RANDOM MATRIX THEORY AND TOEPLITZ DETERMINANTS

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OVERVIEW

RANDOM MATRIX THEORY

INTRODUCTION

The study of the asymptotics of statistics of the eigenvalues of random matrices.

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- 2d Ising model: correlation functions along lines are determinants of Toeplitz matrices.
- Montgomery's conjecture: the pair correlation between zeros of the Riemann zeta function is the same as the pair correlation between eigenvalues of the Gaussian Unitary Ensemble.
- Baik-Deift-Johansson theorem: the length of the longest increasing subsequence of a random permutation of N numbers converges in distribution to the Tracy-Widom distribution of the largest eigenvalue of a random matrix from the Gaussian Unitary Ensemble.
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We study *matrix ensembles*, which are spaces of matrices, often with certain symmetries, with a probability measure defined on the entries of the matrices.

MATRIX ENSEMBLES

The most studied ensembles are the *invariant* ensembles

- $\beta = 1$ Gaussian Orthogonal Ensemble (GOE), real symmetric matrices
- $\beta = 2$ Gaussian Unitary Ensemble (GUE), complex hermitian matrices
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together with a probability measure invariant under orthogonal, unitary or symplectic transformations respectively, and such that the entries of the matrices are independent random variables. This makes the p.d.f. to be of the form

$$dP(M) = \frac{1}{Z_{N,\beta}} \exp(-a\mathrm{tr}(M^2) + b\mathrm{tr}(M) + c)dM \qquad (a > 0; b, c \in \mathbb{R});$$

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$$dP(x_1, \ldots, x_N) = \frac{1}{Z_{N,\beta}} \exp\left(-a \sum_{j=1}^{N} x_j^2 + b \sum_{j=1}^{N} x_j + c\right) \prod_{i<j} |x_j - x_i|^{\beta} dx_1 \ldots dx_N.$$

THE JOINT PROBABILITY FUNCTION: DETERMINANTAL FORM

Starting from the Vandermonde determinant

$$\prod_{i<j}(x_j - x_i) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ x_1^2 & x_2^2 & \dots & x_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \end{vmatrix} \propto \begin{vmatrix} p_0(x_1) & p_0(x_2) & \dots & p_0(x_N) \\ p_1(x_1) & p_1(x_2) & \dots & p_1(x_N) \\ \vdots & \vdots & \ddots & \vdots \\ p_{N-1}(x_1) & p_{N-1}(x_2) & \dots & p_{N-1}(x_N) \end{vmatrix},$$

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we see that the joint p.d.f. has the determinantal form

$$P(x_1, \dots, x_N) = \frac{1}{Z_{N,\beta}} \det \begin{pmatrix} K_N(x_1, x_1) & K_N(x_1, x_2) & \dots & K_N(x_1, x_N) \\ K_N(x_2, x_1) & K_N(x_2, x_2) & \dots & K_N(x_2, x_N) \\ \vdots & \vdots & \ddots & \vdots \\ K_N(x_N, x_1) & K_N(x_N, x_2) & \dots & K_N(x_N, x_N) \end{pmatrix},$$

where

$$K_N(x_i, x_j) = e^{-x_i^2/2} e^{-x_j^2/2} \sum_{k=0}^{N-1} p_k(x_i) p_k(x_j).$$

A SAMPLE COMPUTATION: THE DENSITY OF THE EIGENVALUES FOR GUE

If we choose the polynomials p_k to satisfy

$$\int_{-\infty}^{\infty} p_j(x) p_k(x) e^{-x^2} dx = c_j \delta_{jk},$$

the Hermite polynomials, then K_N is a reproducing kernel, and thus the density of the eigenvalues becomes

$$\sigma_N(x) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} P(x, x_2, \dots, x_N) dx_2 \dots dx_N = K_N(x, x) = e^{-x^2} \sum_{k=0}^{N-1} p_k^2(x).$$

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Thanks to Christoffel-Darboux formula

$$\sum_{k=0}^{N-1} p_k(x)p_k(y) = \frac{p_{N-1}}{p_N}\frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y},$$

the analysis of this expression is much simplified. In the end, this leads to *Wigner's semicircle law*:

$$\sigma_N(x) \sim \frac{1}{\pi}\sqrt{2N - x^2} \qquad (|x| < \sqrt{2N}).$$

TOEPLITZ DETERMINANTS

The Fourier coefficients of functions $f \in L^1(\mathbb{T})$ are defined as

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If $f \in L^\infty(\mathbb{T})$, the following expression defines a bounded operator on the Hardy space H^2 (P denotes the projection $P : L^2 \to H^2$)

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This is called a Toeplitz operator, and it has the following matrix expression

$$T_f = (f_{i-j})_{i,j=1}^{\infty} = \begin{pmatrix} f_0 & f_{-1} & f_{-2} & \\ f_1 & f_0 & f_{-1} & \ddots \\ f_2 & f_1 & f_0 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} .$$

Its finite truncations $T_f^{(N)}$ are called Toeplitz matrices.

THE STRONG SZEGŐ LIMIT THEOREM

Theorem (Szegő, 1952)

Let $f(e^{i\theta}) = \exp(\sum_{k=-\infty}^{\infty} c_k e^{ik\theta})$, *with*

$$\sum_{k=-\infty}^{\infty} |k||c_k|^2 < \infty;$$

then

$$\lim_{N\to\infty} \left(\det T_f^{(N)} / e^{Nc_0}\right) = \exp\left(\sum_{k=1}^{\infty} k c_k c_{-k}\right).$$

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A consequence: for positive f, the eigenvalues are uniformly distributed

$$\frac{1}{N}\left(\log \lambda_1^{(N)} + \cdots + \log \lambda_N^{(N)}\right) \sim c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} c_k(e^{ik\theta})\right) d\theta.$$

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There are at least 9 proofs of this result!

MINORS OF TOEPLITZ MATRICES

If we consider a minor of a Toeplitz matrix, such as

$$\begin{pmatrix} f_0 & f_{-1} & f_{-2} & f_{-3} & f_{-4} & f_{-5} & \dots \\ f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} & f_{-4} & \dots \\ f_2 & f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} & \dots \\ f_3 & f_2 & f_1 & f_0 & f_{-1} & f_{-2} & \dots \\ f_4 & f_3 & f_2 & f_1 & f_0 & f_{-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} = (f_{i-j} \qquad)_{i,j=1}^{N},$$

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the following formula holds

$$\lim_{N\to\infty} \left(\det T_{f,\mu}^{(N)} / e^{Nc_0}\right) = \frac{1}{m!} \sum_{\pi \in S_m} \chi^{\mu}(\pi) \Delta(\sigma, \pi) \exp\left(\sum_{k=1}^{\infty} k c_k c_{-k}\right),$$

where μ is a partition of m, and

$$\Delta(\sigma, \pi) = \prod_{k=1}^{\infty} (k c_k)^{\gamma_k},$$

where π has γ_k cycles of order k.

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THANK YOU!