Universality of dimers via Imaginary geometry

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(ongoing) Joint project with

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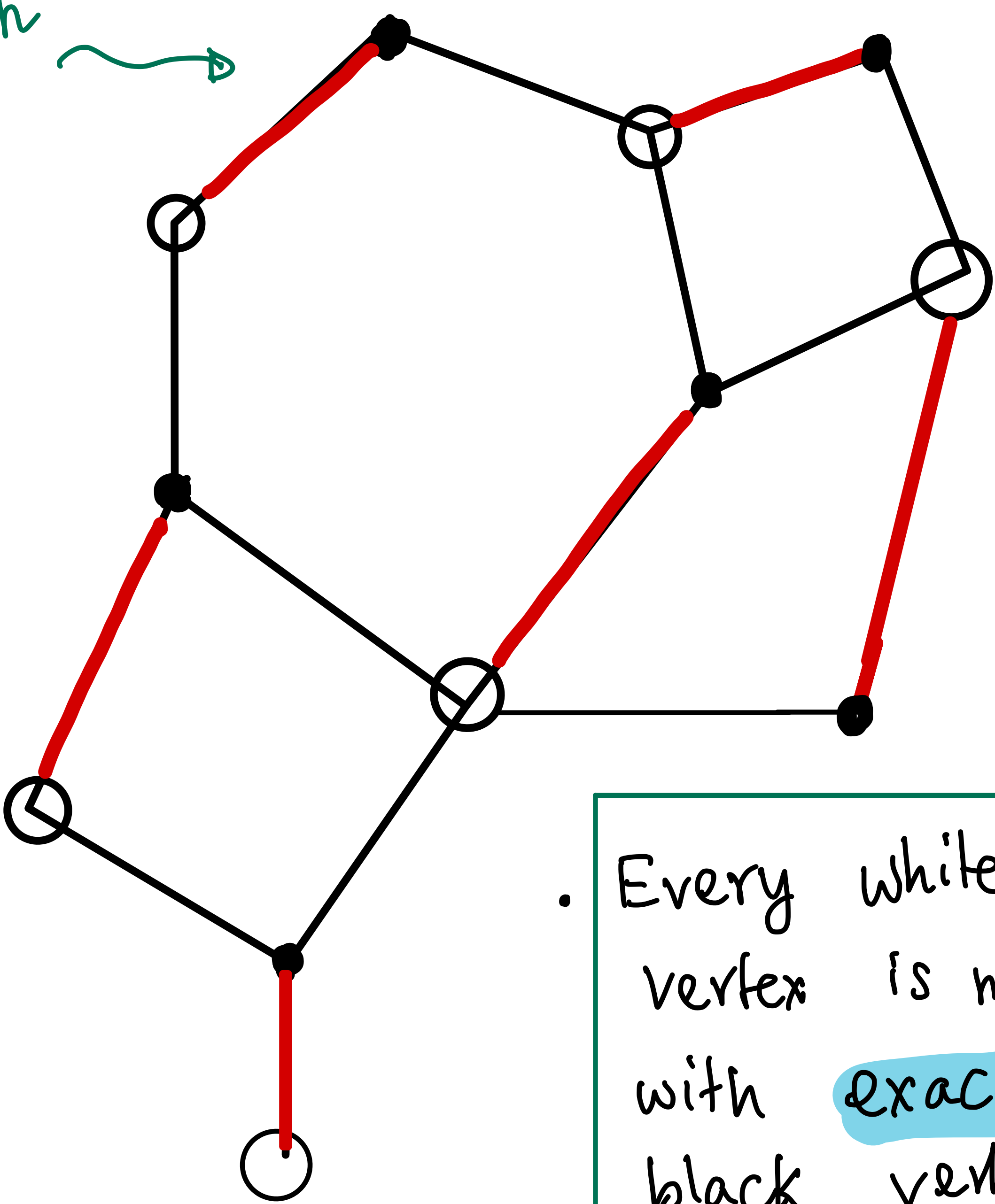
- B. Laslier (U. Paris-Diderot)

- A dimer model is a model of

$\boxed{\text{uniform perfect matching}}$

Bipartite graph



- Every white vertex is matched with exactly one black vertex

Dimer model

G: A bipartite graph.

m: A perfect matching.

$\mathcal{M}$: Set of all perfect matchings.

$$\boxed{\mu(m) = \frac{1}{|\mathcal{M}|}}$$

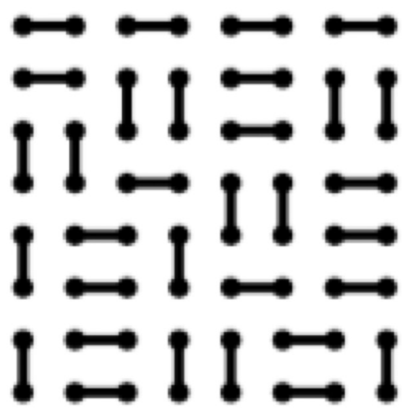
partition function.

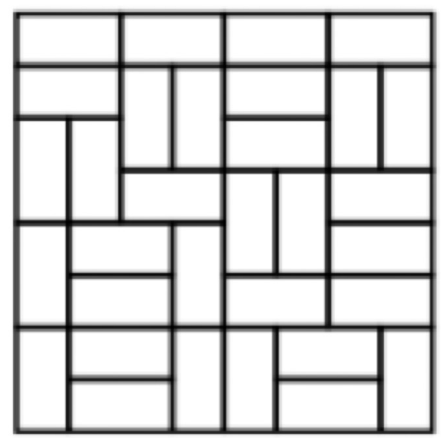
μ: Prob. measure on perfect matchings.

Remark: - One can also add weights to edges.

- We will be mostly concerned with planar bipartite graphs

• Some special cases (with visual aspects)





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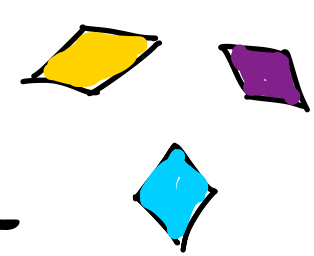
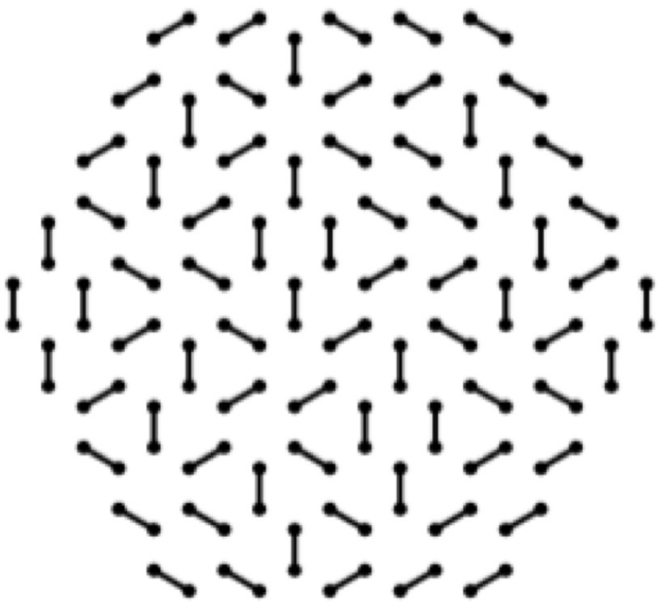
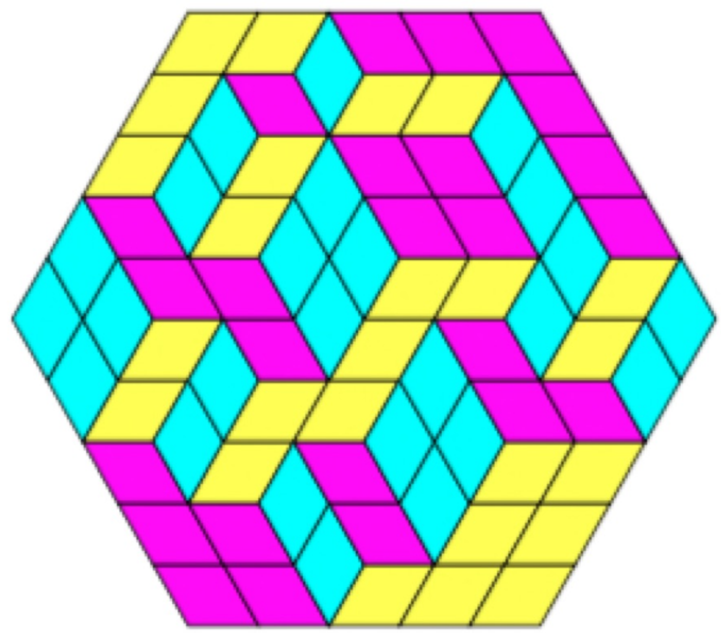
dimers on square lattice = Domino tilings.

- Some special cases (with visual aspects)

Lozenges







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Dimers on hexagonal lattice = Lozenge tilings

Some background

- This is a classical model in statistical mechanics, studied by
  - Kenyon, Propp, Lieb, Okounkov, Sheffield, Dubédat, De-Tiliére .... and many more.
- Usual approach.
  - Study Kasteleyn matrix. K (similar to the adjacency matrix).

- $Z = \det(K)$

(partition function)

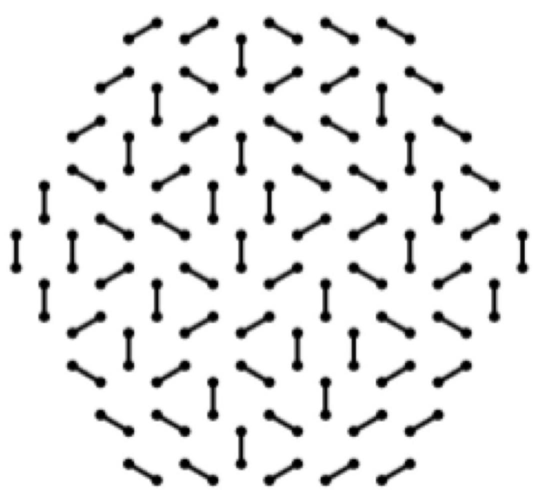
Kasteleyn matrix.

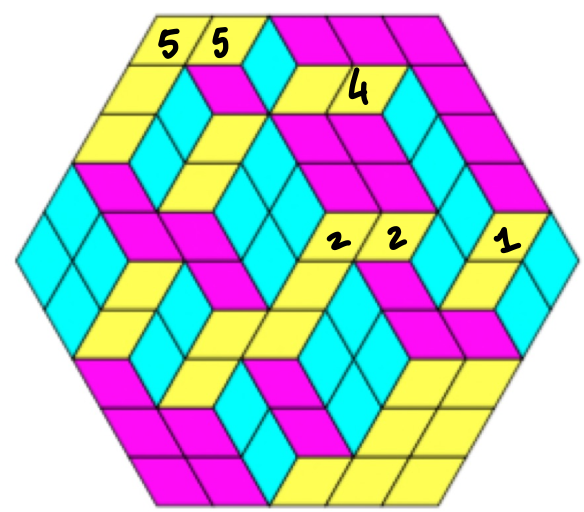
- (Exact solvability).

$$\boxed{Z_{m,n} = \prod_{j=1}^{m} \prod_{k=1}^{n} \left| 2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right) \right|^{\frac{1}{2}}}$$

dimers on $\mathbb{Z}_m \times \mathbb{Z}_n$ (m×n torus)

- Dimers on planar bipartite graphs can also be encoded by height functions (introduced by Bill Thurston).





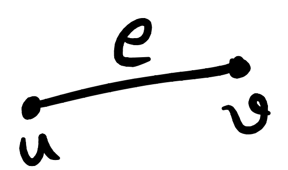
height function = height of cubes.

This coding is particularly visual for Lozenge tilings, ≡ stack of cubes.

- A general way to define height function of dimers.

(For general bipartite planar graphs)

- It is a function $h : \text{Faces}(G) \mapsto \mathbb{R}$

- It is defined through its (discrete) gradient, ($h(v) - h(u)$ for 

.)

(so defined upto global additive constant unless some value is fixed).

$\vec{E}$: oriented edges.

$$e^- \xrightarrow{\vec{e}} e^+$$

- Define $w(\vec{e}) = 1$ if $e^- = \bullet$, $e^+ = \circ$

dimer present.

$\phantom{w(\vec{e})} = 0$ if dimer absent.

(flow from black to white)

dimer flow

$$w(\vec{e}) = -w(-\vec{e}) \quad \text{(antisymmetric)}$$

- Notice $\displaystyle\sum_{e^- = u} w(\vec{e}) = 1$ if $u = \bullet$

$\phantom{\sum w(\vec{e})} = -1$ if $u = \circ$

- Let $w_0 : \vec{E} \mapsto \mathbb{R}$ be **any** function

with
$$w_0(\vec{e}) = \begin{cases} 1 & \text{if } u = \bullet \\ -1 & \text{if } u = \circ \end{cases}$$

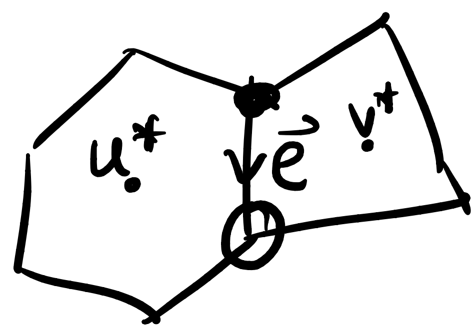
$$w_0(\vec{e}) = -w_0(-\vec{e}).$$

Then $\nabla = \boxed{w - w_0}$ is a **gradient flow**,

ie, $\displaystyle\sum_{e^- = u} \nabla(\vec{e}) = 0$ for any u.

Define **height function**

$$\boxed{h(v^*) - h(u^*) = \nabla(\vec{e})}$$

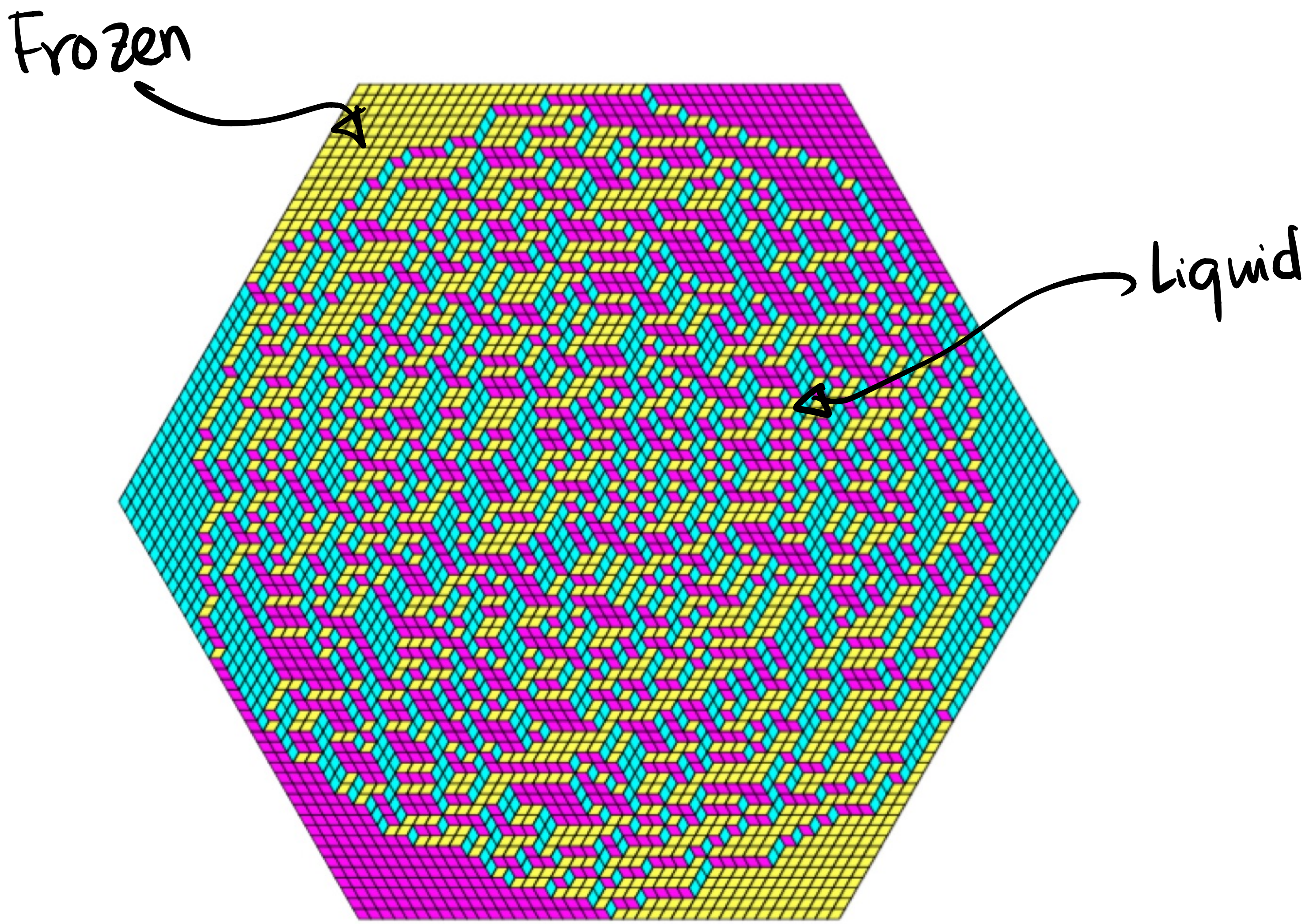


Goal: To understand large scale behaviour of $\boxed{\bar{h} = h - \mathbb{E}(h)}$

$$\bar{h} = \underbrace{h - \mathbb{E}(h)}_{\text{fluctuation}}$$

$\mathbb{E}$: Expectation/mean.

- Note $\bar{h}$ does NOT depend on the choice of reference flow.
- Fixing the boundary height can have a drastic effect (everything could be frozen).



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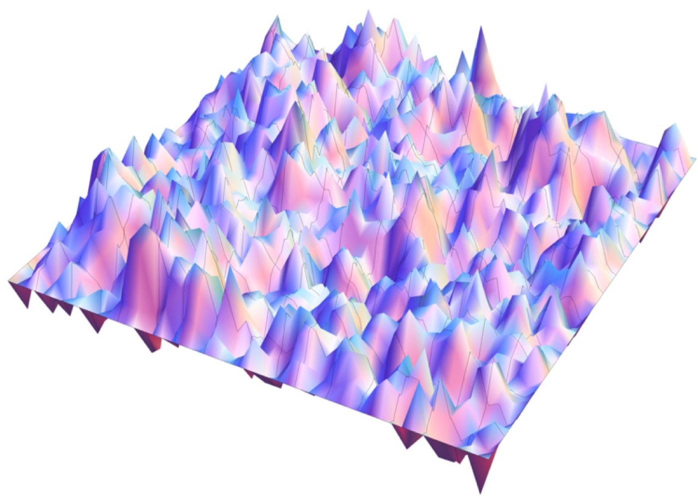
Conjecture: In the liquid region, the height function converges to the Gaussian free field in the scaling limit.

$\boxed{\text{GFF}}$ (in $D \subset \mathbb{C}$)

- Gaussian process $(h_x)_{x \in D}$
- Conformally invariant
- Random distribution

$$(h, f) \sim N\left(0, \int_{D^2} f(x)\, G^D(x,y)\, f(y)\, dx\, dy\right)$$

$G^D = -\frac{1}{2\pi} \Delta^{-1}$: Green's function in D.



NOT a random function!

Problem: Finding asymptotics of Kasteleyn matrix is hard for graphs with microscopic irregularities.

(e.g.: • A GFF scaling limit was shown by Kenyon in $\mathbb{Z}^2$.

• "Isoradial graphs": De Tilière

• Some related works by Bufetov, Gorin, Petrov for Lozenge tilings.

Our Goal

- Extend the convergence result to graphs satisfying an invariance principle, namely

$$\text{Random walk} \xrightarrow[\text{limit}]{\text{scaling}} \text{Brownian motion.}$$

- Equivalently, discrete harmonic $\longrightarrow$ cont. harmonic.

- Kasteleyn approach: "derivative of discrete harmonic $\longrightarrow$ "Derivative" of cont. harmonic.

Key Idea: Use a well known correspondence between dimers and Uniform Spanning trees

"free fermion point"

- Combine this with → ◦ Scaling limit result of UST (branches are SLE_2).

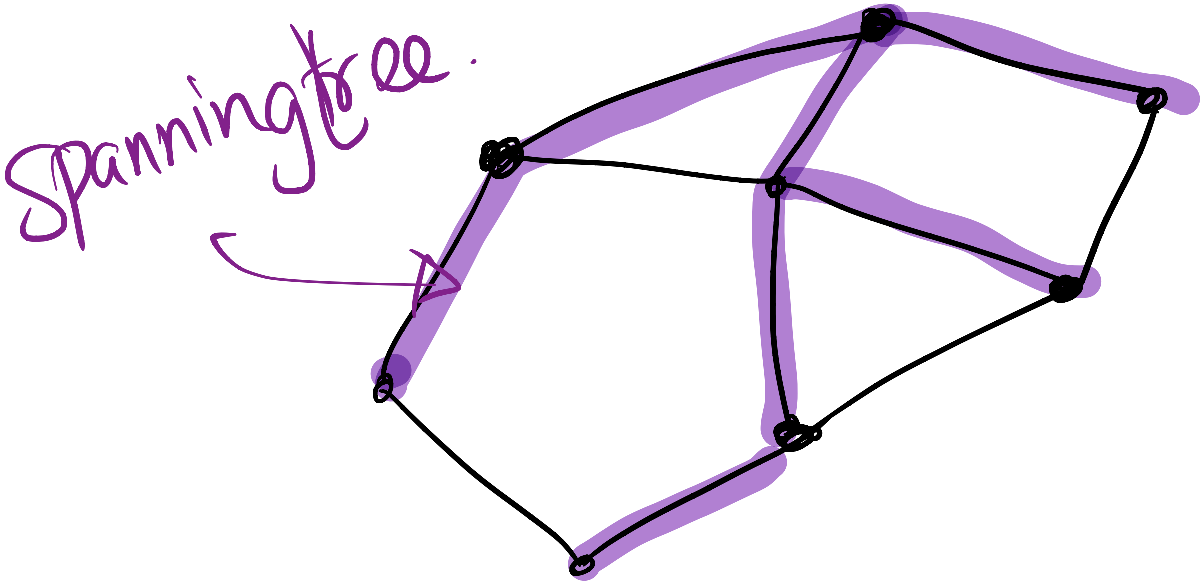
• Imaginary geometry

(coupling between GFF and SLE via "winding")

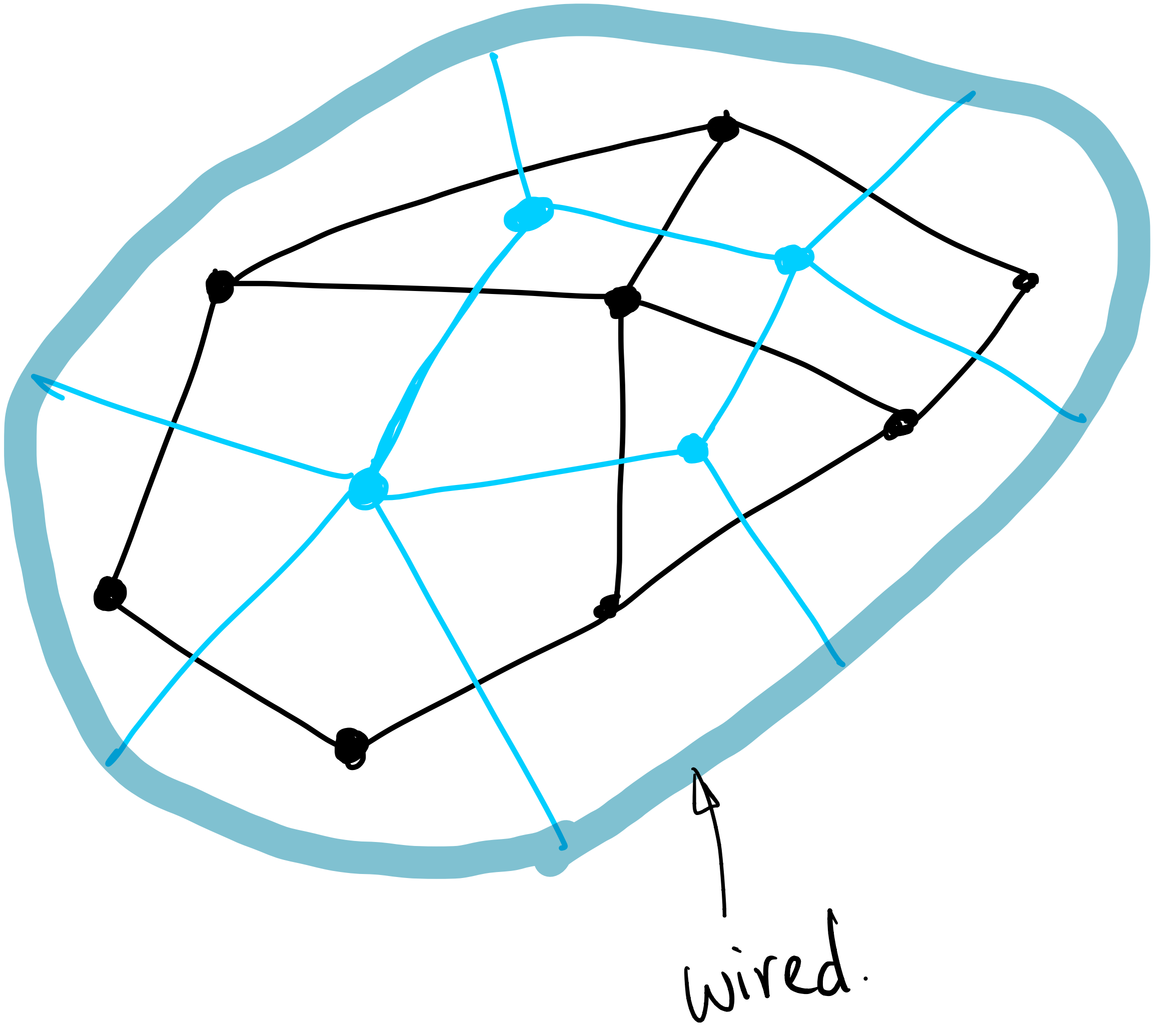
Uniform spanning tree:

Uniformly picked.

spanning tree



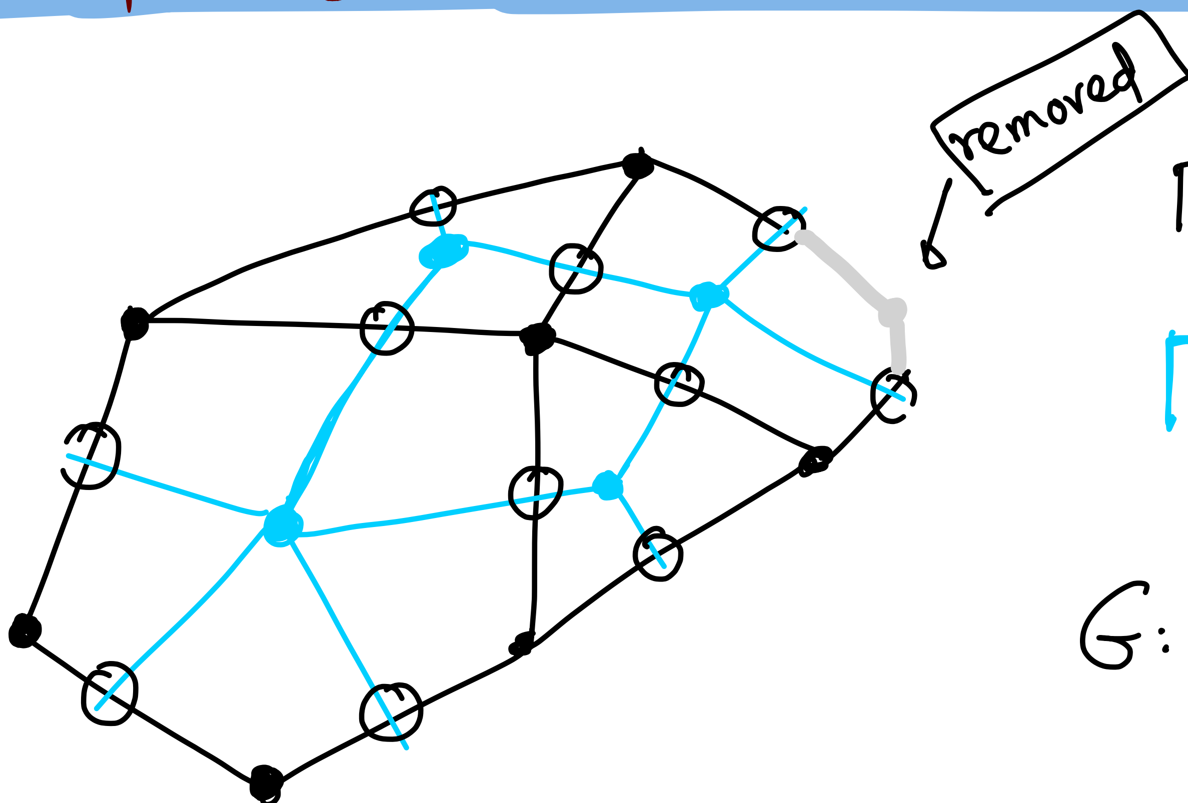
Temperley Fisher bijection



Γ: black graph.

$\Gamma^\dagger$: Dual graph.

Temperley Fisher bijection



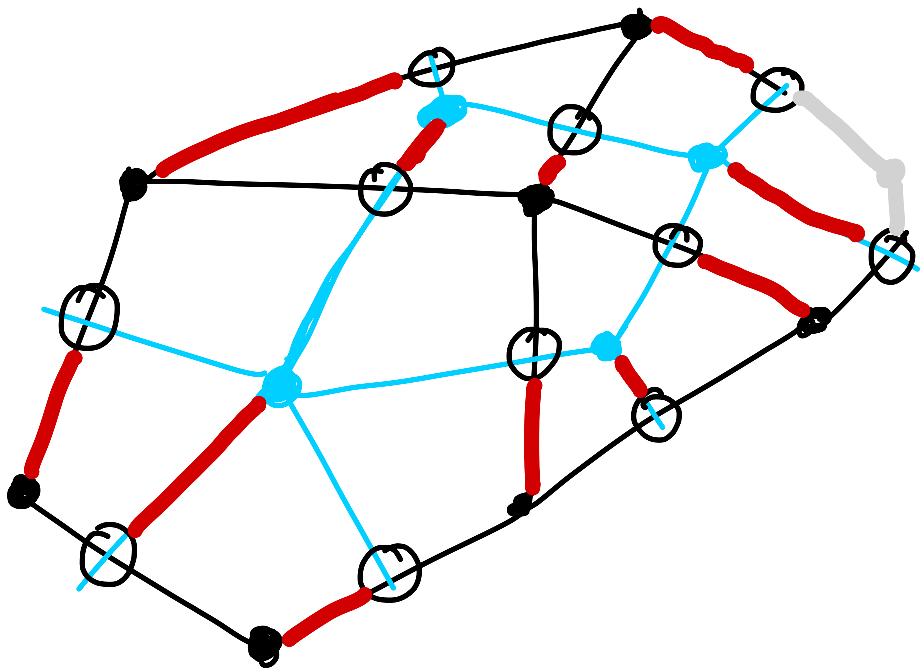
Γ: black graph.

Γ^+: Dual graph.

G: $\Gamma \cup \Gamma^+ \cup$ white vertices.

$\setminus$ {one grey vertex} on boundary.

Temperley Fisher bijection



Γ: black graph.

Γ^+: Dual graph.

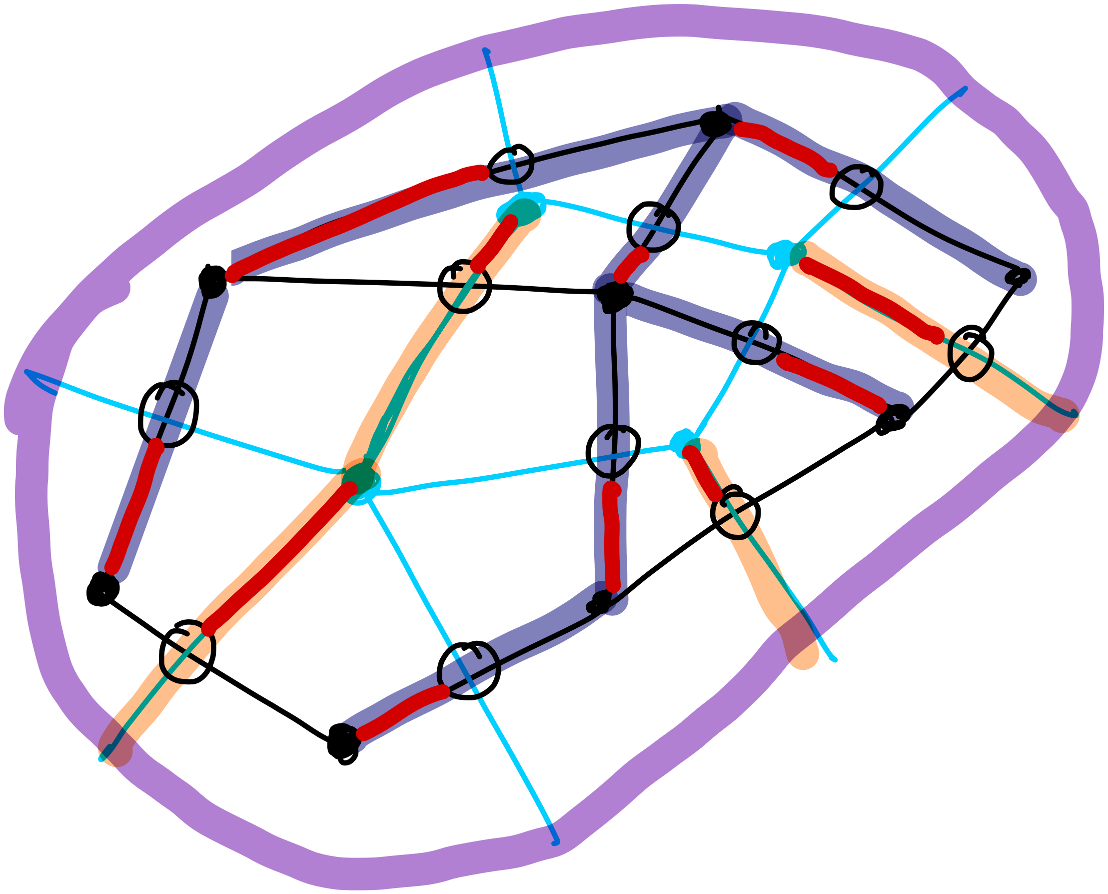
G: $\Gamma \cup \Gamma^+ \cup$ white vertices.

$-$ {one black vertex} on boundary

(bipartite)

$-$: dimer on G.

Temperley Fisher bijection



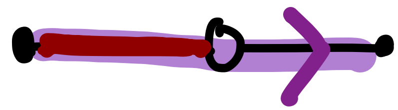
Γ: black graph.

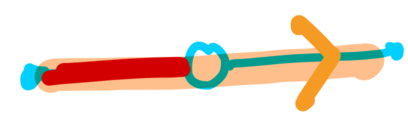
Γ^+: Dual graph.

G: $\Gamma \cup \Gamma^+ \cup$ white vertices.

(bipartite)

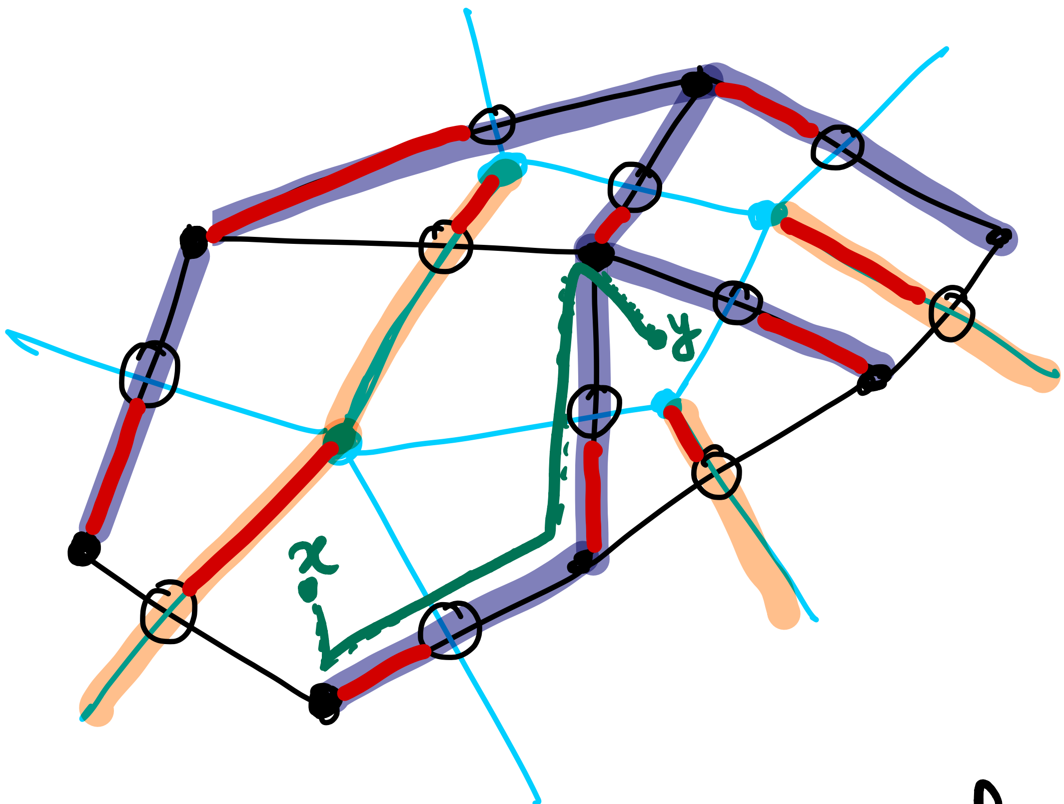
Local map:





spanning tree of Γ + spanning tree of Γ^+

(Remarkable) Observation by Benjamini.

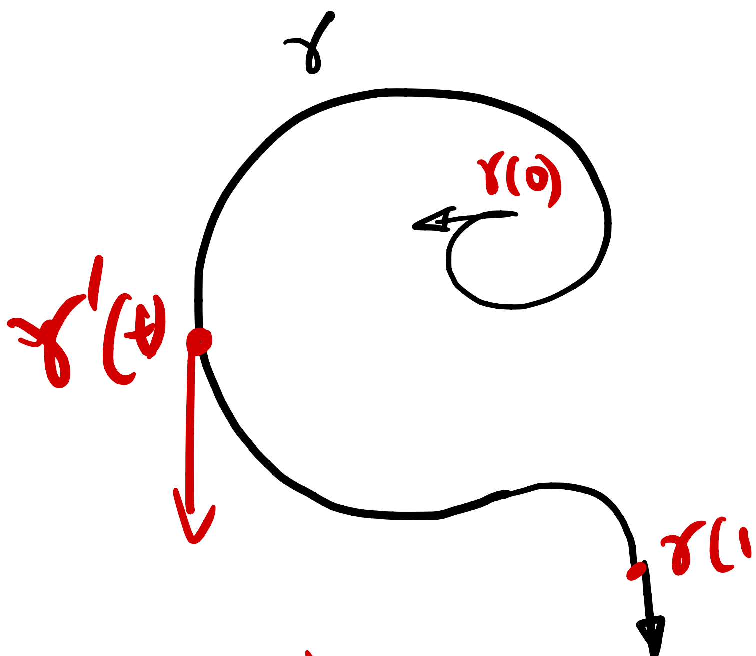


$h_x - h_y$

= "Winding" of the green path.

for a well chosen reference flow.

- Winding (for smooth curves) is the "amount a curve has turned"



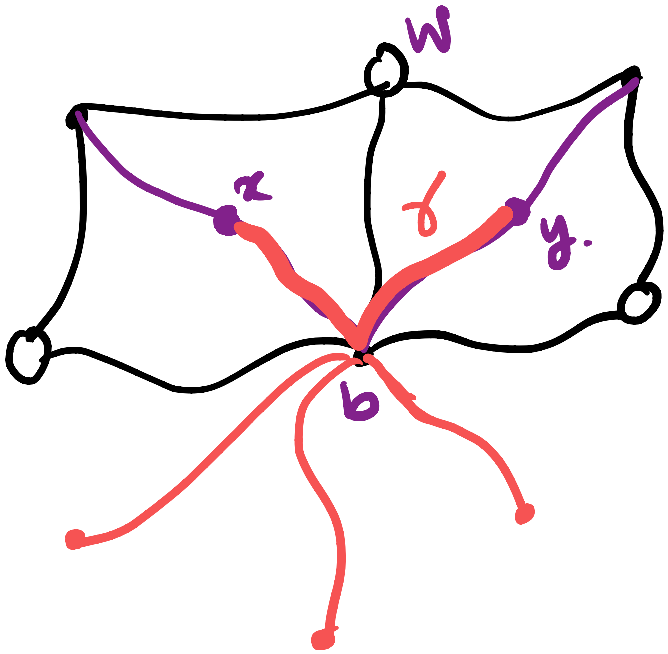
$$\gamma'(t) = r(t)e^{i\theta(t)}$$

$$\text{Winding}(\gamma) = 3\pi - \frac{\pi}{2} = \frac{5\pi}{2}.$$

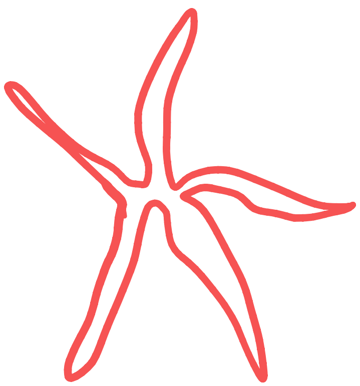
$$= \int_0^1 \theta(t)\,dt$$

$$= \int_0^1 \arg(\gamma'(t))\,dt.$$

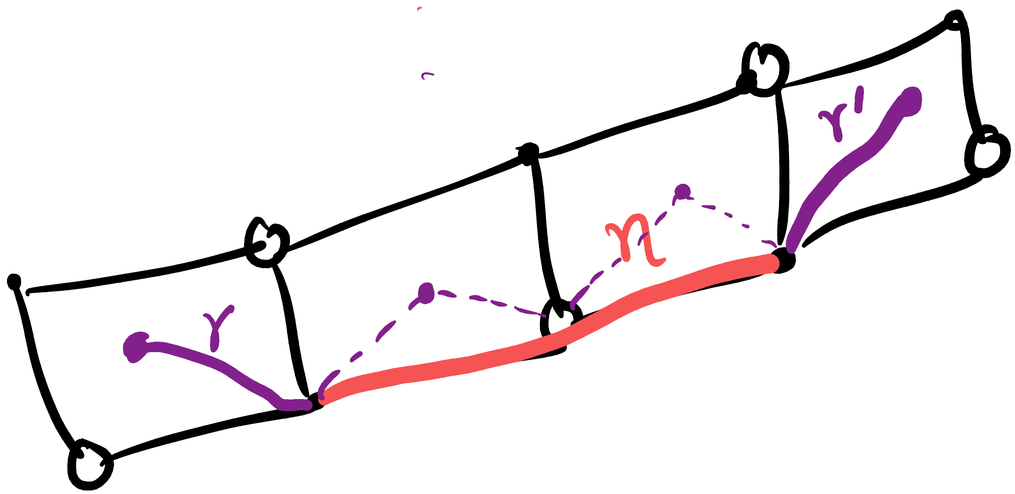
Our version of this reference flow



$$\omega_{ref}(b \to w) = \frac{1}{2\pi}\left(W(\gamma) - \pi\right)$$



$$\sum_{w \sim b} \omega_{ref}(b \to w) = -1.$$



Winding = height function.

white and black alternately add $\pm\pi$.

Overall:

Dimers $\xleftrightarrow{1\text{-}1}$ spanning tree + dual spanning tree pair.

$\Leftrightarrow$ Uniform dimer $\longleftrightarrow$ Uniform spanning tree.

measure preserving map

Thm (Yadin-Yehudayoff).

If Random walk $\longrightarrow$ Brownian motion

then branches of UST $\longrightarrow$ SLE_2 curves

$\downarrow$

$\boxed{\text{fractal curves, Hausdorff dim. } 5/4. \text{ (Beffara)}}$

Schramm-Loewner Evolution

height function $\xleftarrow{\text{winding}}$ Unif. Spanning tree

height function $\overset{?}{\dashrightarrow}$ GFF

Unif. Spanning tree $\xrightarrow{\text{scaling lim.}}$ Cont. Spanning tree

GFF $\overset{?}{\dashleftarrow}$ Cont. Spanning tree

Problem: Each branch winds ∞-often in both positive and negative direction

(as it should since it should NOT be a random function).

- **Program**

Step1: Show that winding of UST $\longrightarrow$ Conf. invariant "continuum winding field".

Step2: Identify the limit using Imaginary geometry (Miller-Sheffield).

Theorem: (Berestycki, Laslier, R '20)

For a sequence of graphs in $\mathbb{C}$, where random walk → Brownian motion (+ other "mild" assumptions)

$$\left(\text{Winding}\left(\gamma_{x \to y}\right)\right)_{x,y \in D} \longrightarrow \left(h_x - h_y\right)_{x,y \in D}.$$

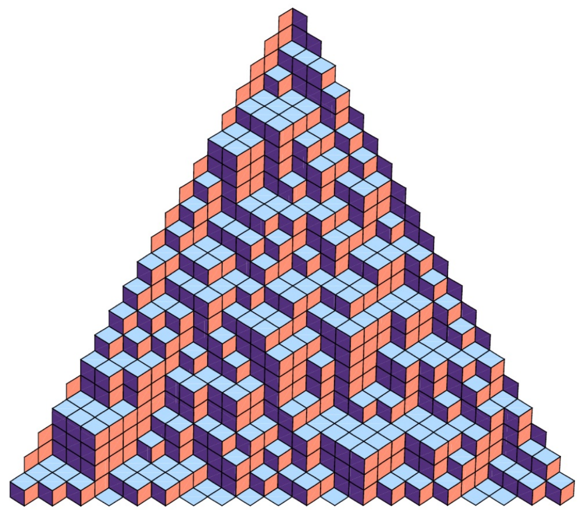
Discrete UST branch from $x \to y$

Where h = GFF. (with zero boundary condition)

- **Consequences:**

We get GFF convergence for

- hexagonal lattice with non-zero slope



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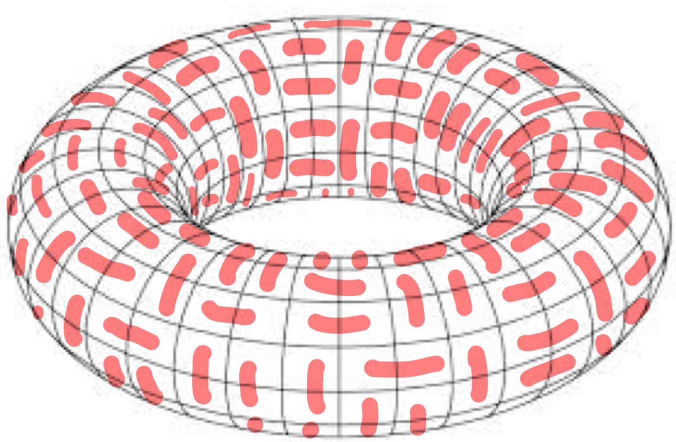
- Unif. elliptic random environment in $\mathbb{Z}^2$.

Thm: (Berestycki, Laslier, R, 20+).

Conf Invariance for dimer on Riemann surfaces.

(Identification of limit still elusive due to absence of Imaginary geometry).



Imaginary geometry (Miller, Sheffield)

Idea: Think of the vector field

$$\left(\text{"} e^{i c h_x} \text{"} \right)_x .$$

c: constant.

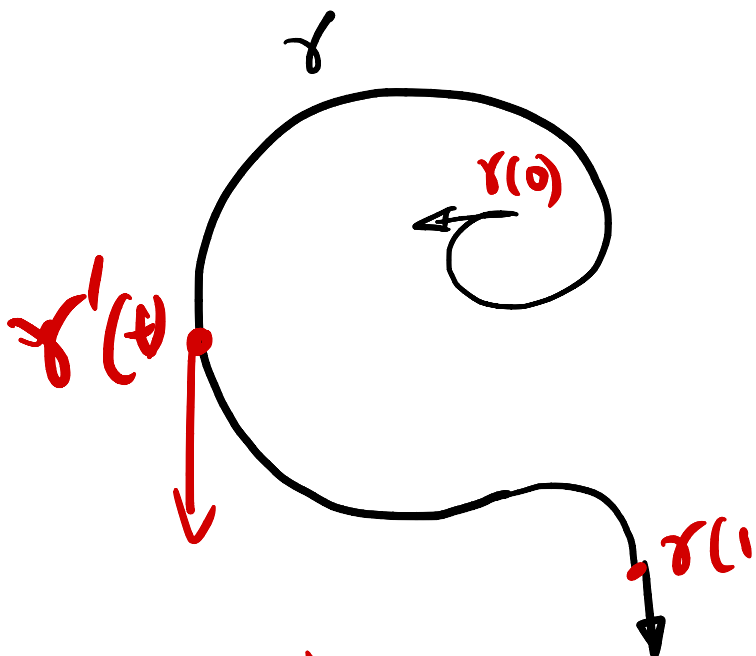
h: GFF

The "flow lines" should be SLE_κ curves

$$\kappa = f(c).$$

RECALL

- Winding (for smooth curves) is the "amount a curve has turned"



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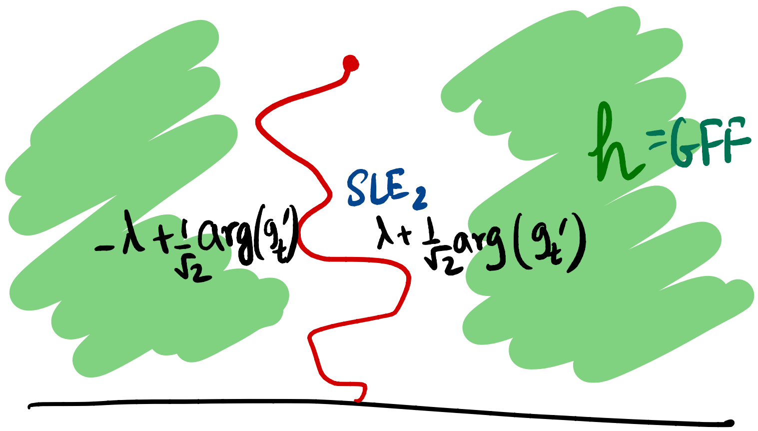
Imaginary geometry (Miller, Sheffield)



Flow line of GFF$/\sqrt{2}$

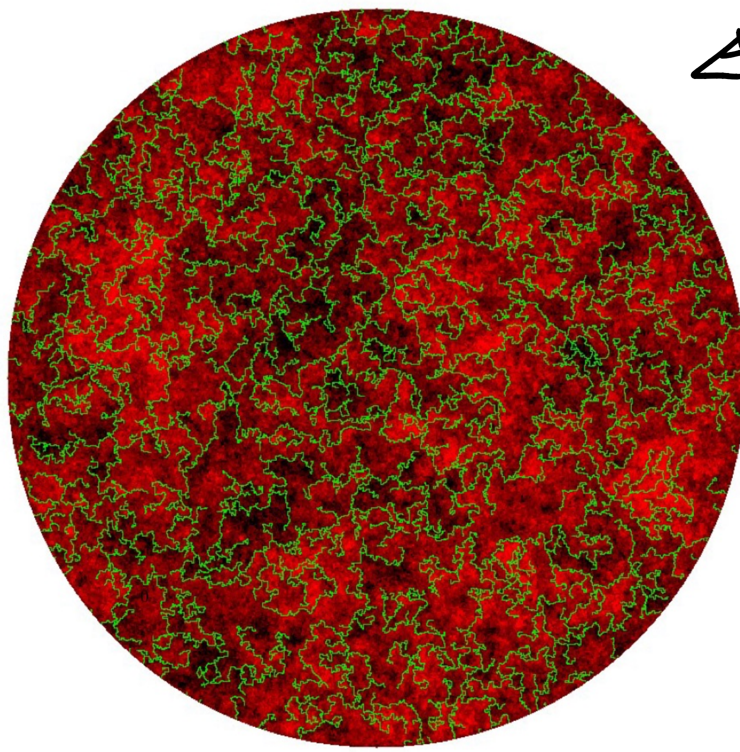
One can couple SLE$_2$ and $\sqrt{2}\cdot$ GFF. such that:

GFF conditioned on SLE$_2(0,t)$.

= GFF on $\mathbb{H} \setminus$ SLE$_2(0,t)$ with boundary values $\boxed{\frac{1}{\sqrt{2}} \arg(g_t')}$.

g_t': $\mathbb{H} \setminus \text{SLE}_2(0,t) \xrightarrow{\text{conformal}} \mathbb{H}$

- Can be extended to multiple branches
- Can be extended to other SLE curves.
- Here is a simulation for UST/GFF imaginary geom. coupling

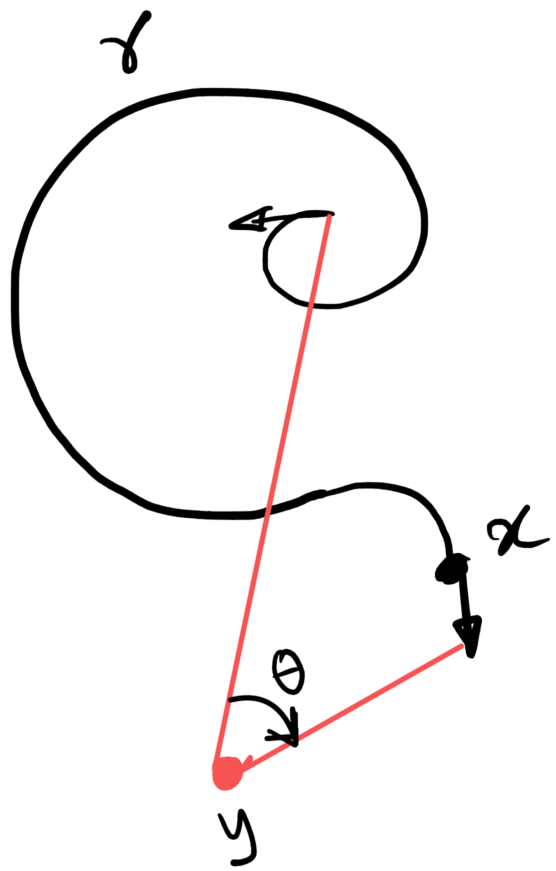


Remark:

UST is a function of GFF in this coupling.

Issue: Imaginary geometry does not calculate winding "hands on".

- We do this directly to connect with the height functions.

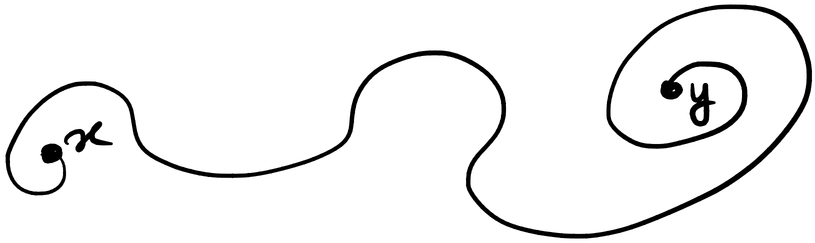


Key (but elementary) idea

"Calculate winding seen from an external point". y.

Observation: This makes the winding "continuous" as long as $x \neq y$.

Lemma



$$W(\gamma,x) + W(\gamma,y) = W(\gamma)$$

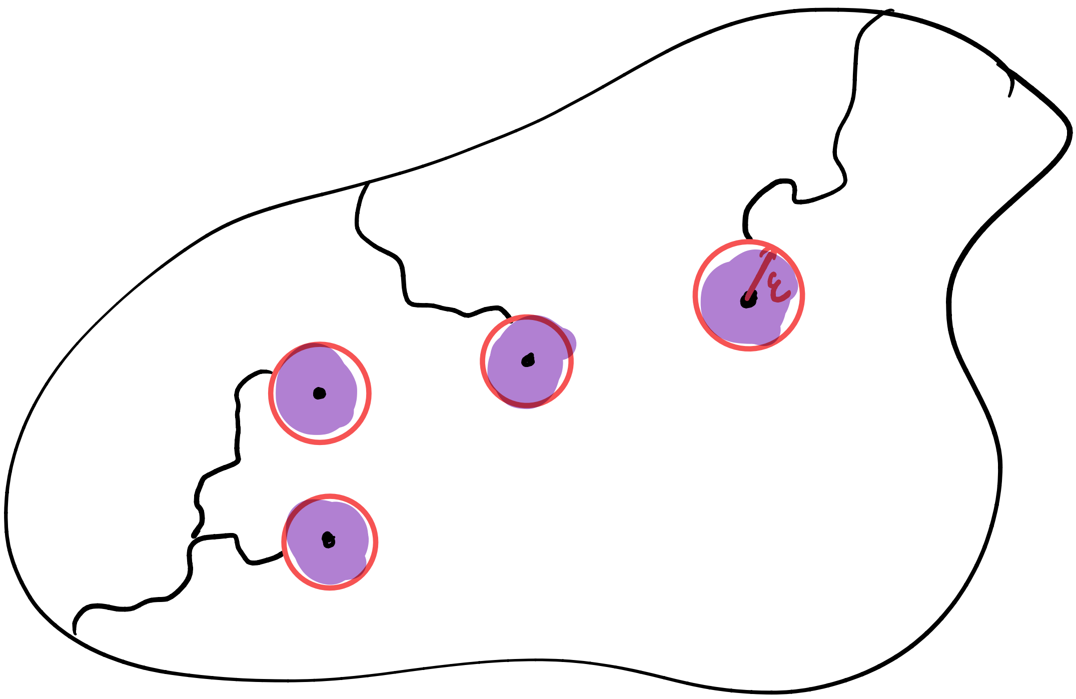
winding seen from x

winding seen from y.

winding

approximation:

Cut off the branch.



Regularization of winding field.

Calculation

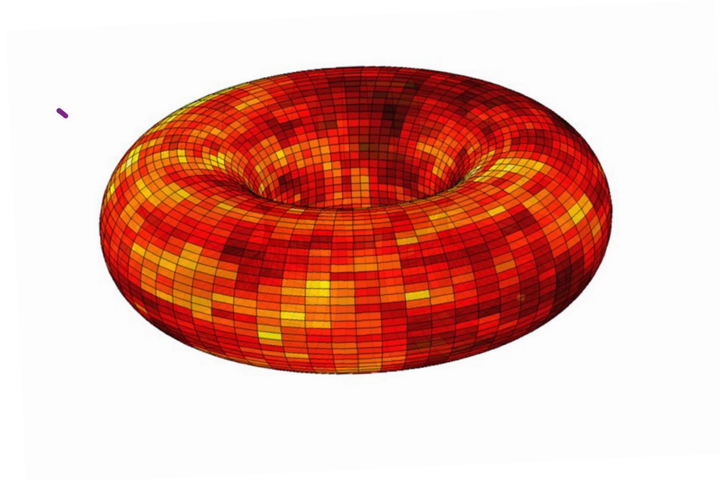
(UST, $\overset{\text{(regularized)}}{\text{winding of UST}}$)

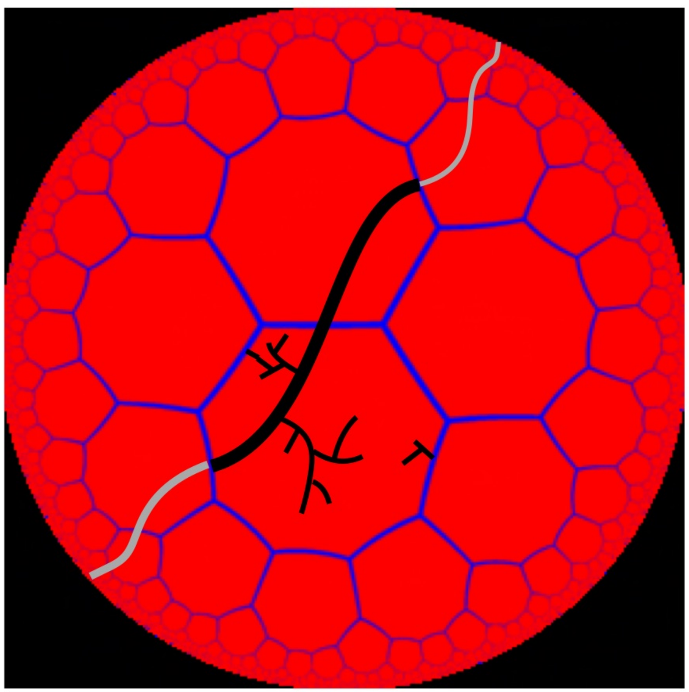
$\approx$ $(\underbrace{\text{flowline, GFF}}_{\substack{\text{coupled via} \\ \text{Imaginary geom.}}})$

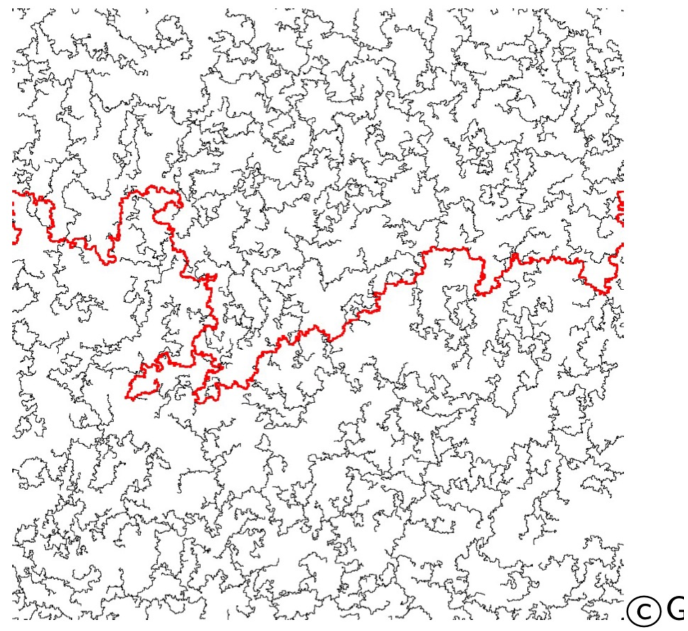
- Show that the regularization works! (purple regions roughly independent!)

Other directions

Conformal invariance on Riemann surfaces.







Future:

(1) Extend Imaginary geom. to other geometries.

(2) Interacting dimers (?) other SLE curves.?

Thank You for your attention!