On the Yau–Tian–Donaldson conjecture for spherical varieties

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Plan: 1) YTD conjecture
2) Spherical varieties
3) Main results

The Yau–Tian–Donaldson conjecture

\[ \exists \text{csck in } c_1(L) \iff (X,L) \text{ is K-stable} \]

Central phenomena in Kähler geometry:

given a compact complex manifold \( X \)

\( \omega \) Kähler class on \( X \)

\( \exists \) a constant scalar curvature Kähler metric \( \omega \)

Recall:

\( \omega \) Kähler form \( \iff g(\cdot, \cdot) = \omega(\cdot, \cdot) \)

\( \uparrow \)

real 2-form on \( X \) which can be written, in local holomorphic coordinates as

\[ \omega = \sum_{j,k} \frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_k} \, dz_j \wedge d\bar{z}_k \]

\( \Phi : U \to \mathbb{R} \)

positive definite
\[ \omega = i \tilde{\partial} \tilde{\bar{\partial}} \ln (1 + |z_1|^2 + \ldots + |z_n|^2) \]

on affine chart \([1 : z_1 : \ldots : z_n]\)

\[ \omega' \in [\omega] \quad \text{if} \quad \omega' - \omega = i \tilde{\partial} \tilde{\bar{\partial}} \psi \quad \psi : X \to \mathbb{R} \]

**Ricci curvature**: \( \text{Ric}(\omega) \) global real 2-form defined locally by

\[
\text{Ric}(\omega) = i \tilde{\partial} \tilde{\bar{\partial}} (- \ln \det (\frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k}))
\]

\( \nabla \) \( \twoheadrightarrow \) Kähler Einstein metric: \( \text{Ric}(\omega) = \lambda \omega \) for some \( \lambda \in \mathbb{R} \)

\( \text{e.g. Fubini-Study metric} \)

**Scalar curvature**: \( S(\omega) = \frac{n \text{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n} \)

function \( X \to \mathbb{R} \)

\( \text{csch} \) metric \( S(\omega) = n \frac{c_1(X) \cdot [\omega]^{n-1}}{[\omega]^n} \) chronological constant.
K-stability

$[\omega] = c_1(L)$ Lample line bundle in part. $X$ is projective.

Test configuration (TC)

$[X, L]$ flat $\mathbb{C}^*$-equivariant

family $\pi: X \to \mathbb{C}$ s.t

$L$ π-ample line bundle on $X$

and $(X_\pi, L_\pi) = \pi^{-1}(\eta) \cong (X, L^\mu)$ for some $\mu \in \mathbb{N}^*$

central fiber $(X_0, L_0)$ can be non-reduced

equipped with $\mathbb{C}^*$-action

Product TC: when $X_0 = X$.

Special TC: when $X_0$ is normal variety.
Donaldson-Futaki invariant

\[ H^0_0 (X_0, L^k) = \bigoplus_{i=1}^{d_k} C_{x_{ik}} \]

as \( \mathbb{C}^* \) representable

Weight of the action of \( \mathbb{C}^* \) on \( C_{x_{ik}} \)

\[ z \cdot s = z^{x_{ik}} s \]

\[ \frac{\sum \lambda_{x_{ik}}}{k \cdot d_k} = F_0 + F_1 k^{-1} + o(k^{-1}) \]

\[ \text{DF}(X, L) := -F_1 \]

Defn: \((X, L)\) is K-stable if

\[ \text{DF}(X, L) > 0 \quad \text{for all TC} \]

\[ = 0 \quad \text{if product TC} \]

Rem: when a group \( G \) acts on \( X \), one can consider only \( G \)-equivariant test configurations.
Best partial results on the YTD conjecture:

1) Fano case

Chen-Donaldson-Sun ~2015

FKE on a Fano mfd $\iff (X, c_1(X))$ is K-stable wrt special TC

2) Donaldson 2009: YTD conj holds for tur surfaces

+ simplification & convex geometric translation

of the K-stability condition.
1) Spherical varieties

2) Toric manifolds

X is toric if it admits an effective holomorphic action of a compact Lie group \((S^1)^n\) where \(n = \dim_{\mathbb{C}} X\).

Key property:

\[(X^n, L)\) polarized toric manifold \(\iff\) (Delzant) integral convex polytope \(\Delta\) in \(\mathbb{R}^n\) (up to translation).

Integral points in \(k\Delta\) encode \(H^0(X, L^k)\) as a \((S^1)^n\)-representation \((\mathbb{C}^*)^n\)-representation.
e.g.
• $\mathbb{C}P^2$
• $(\mathbb{S}^4)^2$ by choice of an affine chart

$\mathcal{H}_0(\mathbb{P}^2, \mathcal{O}(2))$

• can always blow up codim $\geq 2$ orbits $(\mathbb{C}^*)^n$
  from a biric mfd

Rem: a proj mfd of $\dim X = n$
  cannot admit an effective adelization
  $\mathcal{O}(\mathbb{S}^{n+1})$. 
b) Cohomogeneity one manifolds

If $KGX^n$ with orbits of dimension $2n$, $X$ is homogeneous

$\Rightarrow$ cscK in all Kähler classes

If $KGX^n$ admits at least one orbit of real dim $2n-1$

$\Rightarrow$ "cohomogeneity one manifold"

$\Rightarrow$ Classification by Akhiezer / Huckleberry & Snow

$\mathbb{F} \tilde{X} \rightarrow X$ K-equiv blowup st

$\tilde{X}$ is a K-homogeneous bundle over a homogeneous K-mfd with fiber a "primitive" cohomogeneity 1 mfd.
Primitive cases:

- \( \mathbb{P}^1 \cong S^1 \)
- \( \mathbb{P}^n \times (\mathbb{P}^n)^* \cong SU(n+1) \)
- \( Q^n \) projective quadric \( \cong SO(n+1) \)
- ...

all almost homogeneous under the action of \( \mathbb{K}^* \) can have as open orbit all complexified real symmetric spaces among the primitive cases.
Theorem B

\((X,L)\) cohomogenously one is cscK iff it is K-stable w.r.t. its (essentially unique) equivariant special test configuration.
3) Strategy for the proof

- Differential geometric approach using Chen-Cheng

- via uniform K-stability

\[ \text{Ch}i Li + \text{Yuji Odaka} \]

\[ \Downarrow \]

"uniform" YTD holds for spherical manifolds

true for, cohomogeneity + m-folds
Defn:
XSG complex connected reductive group
is spherical if a Borel subgroup $B \subseteq G$ acts with an open dense orbit on $X$. 
### Main results

<table>
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<tr>
<th>Toric mfd</th>
<th>Spherical mfd</th>
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<tbody>
<tr>
<td>$(X^n, L)$</td>
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<tr>
<td>$G \rightarrow T = (C^*)^n$</td>
<td>$G$ complex reductive group</td>
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<tr>
<td>$U \subseteq T$ maximal torus</td>
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- $\Delta \subseteq R^n$ convex polytope (moment polytope)
- $\mathcal{X}(T) = \text{lattice of characters of } T$
- $\mathcal{X}(T) \otimes R$
- $\Delta_+ \subseteq \mathcal{X}(T) \otimes R$
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<th>spherical</th>
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<tr>
<td>[ T ], piecewise linear rational convex function ( g ) on ( \Delta )</td>
<td>choose ( x \in \Delta_+ \cap M ), set ( \Delta = -x + \Delta_+ ), piecewise linear rational convex function ( g ) on ( \Delta ), with slopes in (-2).</td>
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\[
DF(g) = \int_{\Delta} g \, d\sigma - 2 \int_{\Delta} g \, d\mu \]

\[
\int_{\Delta} g \, P d\sigma - \frac{1}{2} \left( \alpha \cdot P - Q \right) g \, d\mu
\]

\[
P(x) = \prod_{\alpha \in \mathbb{R}^+ - \Delta_+^1} \frac{\langle x, \alpha + x \rangle}{\langle x, \alpha \rangle} \]

Duistermaat-Heckman polynomial

\[
Q(x) = d_{\alpha} P(\rho)
\]

\( a > 0 \) \& \( DF(1) = 0 \).
<table>
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<th>K-stable:</th>
<th>uniformly K-stable:</th>
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<tr>
<td>$DF(f) \geq 0$</td>
<td>$\exists \varepsilon &gt; 0 \text{ s.t. } DF(f) \geq \varepsilon \inf_{\Delta} \int (f + l - \inf(f + l)) , d\mu_{\text{linear}}$</td>
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<tr>
<td>$= 0 \text{ if } f \text{ linear}$</td>
<td>$\exists \varepsilon &gt; 0 \text{ s.t. } DF(f) \geq \varepsilon \inf_{\Delta} \int_P (f + l - \inf(f + l)) , d\mu_{\text{linear}(\mathcal{P})}$</td>
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Cohomogeneity one manifold $\Leftrightarrow$ spherical with $\dim(\Delta_+)=1$

$$\Delta = [s_-, s_+] \subset \mathbb{R}$$

$$\gamma : [s_-, s_+] \rightarrow \mathbb{R}$$

$$DF(\gamma) = \int_{s_-}^{s_+} g(s) P(s) \, ds + \int_{s_+}^{s_-} g(s) P(s) \, ds - \int_{s_-}^{s_+} 2g(t)(aP(t) - Q(t)) \, dt$$

**Theorem B**

$K\Gamma^1(X;L)$ cohomogeneity one admits csck if 

$$\begin{cases} 
    DF(t) = 0 & \text{if } 3 \text{ slice under } K\Gamma \\
    DF(t) > 0 & \text{if } 2 \text{ slice under } K\Gamma \\
\end{cases}$$

primitive

$P^* \Gamma^*_C$

all other cases
A general combinatorial condition

Decompose $\Delta$ into pyramids $T_i$ with base the facet $F_i$, vertex the origin $O$.

$\Delta = \{ x | u_i(x) \leq n_i \}$

**Theorem C**

Assume $\forall i,$

$$\frac{1}{n_i}(d_x P(x) + (r+1)P(x)) + 2Q(x) - 2a P(x) \geq 0 \ \forall x \in T_i$$

Then $(X,L)$ cscK iff $K$-stable wrt special TC
Why is it useful?

in toric case, (inspired by Zhan Zhu 2008)

all $n_i = 1 \iff \Delta$ reflexive $\iff X$ Fano

and $L = K_X^{n_i}$

then $2a = \text{scalar curvature} = n \frac{c_i(X) \cdot c_i(X)^{n_i - 1}}{c_i(X)^n} = n$

Thus $\frac{n + 1}{n_i} - 2a > 0$

+ this condition is open as the Kähler class varies.

Expectation: condition holds in neighborhood of $c_i(X)$ always

**Theorem D:** Proof for some families of spherical fibrations