Co-associative fibrations of G_2-manifolds and deformations of singular sets

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Section I. G_2 and G_2 holonomy.

The exceptional Lie group G_2 can be defined in various ways.

- The octonion (or Cayley) algebra is a nonassociative algebra structure on $\mathbf{R}1 \oplus \mathbf{R}^7$. This gives a *cross product* $\times : \mathbf{R}^7 \times \mathbf{R}^7 \to \mathbf{R}^7$, just as the ordinary cross-product on $\mathbf{R}^3$ is related to the quaternions. The group G_2 can be defined as the subgroup of $GL(7, \mathbf{R})$ preserving this cross product.

- The formula $|x|^2 = (-1/6)\,\mathrm{Tr} L_x^2$, where $L_x(y) = x \times y$, defines the Euclidean structure on $\mathbf{R}^7$ and hence an exterior 3-form:

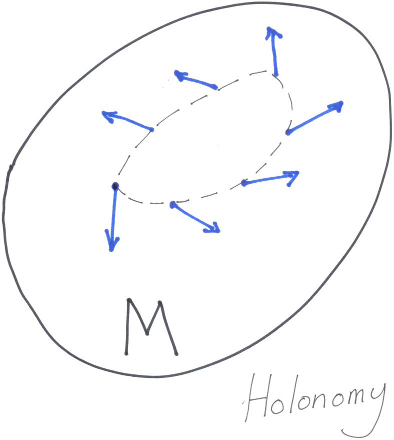
$$\phi(x, y, z) = \langle x \times y, z \rangle.$$

  The cross product can be recovered from ϕ so G_2 can also be defined as the subgroup which preserves ϕ.

  In fact the $GL(7, \mathbf{R})$ orbit of ϕ is *open* in Λ^3 and G_2 has dimension $49 - 35 = 14$.
- The *spin representation* in dimension 7 is a representation of $\mathrm{Spin}(7)$ on $\mathbf{R}^8$, acting transitively on the unit sphere S^7. The group G_2 can be defined as the stabiliser in $\mathrm{Spin}(7)$ of a unit vector. (Check: the dimension is $21 - 7 = 14$.)

The group G_2 is important in Riemannian geometry because it arises as a *holonomy group*.

Recall that a Riemannian metric g on an n-manifold M defines a Levi-Civita connection and hence the notion of parallel transport of tangent vectors along paths. Considering parallel transport around loops gives a holonomy group $\mathrm{Hol}(g) \subset O(n)$.


M
Holonomy

There is a small "Berger" list of possible (connected) holonomy groups which can arise (leaving aside "symmetric spaces").

- $SO(n)$;
- $U(m)$ and $SU(m)$ when $n = 2m$. These are *Kähler* and *Calabi-Yau* metrics.
- $Sp(1).Sp(r)$ and $Sp(r)$ when $n = 4r$. These are *quaternionic Kähler* and *hyperkähler* metrics.
- $G_2 \subset SO(7)$ when $n = 7$ and $\mathrm{Spin}(7) \subset SO(8)$ when $n = 8$.

The exceptional cases $G_2, \mathrm{Spin}(7)$ are important in some branches of theoretical physics. Such manifolds have zero Ricci curvature and parallel spinor fields, which are required for supersymmetry.

A Riemannian manifold with holonomy G_2 has a cross-product on tangent vectors which is preserved by parallel transport. Equivalently it has a parallel 3-form ϕ.

There are some special geometric objects one can consider in a manifold M of holonomy G_2.

A 3-dimensional vector subspace of $\mathbf{R}^7$ is called *associative* if it is closed under cross product. A 4-dimensional subspace is called *co-associative* if its orthogonal complement is associative.

These definitions give rise to the notions of associative and co-associative submanifolds of M. They are minimal and "calibrated" submanifolds with finite-dimensional deformation spaces, analogous to holomorphic curves in complex Kähler manifolds.

Standard questions in G_2-manifold theory.

- Find the 7-manifolds M which admit G_2-structures.
- Describe the moduli spaces of these structures and the relation with the period map $\phi \mapsto [\phi] \in H^3(M; \mathbf{R})$ (which defines a local equivalence)
- Describe the associative and co-associative submanifolds etc.

These questions are all *inaccessible* at present, in any kind of generality.

Section II. Adiabatic constructions and $K3$-surfaces.

Generalities

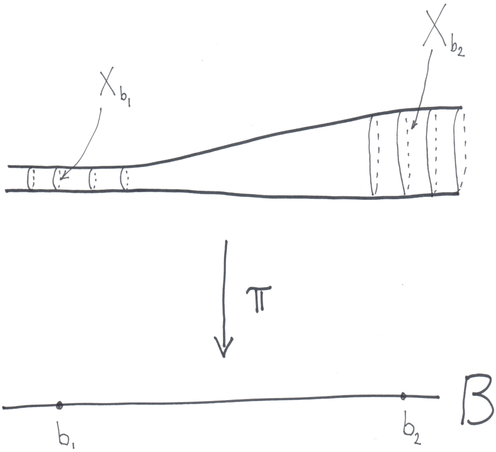
If one is studying some kind of structures on a manifold M it is often useful to consider a situation where there is a fibration $\pi : M \to B$ with very small fibres, with fibres of diameter $O(\epsilon)$ and base of diameter $O(1)$.

One expects that, on a scale $O(\epsilon)$ a neighbourhood of $\pi^{-1}(b)$ is modelled on a product $X_b \times \mathbf{R}^p$ where $p = \dim B$, scaled by a factor ϵ.

But on a scale $O(1)$ the structure X_b on the fibre can vary.

One expects that there should be some adiabatic equation on B which governs this variation in the limit as $\epsilon \to 0$.

In realistic cases there will usually be some discriminant set $\Delta \subset B$ where the fibres are singular, which causes extra complications.



X_{b_1}
X_{b_2}
π
b_1
b_2
B

The topic of this talk is a programme of this kind where M is a G_2-manifold, the base has dimension 3 and the (smooth) fibres are co-associative submanifolds of dimension 4.
The product model on $X \times \mathbf{R}^3$ is given by a 3-form

$$\phi = \sum_{i=1}^{3} \omega_i dt_i + dt_1 dt_2 dt_3,$$

where t_i are co-ordinates on $\mathbf{R}^3$ and $\omega_1, \omega_2, \omega_3$ are closed 2-forms on the 4-manifold X with

$$\omega_i \wedge \omega_j = \delta_{ij} \, \mathrm{vol}_X .$$

This structure on X is equivalent to a *hyperkähler* structure, with three complex structures I_1, I_2, I_3 obeying the quaternion relations and such that (X, I_i, ω_i) is Kähler.
It is known that, for compact X, this can only occur if X is a flat torus or a $K3$-manifold. We focus on the second case (because there are good reasons to think that there will be many examples).

This programme has analogies with the Strominger, Yau, Zaslow proposal to study *Special Lagrangian* fibrations of Calabi-Yau manifolds, near the “large complex structure” limit.

The moduli space of hyperkähler structures on a K3 manifold X is described by the “Torelli Theorem”. Recall that $H^2(X;\mathbf{R})$ has

- a quadratic form defined by the cup-product. This has signature $(3,19)$ and we will write $H^2(X;\mathbf{R}) = \mathbf{R}^{3,19}$.
- an integer lattice $\Lambda \subset \mathbf{R}^{3,19}$.

Let $\omega_1, \omega_2, \omega_3$ be a hyperkähler structure on X as above. The cohomology classes $[\omega_i] \in H^2(X)$ span a *maximal positive subspace* in $\mathbf{R}^{3,19}$. Conversely, a maximal positive subspace $P \subset \mathbf{R}^{3,19}$ arises from a hyperkähler structure (which is essentially unique) if and only if there is no class $\alpha \in \Lambda$ with $\alpha^2 = -2$ orthogonal to P.

The moduli space of hyperkähler structures is the quotient of an open subset $U \subset \mathrm{Gr}^+ = \mathrm{Gr}_3^+(\mathbf{R}^{3,19})$ by a discrete group $\Gamma \subset O(\Lambda)$. The complement of U has codimension 3.

What is the adiabatic equation which governs the variation of the fibre structure?

Locally, on a suitably small open set $B_0 \subset B$, the variation is given by a map

$$f : B_0 \to \mathrm{Gr}^+.$$

The adiabatic equation is that this is the Gauss map of a (parametrised) *maximal positive submanifold* in $\mathbf{R}^{3,19}$.

(Related to work of Baraglia in the case of torus fibres.)

A maximal positive submanifold in $\mathbf{R}^{p,q}$ is a p-dimensional submanifold whose tangent spaces are positive for the quadratic form and which is solution of the Euler-Lagrage equations associated to the volume functional.

Example Take $q = 1$ and consider the graph of a function u on $\mathbf{R}^p$.

- In Euclidean space $\mathbf{R}^{p+1}$ the minimal submanifold equation is

$$\sum_i \frac{\partial}{\partial x_i} \left(\frac{1}{\sqrt{1 + |\nabla u|^2}} \frac{\partial u}{\partial x_i} \right).$$

- In Lorentzian space $\mathbf{R}^{p,1}$ the maximal submanifold equation is

$$\sum_i \frac{\partial}{\partial x_i} \left(\frac{1}{\sqrt{1 - |\nabla u|^2}} \frac{\partial u}{\partial x_i} \right),$$

for functions u with $|\nabla u| < 1$

Section III. Singular fibres

Recall that in complex geometry a *Lefschetz fibration* of a compact complex manifold V of complex dimension m is a holomorphic map

$$p : V \to \mathbf{CP}^1$$

with a finite set of critical values $\Delta_p \subset \mathbf{CP}^1$. Each critical value is the image of a unique critical point and in local complex co-ordinates centred on those points the map is given by a nondegenerate quadratic form $\sum z_i^2$.

They are "meromorphic Morse functions".

Fix attention on the case $m = 3$, so the general fibres are complex surfaces. Going round a small loop about a point in Δ_p we get a *monodromy action* on the 2-dimensional homology of the fibre F. There is a *vanishing cycle* $\alpha \in H_2(F)$ with $\alpha^2 = -2$ and the monodromy is given by the Picard-Lefschetz reflection:

$$R_\alpha(h) = h + (\alpha.h)h.$$

We consider *Kovalev-Lefschetz fibrations* which, topologically, are smooth maps

$$\pi : M^7 \to S^3$$

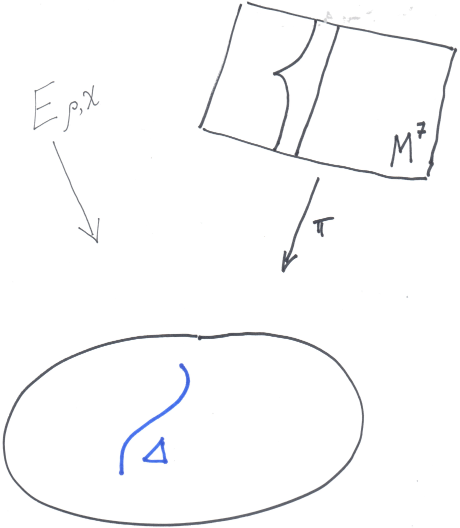
which are fibrations with $K3$ fibres outside a discriminant set $\Delta \subset S^3$ which is a *link* (a union of disjoint embedded circles). Over the link the local model is the product of a Lefschetz fibration with $\mathbf{R}$.
The cohomology of the fibres defines a flat vector bundle with fibre $\mathbf{R}^{3,19}$ over $S^3 \setminus \Delta$ and structure group $\Gamma \subset O(\Lambda)$. Equivalently, we have a homomorphism $\rho : \pi_1(S^3 \setminus \Delta) \to \Gamma$. A small loop around a component of Δ maps to a reflection.

There is one more piece of data, which is a lift $\hat{\rho}$ of ρ to the affine extension:

$$0 \to (\mathbf{R}^{3,19}, +) \to \hat{\Gamma} \to \Gamma \to 1.$$

This lift corresponds to a class χ in a certain cohomology group (a real vector space) H^1_ρ. It defines a flat affine bundle $E_{\rho,\chi}$ over $S^3 \setminus \Delta$ with fibre $\mathbf{R}^{3,19}$.

The data $\chi \in H^1_\rho$ is related to the cohomology class $[\phi] \in H^3(M^7; \mathbf{R})$.


$E_{\rho,\chi}$
M^7
π
Δ

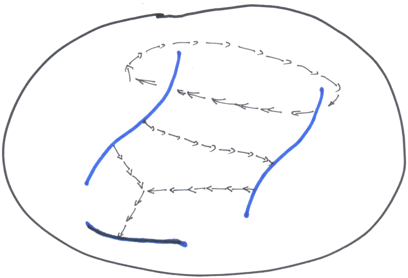
A global solution of the adiabatic equation is given by a section U of $E_{\rho,\chi}$ which, in a local trivialisation, maps to a maximal positive submanifold.

Around a point of Δ we require that the section is a "branched" solution, modelled transverse to Δ on the graph of the multivalued function

$$u(z) = \mathrm{Re}(z^{3/2}),$$

in $\mathbf{R}^{2,1}$ or $\mathbf{R}^3$.

One can also study adiabatic limits of the associative and co-associative submanifold equations.
"Adiabatic associatives" are integral curves of certain vector fields on S^3.



Adiabatic
associatives.

We get adiabatic versions of our "standard questions".

- Find data (Δ, ρ, χ) which admit maximal positive sections.
- Describe the moduli spaces of these solution and the relation with the class χ (which defines a local equivalence—see below).
- Describe the adiabatic associatives and co-associatives.

These questions are perhaps somewhat more accessible.

In reality there is much foundational work to do in setting up this theory.

Section IV. Deforming the singular set

One of the foundational analysis problems is that of deforming a solution $(U_0, \Delta_0, \rho, \chi_0)$ for a prescribed small deformation in the continuous parameter χ.

This is a prototype for a circle of related questions such as "gluing problems".

Questions of a similar nature have arisen in other areas in differential geometry in the past few years. Taubes, Takahashi, Wu, Zhang, Haydys, Walpuski and others have studied multivalued harmonic functions, spinors and 1-forms, related to compactness of solutions in gauge theory over 3-manifolds and 4-manifolds.

These questions are analogous to “free boundary problems”, with the difference that the singular set Δ has codimension 2 rather than 1.
It seems important to develop techniques to handle these questions.

We would expect to solve the deformation problem by some version of an implicit function theorem, which depends on inverting the linearised operator.

The linearisation of the maximal submanifold equation at the solution U_0 is a Laplace-type operator L. With a fixed Δ_0 we can solve the equation $Lf = \rho$ for f but in general f will have asymptotic behaviour

$$f \sim \mathrm{Re}(A(t)z^{1/2})$$

near Δ.
Here (z, t) with $z \in \mathbf{C}$ and $t \in \mathbf{R}$ are local co-ordinates around a point of Δ
Since $|\nabla f| \to \infty$ as $z \to 0$ the section $U_0 + f$ does not make sense as an approximation to a solution of the maximal submanifold equation near Δ_0.

The explanation is the formula

$$\frac{d}{dz}z^{3/2} = \frac{3}{2}z^{1/2},$$

in other words

$$(z+h)^{3/2} \sim z^{3/2} + \frac{3}{2}z^{1/2}h,$$

for $|h| << |z|$.
A deformation $U_0 + f$ away from Δ_0 needs to be matched with a deformation of Δ_0. If $U_0 \sim \mathrm{Re}(B(t)z^{3/2})$ and $f \sim \mathrm{Re}(A(t)z^{1/2})$ then taking $h(t) = \frac{2A(t)}{3B(t)}$

$$U_0(z+h(t),t) \approx U_0(z,t) + f(z,t)$$

provided that $|h| << |z| << 1$.

The problem is to control the derivatives of the deformation of Δ_0.
This can be attacked using Nash-Moser theory. We will outline a more elementary approach which perhaps gives more detailed information.

Notation:

- For a small perturbation χ of χ_0, write the maximal submanifold equation for a section V schematically as $\mathcal{F}(V) = 0$.
- For a link Δ close to Δ_0 and small radius r write $\Omega^+(r, \Delta)$ for the set of points of distance greater that $r/2$ from Δ and $\Omega^-(r, \Delta)$ for the r-neighbourhood of Δ. Write $\Omega(r, \Delta)$ for intersection, *i.e.* the annular region of points of distance between $r/2$ and r from Δ.

We can suppose that we start with an "approximate solution" V_0 such that $\mathcal{F}(V_0)$ is supported in $\Omega(1, \Delta_0)$ and

$$|\mathcal{F}(V_0)| \leq \eta_0.$$

We can assume that η_0 as small as we like.

The strategy is to generate a sequence of pairs (V_k, Δ_k) such that $\mathcal{F}(V_k)$ is (essentially) supported in $\Omega(\Delta_k, \lambda^k)$ with $|\mathcal{F}(V_k)| \leq \eta_k$. Here $\lambda < 1/2$ is to be determined later.

If the Δ_k converge in C^∞ to some Δ_∞ and if $\eta_k \to 0$ then one can show that the V_i converge to a limit V_∞ with singular set Δ_∞ which gives a solution to the problem.

Make the HYPOTHESIS that in our strategy the Δ_k are C^∞-small perturbations of Δ_0.

Then at stage k one shows that the equation $\mathcal{F}(V_k + f) = 0$ can be solved by small perturbation f over $\Omega^+(\lambda^{k+1}, \Delta_k)$, provided that η_k is small enough.

(i.e. the nonlinear problem is effectively linearised on the outer region $\Omega^+(\lambda^{k+1}, \Delta_k)$.)

The perturbation f has a leading term $f \sim \mathrm{Re}(A_k z^{1/2})$ in a local co-ordinates z transverse to Δ_k

We define Δ_{k+1} by deforming Δ_k according to the leading term A_k in f, as above. One finds that

$$|A_k| \leq C_0 \lambda^{k/2} \eta_k$$

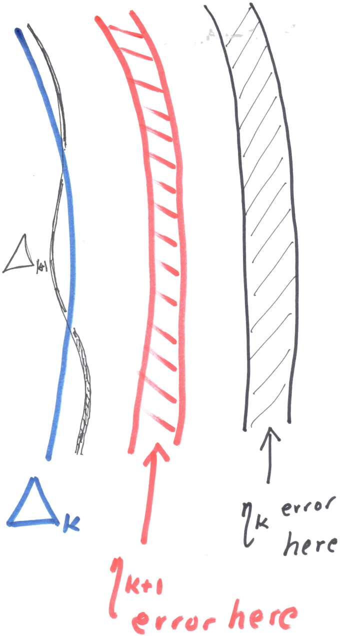
We define V_{k+1} by patching $V_k + f$ over the outer region $\Omega^+(\lambda^{k+1}, \Delta_k)$ with the translate of V_k over the inner region $\Omega^-(\lambda^{k+1}, \Delta_k)$. This patching is done using cut-off functions on the annular intersection region $\Omega(\lambda^{k+1}, \Delta_k)$.

Calculations show that this produces an error $|\mathcal{F}(V_{k+1})| \leq \eta_{k+1}$ over $\Omega(\lambda^{k+1}, \Delta_{k+1})$ where

$$\eta_{k+1} \leq (C_1 \lambda^{1/2}) \eta_k ,$$

plus an additional error term which turns out to be negligible, under our HYPOTHESIS.

Here C_0, C_1 are constants independent of λ.



Δ_{k+1}
Δ_k
η_{k+1} error here
η_k error here

We choose λ so that $C_1\lambda^{1/2} = \mu < 1$. Then

$$\eta_k, |A_k| \leq C_2\mu^k \rightarrow 0,$$

as $k \rightarrow \infty$.
Then under our HYPOTHESIS the iteration scheme converges to a solution.

The problem is to conform to our HYPOTHESIS. To achieve this we modify the scheme by introduce *smoothing operators*.

For functions on **R**, a smoothing operator of scale ϵ could be defined by convolution with S_ϵ where

$$S_\epsilon(x) = \epsilon^{-1} S(x/\epsilon),$$

with S a compactly-supported function of integral 1.

At stage k we smooth $A_k(t)$ over a scale k^{-1} in the variable t. This gives $\underline{A}_k$ with

$$|\nabla^m \underline{A}_k| \leq \text{const.}k^m \|A_i\|_{L^\infty} \leq \text{const.}k^m \eta_k$$

Then one finds that this introduces only small extra errors in the matching construction and we have

$$|\nabla^m \underline{A}_k| \leq \text{const.}k^m \mu^k.$$

Since $\mu < 1$ we have $\sum_{k=0}^{\infty} \|\nabla^m \underline{A}_k\| \leq \text{const.}_m$ and with this modification the HYPOTHESIS holds.