The Gelfand-Graev character of $GL(n,q)$

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Definition

*Let G be a group and V a finite dimensional vector space over some field K. A K-**representation** of G over V is a homomorphism $\rho : G \to GL(V)$.*

Definition

*$\rho : G \to GL(V)$ is said to be **equivalent** to $\sigma : G \to GL(W)$ if there exists a $K-$isomorphism $\varphi : V \to W$ such that:*

$$\varphi(\rho(g)(v)) = \sigma(g)(\varphi(v)), \quad v \in V, \ g \in G$$

Preliminaries

Sum of representations

Given $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$, then G has a natural $K-$representation

$$\begin{aligned} \rho \oplus \sigma : G &\to GL(V \oplus W) \\ g &\mapsto \rho(g) \oplus \sigma(g) \end{aligned}$$

where $\rho(g) \oplus \sigma(g)(v, w) := (\rho(g)v, \sigma(g)w)$.

Remark

If $W \leq V$ is $G-$invariant, then $\rho|_W : G \to GL(W)$ is a $K-$representation of G over W, which we call a **subrepresentation** of ρ. It is called **irreducible** (and say that W is an **irreducible** subspace) if W has no other proper $G-$invariant subspaces besides $\{0\}$.

Preliminaries

Definition

*$\rho : G \to GL(V)$ is **completely reducible** if every $G-$invariant subspace of V has a $G-$invariant complement. In particular V is a finite direct sum of irreducible subspaces.*

Maschke's Theorem

If G is finite and car $K = 0$, then any $K-$representation of G is completely reducible.

Theorem

If G is finite, car $K = 0$ and K is algebraically closed, then:

$$\#\{\text{iso classes of irr } K\text{-representations of } G\} = \#\{\text{conjugacy classes of } G\}$$

Preliminaries

Remark

Let $\mathcal{B} = (v_1, \cdots, v_n)$ be a basis for V, and suppose $R(g)$ is the matrix of $\rho(g)$ relative to $\mathcal{B}$. Then $R : G \to GL_n(K)$, $g \mapsto R(g)$ is a homomorphism, a **matrix representation of G of degree n**.

Definition

*Given matrix representations $T, T' : G \to GL_n(K)$, we say that T is **equivalent** to S whenever there exists some $S \in GL_n(K)$ such that:*

$$T'(g) = ST(g)S^{-1}, \; g \in G$$

Preliminaries

Definition

*Suppose that a representation ρ affords a matrix representation $R : G \to GL_n(K)$. The **character** afforded by R is the function:*

$$\chi_R : G \to K, \qquad g \mapsto Tr(R(g))$$

Remark

χ_R is a class function i.e. $\chi_R(hgh^{-1}) = \chi_R(g)$.

Definition

*We define the **character** afforded by ρ as the character $\chi_\rho := \chi_R$ for some matrix representation R afforded by ρ. We refer to any of these as $K-$**characters of** G.*

Proposition

Isomorphic $K-$representations afford the same character. The converse holds if G is finite and car $K = 0$.

Definition

- *if ρ irreducible, we say χ_ρ is **irreducible**. Also, $Irr_K(G)$ denotes the set of irreducible characters of G.*
- *If $n = 1$, $\chi_\rho : G \to K^\times$ is a homomorphism. We call these **linear characters** of G, and denote the set of these by $Lin_K(G)$.*
- *$\chi_\rho(1_G) = n$ is called the **degree** of χ_ρ.*

Preliminaries

Remarks

- $\rho \oplus \sigma \Rightarrow \chi_{\rho\oplus\sigma} = \chi_\rho + \chi_\sigma$.
- If ρ is completely reducible, then χ_ρ is a sum of irreducible characters.
- If G is finite, car $K = 0$ and K is algebraically closed, then:

$$\#\mathrm{Irr}_K(G) = \#\{\text{conjugacy classes of} G\}$$

- In the case above, let $\mathrm{Irr}_K(G) = \{\chi_1, \cdots, \chi_t\}$. Then any $K-$character χ of G is of the form:

$$\chi = n_1\chi_1 + \cdots + n_t\chi_t$$

where $n_i \in \mathbb{N}_0$ is called the **multiplicity** of χ_i in χ. We say that χ_i is a **component** of χ if $n_i \neq 0$.

Preliminaries

Setting

G a finite group, $K = \mathbb{C}$. Let $\mathrm{cl}(G)$ be the vector space of $\mathbb{C}-$valued class functions of G. It can be shown that:

- $\mathrm{Irr}_K(G)$ is a basis for $\mathrm{cl}(G)$.
- $$\langle \chi, \psi \rangle_G := \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}$$

  defines an inner product in $\mathrm{cl}(G)$ for which $\mathrm{Irr}_K(G)$ is an o.n. basis such that $n_i = \langle \chi, \chi_i \rangle_G$.

- Representation theory of finite matrix groups G as an interest of its own (*e.g.* description of conjugacy classes of $U(n,q)$ is known to be a wild problem).
- A possible strategy: construction of representations of G which has almost every irreducible components in its decomposition, with multiplicity at most one ("models"(Gelfand-Bernstein,1974)).
- Typical approach: induction (to G) of certain linear characters from a given $p-$Sylow subgroup U (Gelfand-Graev (1962) used this approach for $SL(n,q)$).
- Our case: $G = GL(n,q)$ and $U = U(n,q)$.

Outline

1. Construction of the Gelfand-Graev character of GL(n,q)
2. Some more Representation Theory
3. Multiplicity free Theorem
4. The greater framework: finite groups of Lie type

Linear characters of U

Remark

Let H be a group. For any $\chi \in \mathrm{Lin}(H)$, we have $[H, H] \subseteq \ker\chi$.

$$Lin(H) \longleftrightarrow Irr(H/[H, H])$$

Linear characters of U

$$\begin{pmatrix} 1 & a & \cdots & * & * \\ 0 & 1 & b & \cdots & * \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & c \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & \cdots & * & * \\ 0 & 1 & b' & \cdots & * \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & c' \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & a+a' & \cdots & * & * \\ 0 & 1 & b+b' & \cdots & * \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & c+c' \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Linear characters of U

$$\begin{pmatrix} 1 & a & \cdots & * & * \\ 0 & 1 & b & \cdots & * \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & c \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & \cdots & * & * \\ 0 & 1 & -b & \cdots & * \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -c \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Linear characters of U

General form of a commutator in U:

$$\begin{pmatrix} 1 & 0 & \cdots & * & * \\ 0 & 1 & 0 & \cdots & * \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Linear characters of U

$$U/[U,U] \simeq \underbrace{\mathbb{F}_q^+ \times \cdots \times \mathbb{F}_q^+}_{n-1}$$

$$Lin(U) \longleftrightarrow \mathrm{Irr}(\underbrace{\mathbb{F}_q^+ \times \cdots \times \mathbb{F}_q^+}_{n-1})$$

Linear characters of U

Fix a non trivial $\theta \in \mathrm{Irr}(\mathbb{F}_q^+)$.

$$\begin{aligned} \chi_{(\alpha_1,\cdots,\alpha_{n-1})} : U &\longrightarrow \mathbb{C}^\times \\ u &\longmapsto \theta\left(\sum_{i=1}^{n-1} \alpha_i u_{i,i+1}\right) \end{aligned}$$

$$\mathrm{Lin}(U) = \{\ \chi_{(\alpha_1,\cdots,\alpha_{n-1})} \mid \alpha_i \in \mathbb{F}_q\ \}$$

Action of T on $\mathrm{Lin}(U)$

Remark

Let H be a group, $K \leq H$ and $\varphi \in \mathrm{Irr}(K)$. Then for each $h \in H$:

- $\varphi^h \in \mathrm{Irr}(K^h)$ where $K^h := h^{-1}Kh$.
- $\mathrm{Ind}_K^H \varphi = \mathrm{Ind}_{K^h}^H (\varphi^h)$.

Remark

T normalizes U. Hence T acts on $\mathrm{Lin}(U)$ by conjugation:

$$\chi^t(u) := \chi(tut^-1), \; u \in U$$

Action of T on Lin(U)

$$\chi_{(\alpha_1,\cdots,\alpha_{n-1})} \in Lin(U), \quad t = diag(t_1, \cdots, t_n) \in T$$

Action of T on Lin(U)

$$\chi^t_{(\alpha_1,\cdots,\alpha_{n-1})} = \chi_{(t_1\alpha_1(t_2)^{-1},\cdots,t_{n-1}\alpha_{n-1}(t_n)^{-1})}$$

Action of T on Lin(U)

Action of T on Lin(U)

$$\chi^{t}_{(\alpha_1,\cdots,\alpha_{n-1})} = \chi_{(t_1\alpha_1(t_2)^{-1},\cdots,t_{n-1}\alpha_{n-1}(t_n)^{-1})}$$

Definition

We say $\chi_{(\alpha_1,\cdots,\alpha_{n-1})} \in Lin(U)$ is ***nondegenerate*** *if $\alpha_i \neq 0$ for every $i \in \{1,\cdots,n-1\}$*

Proposition

T acts transitively on the set of nondegenerate linear characters of U.

Definition

Let $\sigma \in Lin(U)$ nondegenerate. Then $\Gamma := Ind_U^G \sigma$ is called the ***Gelfand-Graev character*** *of $G = GL(n, q)$.*

1. Construction of the Gelfand-Graev character of GL(n,q)
2. Some more Representation Theory
3. Multiplicity free Theorem
4. The greater framework: finite groups of Lie type

Representations and characters of f.d. $K-$algebras

Recall

Let A be a f.d. $K-$algebra and V an $A-$module. Then V affords a $K-$algebra homomorphism $\rho : A \to \mathrm{End}_K(V)$ i.e. a **representation** of A over V. Fixing a $K-$basis $\mathcal{B} = \{v_1, \cdots, v_n\}$ for V, we get a **matrix representation** $\rho : A \to \mathrm{M}_n(K)$.

Definition

Let V be an $A-$module, and consider the representation

$$\rho : A \to End_K(A), \quad a \mapsto a_L$$

where $a_L(v) := a \cdot v$. Suppose it affords some matrix representation $R : A \to M_n(K)$. The ***character*** *afforded by V is the $K-$linear map*

$$\chi_M : A \to K, \quad a \mapsto Tr(R(a))$$

Study of $KG-$modules

Group algebra

Define the **group algebra** KG of G as the ring of formal $K-$linear combinations $\sum_{g \in G} \alpha(g)g, \; \alpha(g) \in K$. Then KG is a $K-$algebra with basis G.

Equivalent study of $KG-$modules

- Any representation of G extends linearly to a representation of KG. Conversely, any representation of KG restricts to a representation of G ($K-$rep of $G \leftrightarrow KG$-modules).
- Subrepresentations $\leftrightarrow$ submodules.
- Completely reducible rep of $G \leftrightarrow$ semisimple $KG-$modules.
- Irr representations of $G \leftrightarrow$ simple $KG-$modules.
- Characters of $KG \leftrightarrow K-$characters of G.

Semisimple rings

Definition

*A ring R with unity is **semisimple** if R is a finite direct sum of minimal left ideals i.e. if ${}_RR$ is a semisimple module.*

Proposition

- *R is a semisimple ring iff every f.d. $R-$module is semisimple.*
- *If R is semisimple, then every simple $R-$module is isomorphic to some minimal left ideal of R.*
- *If R semisimple, then left ideals of R are generated by idempotent elements.*

May as well fix a finite set $\{M_1, \cdots, M_t\}$ of representatives of the iso classes of simple $R-$modules.

Semisimple rings

$$R = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_t$$

$$\mathcal{B}_i := \sum_{\mathcal{L}_j \cong_R M_i} \mathcal{L}_j, \quad R = \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_t$$

Wedderburn-Artin Structure Theorem

If R is a semisimple ring, then there exist unique $n_1, \cdots, n_t \in \mathbb{N}$ and division rings $D_1, \cdots, D_t$ such that:

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$$

Each n_i corresponds to the number of times M_i "appears"in the decomposition of KG i.e. number of minimal left ideals iso to M_i.

Reformulation of Maschke's Theorem and its consequences

Maschke's Theorem

If G is finite and car $K = 0$, then KG is semisimple.

Theorem

In the above conditions, if K is also algebraically closed then:

$$KG \cong M_{n_1}(K) \times \cdots \times M_{n_t}(K)$$

Remark

Since KG is semisimple, irreducible $K-$representations of G are afforded by minimal left ideals of KG, hence are related to certain idempotents of KG.

Induction of characters

Goal

Given $H \leq G$ and a representation ρ of H over V, want to build from it a new representation of G over some new vector space.

Definition

Let $H \leq G$ and suppose V is a (left) $KH-$module. We define the ***induced module*** *$V^G := KG \otimes_{KH} V$ (module multiplication is given by $g \cdot (z \otimes v) := (g \cdot z) \otimes v$). If V affords the character χ of H, then we denote by $\mathrm{Ind}_H^G(\chi)$ the character of G afforded by V^G, which is called the* ***induced character*** *of χ from H to G.*

Hecke algebras and decomposition of induced characters

Remarks

Let $H \leq G$. If $e \in KG$ is idempotent such that KHe affords a character χ of H, then:

- $KG \otimes_{KH} KHe \cong_{KG} KGe$ affords $\mathrm{Ind}_H^G(\chi)$.
- $eKGe$ is a ring which does not depend (up to iso) on the idempotent that affords χ (**Hecke algebra** of χ).
- If KG is semisimple, it can be shown that $eKGe$ is also semisimple.

Hecke algebras and decomposition of induced characters

Theorem

Assume G finite, car $K = 0$ and K algebraically closed. Let $Irr_K(G) = \{\chi_1 \cdots \chi_t\}$. Furthermore, let $H \leq G$, ψ a $K-$character of H and $\mathcal{H}$ its Hecke algebra. Then:

- $\chi_i|_{\mathcal{H}} \neq 0$ *if and only if* $\langle Ind_H^G(\psi), \chi_i \rangle_G \neq 0$.
- *The map* $\chi \mapsto \chi|_{\mathcal{H}}$ *is a bijection between irreducible characters of G s.t.* $\langle Ind_H^G(\psi), \chi_i \rangle_G \neq 0$ *and irreducible characters of* $\mathcal{H}$.

Key remark

$$\mathcal{H} \cong M_{n_1}(K) \times \cdots \times M_{n_t}(K)$$

Theorem

Γ *is multiplicity free i.e. its irreducible components appear with multiplicity at most one.*

Proof of the multiplicity free Theorem

Setting

- Fix $\sigma \in \mathrm{Lin}(U)$ nondegenerate.
- Take an idempotent $e \in \mathbb{C}U$ such that $\mathbb{C}Ue$ affords σ.
- $\mathbb{C}Ge$ affords Γ.
- The Hecke algebra of σ is $\mathcal{H} = e\mathbb{C}Ge$.

$$\mathcal{H} \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_t}(\mathbb{C})$$

Hence just need to show that H is commutative!

Proof of the multiplicity free Theorem

Lemma

Let $N \leq G := GL(n,q)$ be the subgroup of monomial matrices, and $B \leq GL(n,q)$ the subgroup of upper triangular matrices. Then:

- $B = U \rtimes T$.
- $G = BNB = UNU$.

$$ege \in \mathcal{H} \Rightarrow ege = eunu'e = \sigma(u)\sigma(u')ene \quad u, u' \in U, n \in N$$

Conclusion

$$\mathcal{H} = \langle \{ene \mid n \in N\} \rangle_{\mathbb{C}}$$

Proof of the multiplicity free Theorem

Lemma

$\mathcal{H}$ is commutative if there exists a linear isomorphism $\psi : \mathbb{C}G \to \mathbb{C}G$ satisfying the following conditions:

1. $\forall\ a, b \in \mathbb{C}G, \psi(ab) = \psi(b)\psi(a)$.
2. $\psi(U) = U$.
3. $\forall\ u \in U,\ \sigma(\psi(u)) = \sigma(u)$.
4. $\psi(n) = n$ *for all* $n \in N$ *such that* $ene \neq 0$.

Proof of the multiplicity free Theorem

$$e = \frac{1}{|U|}\sum_{u\in U}\sigma(u^{-1})u$$

$$\psi(e) = \psi\left(\frac{1}{|U|}\sum_{u\in U}\sigma(u^{-1})u\right) = \frac{1}{|U|}\sum_{u\in U}\sigma(u^{-1})\psi(u) =$$

$$= \frac{1}{|U|}\sum_{u\in U}\sigma(\psi^{-1}(u^{-1}))u = \frac{1}{|U|}\sum_{u\in U}\sigma(u^{-1})u = e$$

Proof of the multiplicity free Theorem

$$ene \neq 0, \ \psi(ene) = \psi(e)\psi(n)\psi(e) = ene$$

Conclusion

$$\psi|_{\mathcal{H}} = id_{\mathcal{H}}$$

$$a, b \in \mathcal{H}, \ ab = \psi(ab) = \psi(b)\psi(a) = ba$$

Proof of the multiplicity free Theorem

Involution of G

$$\begin{aligned} \gamma : G &\to G \\ g &\mapsto n_0 (g^{-1})^T n_0 \end{aligned}$$

where

$$n_0 = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$$

Proof of the multiplicity free Theorem

Remark

The map:

$$\widetilde{\sigma} : U \to \mathbb{C}^{\times}$$
$$u \mapsto \sigma(\gamma(u^{-1}))$$

is also a nondegenerate linear character of U.

Since T acts transitively on nondegenerate linear characters of U, choose $t \in T$ such that $\widetilde{\sigma} = \sigma^{t}$.

Proof of the multiplicity free Theorem

$$\sigma(\psi(u)) = \sigma(t\gamma(u^{-1})t^{-1}) =$$
$$= \sigma^t(\gamma(u^{-1})) = \widetilde{\sigma}(\gamma(u^{-1})) =$$
$$= \sigma(\gamma(\gamma(u^{-1})^{-1})) = \sigma(u)$$

Remarks

For our proof the following facts were essential:

1. $B = U \rtimes T$ and $G = BNB$.
2. Existence of T and its transitive action on $\mathrm{Lin}(U)$.
3. Existence of a certain involution of $GL(n, q)$.
4. (although not seen) Facts on representation theory of S_n.

Connected reductive group "counterparts"

1. Existence of a $BN-$pair (B Borel subgroup, N normalizer of a maximal torus $T \subset B$, U the unipotent radical of B).
2. $Z(G)$ connected implies "good"action of T over simple root subgroups of G.
3. Dynkin diagram automorphism and related opposition graph automorphism of G.
4. Representation theory of Weyl group $W = N/T$ and respective action on $\mathrm{Hom}(T, K^\times)$.

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