Ramified coverings of algebraic varieties

Nguyen Bin

Lismath

May 6, 2016

Outline

Finite maps

We only work over the field $\mathbb{C}$ of complex numbers.

Definition

Let X, Y be affine varieties. A morphism $f : X \longrightarrow Y$ is called finite if

1. f is dominant,
2. $K[X]$ is finitely generated as $K[Y]$ - module.

Definition

Let X, Y be projective varieties. A morphism $f : X \longrightarrow Y$ is called finite if any point $y \in Y$ has an affine neighborhood V such that

1. $f^{-1}(V)$ is affine,
2. $f\mid_{f^{-1}(V)}: f^{-1}(V) \longrightarrow V$ is a finite map between affine varieties.

Definition

1. An affine variety X is called normal if $K[X]$ is normal.
2. A projective variety X is called normal if every point has a normal affine neighborhood.

From now, we assume that the varieties are projective varieties over $\mathbb{C}$.

Covers

Definition

A finite map $f : X \longrightarrow Y$ between projective varieties is called a cover of degree d if the fibre of f over a general point of Y consists of d points.

Example

The following map

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1, (x_0 : x_1) \longmapsto \left(x_0^d : x_1^d\right)$$

is a cover of degree d.

Covers

Remark

The cover of degree d $f : X \longrightarrow Y$ *induces a degree d extension* $\mathbb{C}(Y) \subset \mathbb{C}(X)$

Definition

Let $f : X \longrightarrow Y$ be a cover of degree d. Denoting

$$Aut(f) := \{g : X \longrightarrow X \mid f \circ g = f\}$$

1. f is called Galois if $|Aut(f)| = d$.
2. A Galois cover f is called abelian if $Aut(f)$ is abelian.

Covers

Example

The following map

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1, (x_0 : x_1) \longmapsto \left(x_0^2 : x_1^2\right)$$

is a cover of degree 2. And

$$Aut\left(f\right) = \left\{id, g\right\},$$

where

$$g : \mathbb{P}^1 \longrightarrow \mathbb{P}^1, (x_0 : x_1) \longmapsto (x_0 : -x_1)$$

First properties of abelian covers

Notation

Let $f : X \longrightarrow Y$ be a Galois cover of degree d. We denote

1. $G := Aut(f)$.
2. $\mathscr{A} := f_* \mathscr{O}_X$.

Theorem

Let $f : X \longrightarrow Y$ be a Galois cover of degree d. Then

$$\mathscr{A} = \bigoplus_{\rho \in Irr(G)} \mathscr{A}_\rho$$

where $Irr(G)$ is the set of irreducible representations of G.

Properties of abelian covers

Corollary

Let $f : X \longrightarrow Y$ be an abelian cover of degree d. Then

$$\mathscr{A} = \bigoplus_{\rho \in G^*} \mathscr{A}_\rho$$

where G^* is the set of characters of G.

In the case where f is an abelian cover of degree d, we can write

$$\mathscr{A} = \mathscr{O}_Y \oplus \left(\bigoplus_{\substack{\chi \in G^* \\ \chi \neq 1}} L_\chi^{-1} \right)$$

where L_χ^{-1} are line bundles.

Building data

Definition

Let $f : X \longrightarrow Y$ be a cover of degree d.

1. The set

$$B := \left\{ y \in Y \mid \left| f^{-1}(y) \right| < d \right\}$$

   is called the branch locus of f.

2. We define the ramification locus $R \subseteq X$ as the set of points where the differential of f fails to be an isomorphism.

Building data

Let f be a Galois cover of degree d. For each component S of R,

$$I_S := \{g \in G \mid g.x = x, \forall x \in S\}$$

is called the inertia subgroup.

Proposition

Let f be an abelian cover of degree d. Then

1. *I_S is a cyclic subgroup.*
2. *The tangent representation of I_S at the point $x \in S$ decomposes as the sum $n-1$ copies of the trivial representation and of a 1-dimensional representation ψ_x. Moreover, ψ_x is generator of I_S^* .*

Building data

Notation

Let $f : X \longrightarrow Y$ *be an abelian cover of degree* d*.*

1. *For each component* D *of* B*, we denote*

$$(I_D, \psi_D) := (I_S, \psi_S)$$

   where S *is a component of* $f^{-1}(D)$*.*

2. *Let* H *be a cyclic subgroup of* G*, and* ψ *be a generator of* H^**. We denote*

$$D_{(H,\psi)} := \sum_{\substack{(I_D,\psi_D)=(H,\psi) \\ D \text{ is a component of } B}} D$$

Building data

Remark

$$B = \sum_{(H,\psi)} D_{(H,\psi)}$$

Definition

Let $f : X \longrightarrow Y$ be an abelian cover of degree d. The sheaves $L_\chi, \chi \in G^* \setminus \{1\}$, and the divisors $D_{(H,\psi)}$ are called the building data of the cover.

Building data

Notation

Let $\chi, \chi' \in G^*$, H *be a cyclic subgroup of* G, *and* ψ *is a generator of* H^*. *Denote*

1. $m_H := ord\,(H)$,
2. $m_\chi^{(H,\psi)} := min\left\{k \in \mathbb{N} \mid \psi^k = \chi|_H\right\}$,
3. $\epsilon_{\chi,\chi'}^{(H,\psi)} := \begin{cases} 1, & if\ m_\chi^{(H,\psi)} + m_{\chi'}^{(H,\psi)} \geq m_H \\ 0 & otherwise \end{cases}$

Theorem (R. Pardini 1991)

Let $f : X \longrightarrow Y$ *be an abelian cover of degree* d, *and* χ, χ' *be in* G^*. *Then*

$$L_\chi + L_{\chi'} \equiv L_{\chi\chi'} + \sum_{(H,\psi)} \varepsilon_{\chi,\chi'}^{(H,\psi)} D_{(H,\psi)} \qquad (1)$$

Theorem (R. Pardini 1991)

Let Y be a smooth projective variety, and G be an abelian group of order d. Let $\{L_\chi\}_{\chi \in G^* \setminus \{1\}}$ be line bundles of Y such that $L_\chi \neq \mathscr{O}_Y$ for every χ, and let $D_{(H,\psi)}$ be effective divisors such that $\sum_{(H,\psi)} D_{(H,\psi)}$ is reduced. Then

1. *$L_\chi, D_{(H,\psi)}$ are the building data of an abelian cover $f : X \longrightarrow Y$ if and only if they satisfy the fundamental relation (1).*
2. *The building data determine $f : X \longrightarrow Y$ up to $G-$ equivariant isomorphisms.*

Theorem (R. Pardini 1991)

Let $f : X \longrightarrow Y$ be an abelian cover with branch locus B, let $y \in Y$ be a point, let $D_1, D_2, \ldots, D_s$ be the irreducible components of B that contain y and let (H_i, ψ_i) be the pair subgroup - character associated to $D_i, i = 1, \ldots s$. Then X is smooth above y if and only if

1. *D_i is smooth at y for every i;*
2. *the D_i meet transversely at y;*
3. *the natural map $H_1 \oplus H_2 \oplus \ldots \oplus H_s \to G$ is injective.*

Theorem (R. Pardini 1991)

Let Y be a smooth projective variety, and G be an abelian group of order d. Let $\chi_1, \chi_2, \ldots, \chi_k \in G^*$ be such that

$$G^* = \langle \chi_1 \rangle \oplus \langle \chi_2 \rangle \oplus \ldots \oplus \langle \chi_k \rangle .$$

Let $\{L_{\chi_i}\}_{i=1,\ldots k}$ be line bundles of Y, and let $D_{(H,\psi)}$ be effective divisors such that $\sum_{(H,\psi)} D_{(H,\psi)}$ is reduced. Then

1. *$\{L_{\chi_i}, D_{(H,\psi)}\}$ can be extended to be a set of building data $\{L_\chi, D_{(H,\psi)}\}$ satisfying (1) if and only if $\{L_{\chi_i}, D_{(H,\psi)}\}$ satisfy*

$$d_i L_{\chi_i} \equiv \sum_{(H,\psi)} \frac{d_i m_{\chi_i}^{(H,\psi)}}{m_H} D_{(H,\psi)} \qquad (2)$$

2. *The $\{L_{\chi_i}, D_{(H,\psi)}\}$ uniquely determine L_χ.*

We assume that Y ia a smooth projective variety.

Double covers

Example

Let $G = \mathbb{Z}_2$.

1. Let χ_1 be the non trivial character of G.
2. Let H_1 be the non-trivial cyclic subgroup of G. And let $\psi_1 = \chi_1$. Then $m_{\chi_1}^{(H_1,\psi_1)} = 1$.

So the building data is $\{L_1, D_1\}$ with the following relation

$$L_1 + L_1 \equiv D_1$$

Bidouble covers

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $g_1 = (1, 0), g_2 = (0, 1), g_3 = (1, 1)$

1. Let χ_1, χ_2, χ_3 be the non trivial characters of G defined by

$$\chi_i (g_i) = 1$$

   for all i.

2. The cyclic subgroups of G are $H_i = \langle g_i \rangle$ for all $i = 1, 2, 3$. And

$$H_i^* = \langle \psi_i \rangle$$

   where $\psi_i (g_i) = -1$.

Bidouble covers

We have

$$\begin{bmatrix} m_{\chi_1}^{(H_1,\psi_1)} & m_{\chi_1}^{(H_2,\psi_2)} & m_{\chi_1}^{(H_3,\psi_3)} \\ m_{\chi_2}^{(H_1,\psi_1)} & m_{\chi_2}^{(H_2,\psi_2)} & m_{\chi_2}^{(H_3,\psi_3)} \\ m_{\chi_3}^{(H_1,\psi_1)} & m_{\chi_3}^{(H_2,\psi_2)} & m_{\chi_3}^{(H_3,\psi_3)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Then the building data is $\{L_1, L_2, L_3, D_1, D_2, D_3\}$ with the following relations

$$\begin{cases} L_1 + L_1 \equiv D_2 + D_3 \\ L_2 + L_2 \equiv D_1 + D_3 \\ L_3 + L_3 \equiv D_1 + D_2 \\ L_1 + L_2 \equiv L_3 + D_3 \\ L_1 + L_3 \equiv L_2 + D_2 \\ L_2 + L_3 \equiv L_1 + D_1 \end{cases}$$

Bidouble covers

The reduced building data is $\{L_1, L_2, D_1, D_2, D_3\}$ with the following relations

$$\begin{cases} L_1 + L_1 \equiv D_2 + D_3 \\ L_2 + L_2 \equiv D_1 + D_3 \end{cases}$$

Cyclic covers

Let $G = \mathbb{Z}_3$.

1. Let χ_1, χ_2 be the non trivial characters of G defined by

$$\chi_1(1) = e^{i\frac{2\pi}{3}}, \chi_2(1) = e^{i\frac{4\pi}{3}}$$

2. There is only one non-trivial cyclic subgroup $H_1 = \langle 1 \rangle$ of G, and two characters generating H^*, χ_1, χ_2. So pairs are (H, χ_1), (H, χ_2).

Cyclic covers

We have

$$\begin{bmatrix} m_{\chi_1}^{(H_1,\psi_1)} & m_{\chi_1}^{(H_2,\psi_2)} \\ m_{\chi_2}^{(H_1,\psi_1)} & m_{\chi_2}^{(H_2,\psi_2)} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Then the building data is $\{L_1, L_2, D_1, D_2\}$ with the following relations

$$\begin{cases} L_1 + L_1 \equiv L_2 + D_2 \\ L_2 + L_2 \equiv L_1 + D_1 \\ L_1 + L_2 \equiv D_1 + D_2 \end{cases}$$

The reduced building data is $\{L_1, D_1, D_2\}$ with the relation

$$3L_1 \equiv D_1 + 2D_2$$

Cyclic covers

If we consider

$$\psi_1 := \chi_1 \in H_1^*,$$

We have

$$\begin{bmatrix} m_{\chi_1}^{(H_1,\psi_1)} \\ m_{\chi_2}^{(H_1,\psi_1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Cyclic covers

Then the building data is $\{L_1, L_2, D_1\}$ with the following relations

$$\begin{cases} L_1 + L_1 \equiv L_2 \\ L_2 + L_2 \equiv L_1 + D_1 \\ L_1 + L_2 \equiv D_1 \end{cases}$$

The reduced building data is $\{L_1, D_1\}$ with the relation

$$3L_1 \equiv D_1$$

References I

- Shafarevich, Igor R., *Basic Algebraic Geometry 1*.
- Hartshorne, Robin, *Algebraic Geometry*.
- Lang, Serge, *Algebra*.
- R.Pardini, *Abelian covers of algebraic varieties*, J. reine angew. Math., 417 (1991), 191 - 213.

Thank you for your attention