_i$).

Define $\mathfrak{g}_k(x) =$ min action of a curve C in X_{sk}^x with neg end on η_k, the k-fold cover of short orbit (and any top partitions) (C - genus zero, J-hol for gener J)

Proposition [MSie2]

- $\mathfrak{g}_k(x)$ is well defined, and monotone under symplectic embeddings: i.e. $\exists\ E(1,x) \overset{s}{\hookrightarrow} \mu E(1,y) \Longrightarrow \mathfrak{g}_k(x) \le \mu \mathfrak{g}_k(y)$;
- For all x such that $\max(3/2, \lceil x \rceil - 1) \le x < m = \lceil x \rceil$, we have
  - $\mathfrak{g}_k(x) = k$ if $1 \le k < \lceil x \rceil$,
  - $\mathfrak{g}_{\lceil x \rceil + 2i}(x) = x + i$ for $i \ge 0$, and
  - $\mathfrak{g}_{\lceil x \rceil + 2i+1}(x) = \lceil x \rceil + i$ for $i \ge 0$.
- for $1 \le x < 3/2$, $\mathfrak{g}_k(x), k \ge 1$, is given by the sequence

  $$1,\ x,\ 2,\ 1{+}x,\ 2x,\ 2{+}x,\ 1{+}2x,\ 3x,\ \cdots\ 2{+}(n{-}2)x,\ 1{+}(n{-}1)x,\ nx,\ \cdots$$

  In particular, $\mathfrak{g}_k(1) = \lceil \frac{k+1}{3} \rceil$, i.e. the numbers are $1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, \ldots$ for $k \ge 1$.

Stabilization of $\mathfrak{g}_k$:

We expect that $\mathfrak{g}_k$ stabilizes: i.e. we should be able to extend the definition of $\mathfrak{g}_k(X)$ to manifolds such as $X_x^Y \times B^{2k}(S)$ where $S >> 0$ and to show $\mathfrak{g}_k(X_x^y) = \mathfrak{g}_k(X_x^y \times B^{2k}(S))$. (proof in progress)

If so, we get a new family of stable capacities that in some situations

- contain more information than the Ekeland–Hofer capacities
- give sharp embedding obstructions for ellipsoids into ellipsoids — Cristofaro-Gardiner, Hind, Siegel have a new folding construction for embeddings $E(1, x, S) \overset{s}{\hookrightarrow} \mu E(1, y, S)$ giving an upper bound that for certain integral x, y agrees with the obstruction from the $\mathfrak{g}_k$.

We calculate $\mathfrak{g}_k(X_{sk}^x)$ by (a) constructing suitable curves in X_{sk}^x, and (b) showing that every other curve with neg end η_k has larger energy.

Suppose $x > 2$. Then

$$\mathfrak{g}_k(x) = 1, 2, \ldots, \lfloor x \rfloor, x, \lceil x \rceil, x + 1, \lceil x \rceil + 1, x + 2, \ldots$$

For $k \leq \lfloor x \rfloor$, $\mathfrak{g}_k(x)$ is represented by the k-fold cover of the cylinder with top on $\beta(x)$ and bottom on $\eta =$ short orbit on ∂E_{sk}. There are no somewh. inj. curves with these ends. (proof uses writhe estimate as in ECH)

The representing curves:

Lemma:

- $\exists$ somewh. inj. cylinder in X_{sk}^x with top on $\widetilde{\beta}$ (the long orbit) and bottom on $\eta_{\lceil x \rceil}(s)$ (note: $\lceil x \rceil = k_0 + 1$, and can take any $s > \lceil x \rceil$)
- $\exists$ somewh. inj. cylinder in X_{sk}^x with top on $\beta_{\lceil x \rceil}$ and bottom on $\eta_{\lceil x \rceil + 1}$.

  Proof: In each case, $\exists$ a cylindrical building with these ends since they have the same index; and this building must have just one level by the well definedness of curve counts with neg end on skinny ellip. In second case, the top $\beta_{\lceil x \rceil}$ has nontrivial monodromy contrib. to the index and the writhe, which allows there to be somewh. inj curve.
- $\exists$ somewh. inj. curve in X_{sk}^x
  with top on $(\beta, \widetilde{\beta})$ and bottom on $\eta_{\lceil x \rceil + 2}(s)$
  Can build this by obstruction bundle gluing.
- similarly $\exists$ somewh. inj. curve in X_{sk}^x
  with top $(\beta_{1 \times m}, \widetilde{\beta})$, bottom $\eta_{\lceil x \rceil + 2m}(s)$
- similarly $\exists$ somewh. inj. curve in X_{sk}^x
  with top $(\beta_{m, 1 \times m})$, bottom on $\eta_{\lceil x \rceil + 1 + 2m}(s)$
  (to get maximal index for a given action, you need one 'fat' top end and all others on β_1.)

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