Periodic billiards within quadrics, extremal polynomials, and partitions

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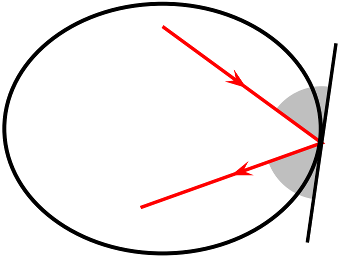
Based on:

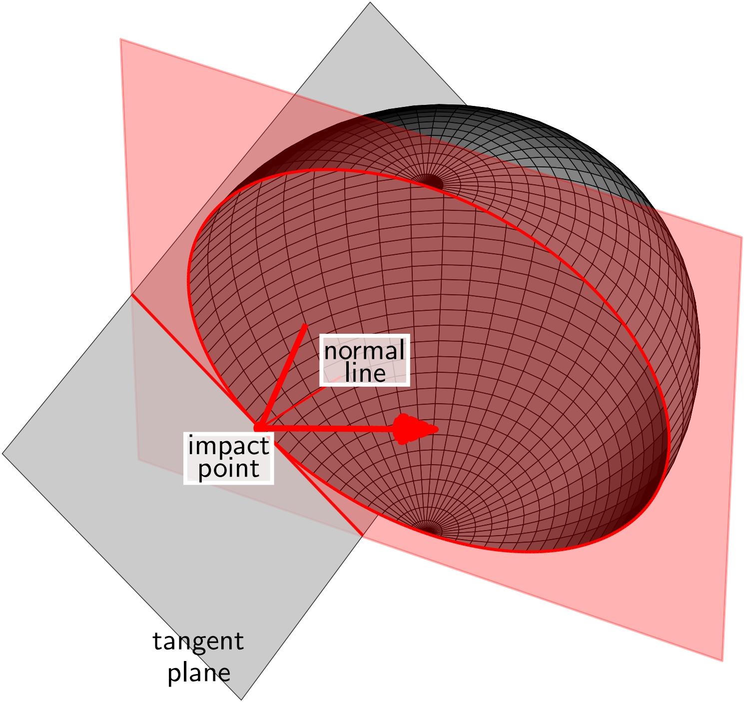
- V. Dragović, M. Radnović, *Periodic ellipsoidal billiard trajectories and extremal polynomials*, arXiv:1804.02515, Comm. Math. Phys. 372 (2019), no. 1, 183–211.
- G. E. Andrews, V. Dragović, M. Radnović, *Combinatorics of periodic billiards within quadrics*, arXiv: 1908.01026, The Ramanujan Journal, DOI: 10.1007/s11139-020-00346-y.

Definition of mathematical billiard

Billiard within a given domain is a dynamical system where a particle is moving without friction inside the domain, and reflecting absolutely elastically on the boundary.

Trajectories are polygonal lines with vertices lying on the domain boundary, with congruent impact and reflection angles at each vertex, while the particle speed remains constant.




normal
line
impact
point
tangent
plane

What about short trajectories of integrable billiards?

in the d-dimensional space

Billiards within ellipsoids

We will show that the shortest essential trajectories are of length $d+1$.

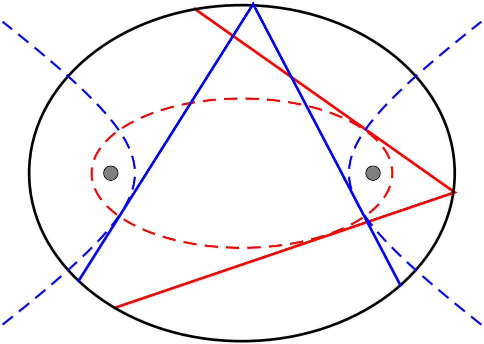
essential trajectories

The trajectories that are not placed in a hyper-plane of symmetry of the ellipsoid.

non-essential trajectories

Can be considered as trajectories within an ellipsoid in a space of lower dimension.

The planar case: billiard within an ellipse



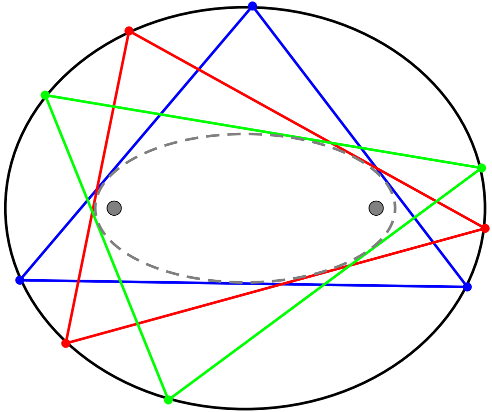
Integrability

The caustics represent geometric manifestation of the integrals of motion.

Poncelet porism

Jean-Victor Poncelet, 1813

Consider one periodic trajectory of the elliptic billiard. Then all trajectories sharing the same caustic are also periodic, and become closed after the same number of reflections.



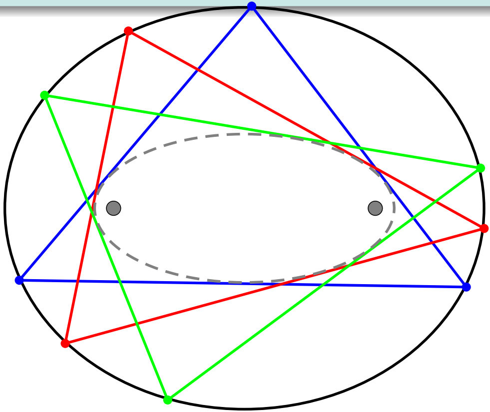
Short periodic trajectories of elliptical billiard

2-periodic trajectories are trivial

Always along one of the axes of the ellipse.

3-periodic trajectories

They have unique caustic, always an ellipse.



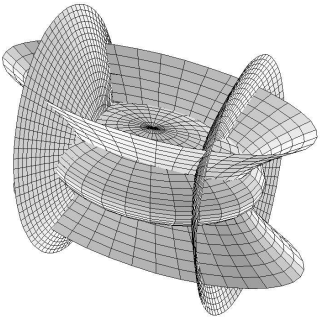
Confocal families in the d-dimensional space

Family of confocal quadrics

$$\frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_d^2}{a_d - \lambda} = 1$$

λ – real parameter

$0 < a_1 < \cdots < a_d$ – real constants



Differential equations
Jacobi

$$\sum_{s=1}^{d} \frac{d\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}} = 0,$$
$$\sum_{s=1}^{d} \frac{\lambda_s d\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}} = 0,$$
$$\dots,$$
$$\sum_{s=1}^{d} \frac{\lambda_s^{d-2} d\lambda_s}{\sqrt{\mathcal{P}(\lambda_s)}} = 0$$

Hyperelliptic curve

$$y^2 = \mathcal{P}(\lambda) = (a_1 - \lambda)\dots(a_d - \lambda)(\alpha_1 - \lambda)\dots(\alpha_{d-1} - \lambda)$$

Solutions of the system

Common tangent lines of the caustics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$.

Behaviour of Elliptic Coordinates along Trajectories

V. D, M. Radnović. (2004), following the ideas of Jacobi and Darboux

Domain Ω

Consists of all points of the billiard trajectories within $\mathcal{Q}_0$ with caustics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$.

In elliptic coordinates, Ω is just a box

Product of d segments in $[0, +\infty)$ where $\mathcal{P}(\lambda)$ is positive.

along any billiard trajectory

The elliptic coordinates change monotonously within their segments, with the extreme points being the segments endpoints.

Endpoints of the segments correspond to

- reflection off the boundary ellipsoid,
- touching a caustic,
- crossing a coordinate hyper-plane.

Periodic Trajectories

V. D, M. Radnović. (2004)

Theorem

A trajectory of the billiard system within with caustics $\mathcal{Q}_{\alpha_1}$, ..., $\mathcal{Q}_{\alpha_{d-1}}$ is periodic with exactly $|n_s|$ points at $\mathcal{Q}_{\gamma_s'}$ and $|n_s|$ points at $\mathcal{Q}_{\gamma_s''}$ $(1 \leq s \leq d)$ if and only if

$$\sum_{s=1}^{d} n_s \left(\mathcal{A}(P_{\gamma_s'}) - \mathcal{A}(P_{\gamma_s''}) \right) = 0$$

on the Jacobian of the curve

$$\Gamma \; : \; y^2 = \mathcal{P}(x) := (a_1 - x) \cdots (a_d - x)(\alpha_1 - x) \cdots (\alpha_{d-1} - x).$$

Here, $\mathcal{A}$ denotes the Abel-Jacobi map, where $P_{\gamma_s'}$, $P_{\gamma_s''}$ are points on Γ with coordinates $x = \gamma_s'$, $x = \gamma_s''$.

Winding numbers

Given an n-periodic trajectory. Denote $m_0 = n$, $m_s = (-1)^s n_s$, $m_d = 0$, and call $(m_0, m_1, \ldots, m_{d-1})$ the winding numbers of the given n-periodic billiard trajectory.

Notation: $b_1, \ldots, b_{2d-1}$

$\{b_1, \ldots, b_{2d-1}\} = \{a_1, \ldots, a_d, \alpha_1, \ldots, \alpha_{d-1}\}$
$b_1 < b_2 < \cdots < b_{2d-1}$, $c_{2d} = 0, c_j = 1/b_j$.

Audin 1994, "the Audin Alternative (AA)"

If $\alpha_1 < \alpha_2 < \cdots < \alpha_{d-1}$, then $\alpha_j \in \{b_{2j-1}, b_{2j}\}$, for $1 \leq j \leq d-1$. We get: $b_{2d-1} = a_d$.

Parity of winding numbers

Let $(m_0, m_1, \ldots, m_{d-1})$ be the winding numbers of a given periodic billiard trajectory. Then:

(i) if the winding number m_j, for $j > 0$ is odd, then b_{2j} and b_{2j+1} are both in the set $\{\alpha_1, \ldots, \alpha_{d-1}\}$;

(ii) two consecutive winding numbers cannot both be odd;

(iii) m_{d-1} is even.

d-dimensional case

Trajectories of period $\leq d$ are placed in coordinate planes.

This is because the divisors of small order do not have the assigned space of functions rich enough to realise the requested equivalence.

trajectories of period $d+1$ have uniquely prescribed types of caustics

- $\left[\dfrac{d-1}{2}\right]$ pairs of the same type caustics;
- and an ellipsoid as caustic if d is even.

Important questions

Uniqueness and existence of a set of caustics generating $d+1$-periodic trajectories.

Monotonicity of the winding numbers.

Injectivity of the frequency map.

To classify/codify the sets of caustics generating n-periodic trajectories in d-dimensional space, for any $n > d$.

From Cayley's type conditions to polynomials

Theorem, V. D, M. Radnović, 2004

A billiard trajectory within $\mathcal{E}$ with caustics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$ has elliptic period m if and only if

$$\operatorname{rank}\begin{pmatrix} C_{d+1} & C_{d+2} & \dots & C_{m+1} \\ C_{d+2} & C_{d+3} & \dots & C_{m+2} \\ & & \dots & \\ C_{m+d-1} & C_{m+d} & \dots & C_{2m-1} \end{pmatrix} < m-d+1,$$

with $\sqrt{\mathcal{P}(x)} = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots$

The above condition is satisfied if and only if there exist a pair of polynomials p_m and q_{m-d} of degrees m and $m-d$ respectively such that: $p_m(x) + q_{m-d}(x)\sqrt{\mathcal{P}(x)} = \mathcal{O}(x^{2m})$.

Pell's Equation

The generalized Cayley's condition $C(n,d)$ is satisfied if and only if there exist a pair of real polynomials $\hat{p}_n$, $\hat{q}_{n-d}$ of degrees n and $n-d$ respectively such that the Pell equation holds:

$$\hat{p}_n^2(s) - \hat{\mathcal{P}}_{2d}(s)\hat{q}_{n-d}^2(s) = 1.$$

$$\hat{\mathcal{P}}_{2d}(s) = s\prod_{j=1}^{2d-1}\left(s - \frac{1}{b_j}\right)$$

The signature $(\tau_1, \ldots, \tau_d)$

τ_j – the number of zeroes of $\hat{q}_{n-d}$ in (c_{2j}, c_{2j-1}).

Generalized Chebyshev polynomials

Polynomials $\hat{p}_n$

Extremal polynomials on the system of d intervals
$[c_{2d}, c_{2d-1}] \cup [c_{2d-2}, c_{2d-3}] \cup \cdots \cup [c_2, c_1]$.

Following the principles formulated by Chebyshev's school and Borel we are going to study the structure of extremal points of $\hat{p}_n$, in particular the set of points of alternance.

Points of alternance

A subset of the solutions of the equation $\hat{p}_n^2(s) = 1$, with the maximal number of elements, such that the signs of $\hat{p}_n$ alter on it. Such a set is not uniquely determined, however the number of its elements is fixed and equal to $n+1$.

Theorem, V. D, M. Radnović, 2019

- The winding numbers satisfy:

$$m_j = m_{j+1} + \tau_j + 1, \quad 1 \leq j \leq d.$$

- The winding numbers are strictly decreasing:

$$m_{d-1} < m_{d-2} < \cdots < m_1 < m_0.$$

The outline of the proof

From Krein, Levin, Nudel'man 1990 (see also Sodin, Yuditskii 1992 overview): the number of points of alternance of the polynomial $\hat{p}_n$ on the segment $[c_{2d}, c_{2j+1}]$ is equal to $1 + m_j$, for $j \in \{1, \dots, d-1\}$. The difference $m_{j-1} - m_j$ is thus equal to the number of points of alternance on the interval $[c_{2j+1}, c_{2j-1}]$. According to the structure of the sets of the alternance, that number equals the sum of the numbers of the double points of the alternance from the interval (c_{2j}, c_{2j-1}) and one simple point of alternance at one of the endpoints of the interval.

Corollary. All zeroes of $\hat{q}_{n-d}$ are real.

The polynomial $\hat{p}_n$ has $n-d$ double extremal points in the interior of the union of the intervals $(c_{2d}, c_{2d-1}) \cup \cdots \cup (c_2, c_1)$. These roots of $\hat{p}'_n$ coincide with the roots of the polynomial $\hat{q}_{n-d}$ of degree $n-d$.

Three 2014 Conjectures of Ramirez Ros

Theorem from the previous slide answers affirmatively to Conjectures 1 and 3 from 2014 Ramirez Ros. The 2014 Ramirez Ros Conjecture 2 is answered affirmatively in the above Corollary.

$d+1$-periodic trajectories

Proposition

- The winding numbers of the trajectories of period $d+1$ within an ellipsoid in the d-dimensional space are

$$(m_0, m_1, \dots, m_{d-1}) = (d+1, d, d-1, \dots, 3, 2).$$

- The signature of such trajectories is $(0, 0, \dots, 0, 1)$.

Proof

Follows from the Theorem, since among m_i, even and odd numbers alternate and decrease, and $m_0 = d+1$.

$d+1$-periodic trajectories

Theorem, V. D, M. Radnović, 2019

For a given ellipsoid from a confocal pencil in $\mathbf{E}^d$, the set of caustics which generates $(d+1)$-periodic trajectories is unique, if it exists.

Proof

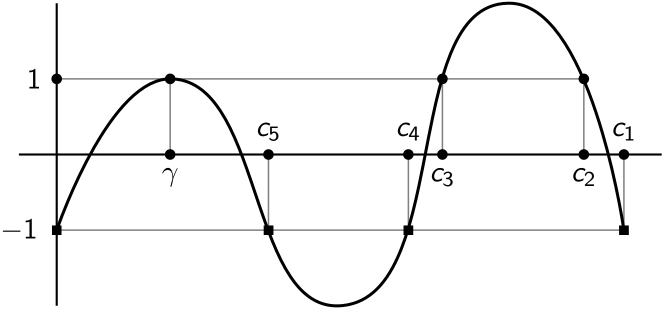
The polynomial $\hat{p}_{d+1}$ has one $(1 = d + 1 - d)$ double point of the alternance, which is the zero of polynomial $\hat{q}_1$: $\hat{q}_1(\gamma) = 0$, with $\gamma \in (c_{2d} = 0, c_{2d-1})$, according to the Propositon above. In addition, $\hat{p}_{d+1}(s)$ has $d+1$ simple points of the alternance at the endpoints of the intervals $[c_{2d}, c_{2d-1}]$,. . . ,$[c_2, c_1]$.

Proof, continuation:

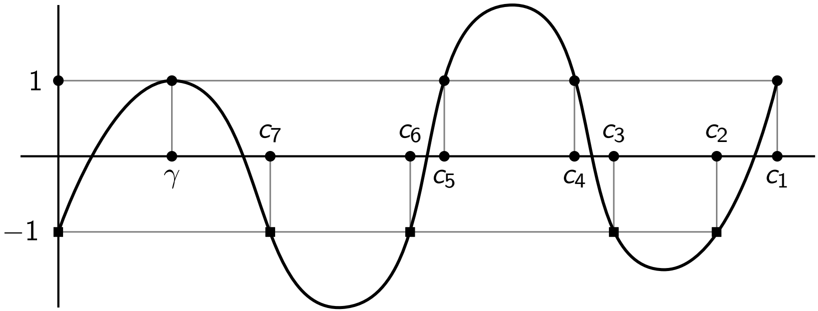
The following properties of the polynomial $\hat{p}_{d+1}$ follow from the structure and distribution of the points of alternance, the winding numbers and the signature:

- $\hat{p}_{d+1}$ takes value -1 at 0, $1/a_1$, ..., $1/a_d$;
- in $(0, 1/a_d)$, $\hat{p}_{d+1}$ has a local maximum equal to unity;
- $\hat{p}_{d+1}$ takes value 1 at $1/\alpha_1$, ..., $1/\alpha_{d-1}$.

Proof, continuation: $\hat{p}_4$



Proof, continuation: $\hat{p}_5$



For each d, there is a unique polynomial satisfying the properties.
γ – the only point of local extremum in $(0, 1/a_d)$ of
$r_{d+1}(s) = s\prod_{j=1}^{d}(s - 1/a_j)$; $\hat{p}_{d+1}(s) = 2r_{d+1}(s)/r_{d+1}(\gamma) - 1$,
$\hat{q}_1(s) = s - \gamma$.

$d + 1$-periodic in d-dim space

$1/\alpha_1, \ldots, 1/\alpha_{d-1}$

are the solutions of the equation $\hat{p}_{d+1} = 1$, different from γ.

If those solutions exist, they are uniquely determined.

Theorem, V. D, M. Radnović, 2019

For a given ellipsoid $\mathcal{E}$, the quadrics $\mathcal{Q}_{\lambda_k}$ are the caustics of $(d+1)$-periodic trajectories if $\lambda_1^{-1}, \ldots, \lambda_{d-1}^{-1}$ are the solutions of the equation $\hat{p}_{d+1}(s) - 1 = 0$, distinct from γ.

n-periodic in d-dim space, any $n > d$

Theorem, V. D, M. Radnović, 2018

Given an ellipsoid $\mathcal{E}$ in d-dimensional space and $n > d$ an integer. There is at most one set of caustics $\{\alpha_1, \dots, \alpha_{d-1}\}$ of the given types, which generates n-periodic trajectories within $\mathcal{E}$ having a prescribed signature.

Lemma A – the Audin Alternative

If $\mathcal{Q}_\alpha$, $\mathcal{Q}_\beta$ are caustics of the same type of a given billiard trajectory within $\mathcal{E}$, then $\{\alpha^{-1}, \beta^{-1}\} = \{c_{2k+1}, c_{2k}\}$, for some k.

n-periodic in d-dim space, any $n > d$

Lemma B: Theorem 2.12 Peherstorfer, Schiefermayr 1999

Let p_n, p_n^* be two polynomials of degree n, which solve the Pell equations. Denote by

$$\mathcal{I}_d = \cup_{j=0}^{d-1}[c_{2(d-j)}, c_{2(d-j)-1}] \quad \text{and} \quad \mathcal{I}_d^* = \cup_{j=0}^{d-1}[c^*_{2(d-j)}, c^*_{2(d-j)-1}]$$

respectively the sets $\{x \mid |p_n(x)| \leq 1\}$ and $\{x \mid |p_n^*(x)| \leq 1\}$. Suppose that:

- i at least one of the intervals from $\mathcal{I}_d$ coincides with one of the intervals from $\mathcal{I}_d^*$;
- ii $j \in \{0, \dots, d-1\}$: $c_{2(d-j)} = c^*_{2(d-j)}$ or $c_{2(d-j)-1} = c^*_{2(d-j)-1}$;
- iii in each pair of the corresponding intervals $[c_{2(d-j)}, c_{2(d-j)-1}]$ and $[c^*_{2(d-j)}, c^*_{2(d-j)-1}]$ the polynomials p_n, p_n^* have the same number of extreme points.

Then the polynomials p_n, p_n^* coincide up to a constant multiplier and sets $\mathcal{I}_d$ and $\mathcal{I}_d^*$ coincide.

3d zoo of short periodic trajectories

3-periodic trajectories are placed in coordinate planes

4-periodic trajectories

- the caustics are 1-sheeted hyperboloids;
- $C_3 = 0$; and $C_0 + C_1\alpha_2 + C_2\alpha_2^2 = 0$,

with

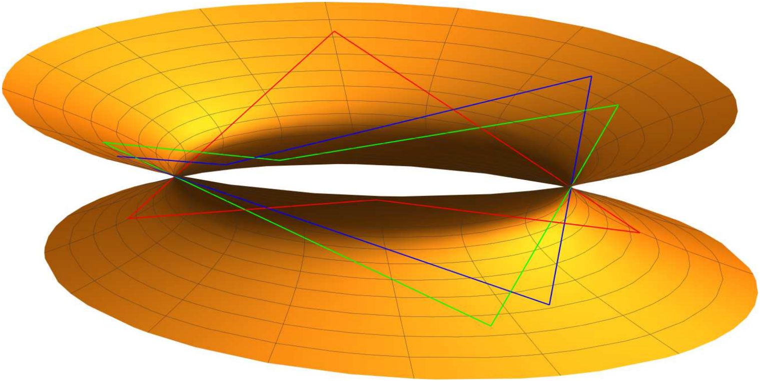
$$\frac{\sqrt{\mathcal{P}(x)}}{\alpha_1 - x} = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots .$$

According to a Theorem above, the winding numbers of such trajectories satisfy $m_0 > m_1 > m_2$, with $m_0 = 4$ and m_2 being even. Thus, $(m_0, m_1, m_2) = (4, 3, 2)$.

Such trajectories also appear as an example in a classification of symmetric periodic trajectories by Casas and Ramirez-Ros (2012).

The case of a double caustic

Each confocal family of quadrics contains a unique pair of ellipsoid and 1-sheeted hyperboloid such that there is a 4-periodic billiard trajectory within the ellipsoid with the segments placed on the hyperboloid.



5-periodic trajectories in 3-dimensional space

A 5-periodic trajectory of the billiard within ellipsoid $\mathcal{E}$, with non-degenerate caustics $\mathcal{Q}_{\alpha_1}$ and $\mathcal{Q}_{\alpha_2}$ if and only if:

- since the period is odd, one of the caustics, say $\mathcal{Q}_{\alpha_1}$, is an ellipsoid, i.e. $\alpha_1 \in (0, a_1)$; and
- $C_3 = C_4 = 0$,

with C_3, C_4 being the coefficients in the Taylor expansion about $x = 0$:

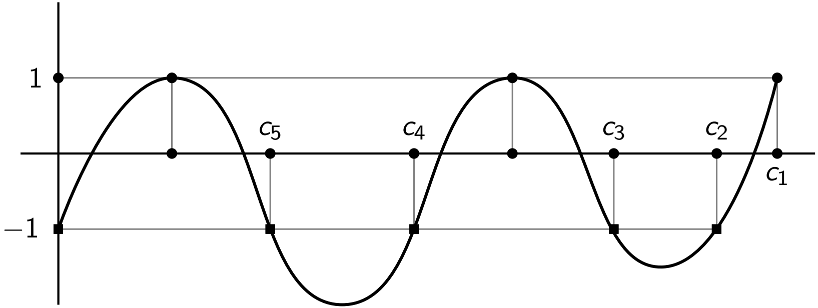
$$\frac{\sqrt{(a_1 - x)(a_2 - x)(a_3 - x)(\alpha_1 - x)(\alpha_2 - x)}}{\alpha_1 - x} = C_0 + C_1 x + C_2 x^2 + \dots$$

5-periodic, continuation

The winding numbers are $(m_0, m_1, m_2) = (5, 4, 2)$

Since $m_0 = 5$, $m_0 > m_1 > m_2$ and m_1, m_2 are even.

The graph of $\hat{p}_5(s)$:



$c_1 = 1/\alpha_1$, $c_2 = 1/a_1$, $\{c_3, c_4\} = \{1/a_2, 1/\alpha_2\}$, $c_5 = 1/a_3$

6-periodic trajectories in dimension three

Winding numbers

4-periodic and 5-periodic trajectories have uniquely determined winding numbers. This is not the case with the trajectories of period 6:

$$(m_0, m_1, m_2) \in \{(6,4,2), (6,5,4), (6,5,2), (6,3,2)\}.$$

Euclidean Billiard Partitions

I. M. Gelfand 1990

The older I get, the more I believe that at the bottom of most deep mathematical problems there is a combinatorial problem.

G. Andrews, V. Dragović, M. Radnović, 2020

Combinatorics of periodic ellipsoidal billiards, The Ramanujan Journal.

Let $\mathcal{D}$ denote the set of all integer partitions into distinct parts where

(E1) the smallest part is even;

(E2) adjacent parts are never both odd.

Let $p_{\mathcal{D}}(n)$ denote the number of partitions of n that are in $\mathcal{D}$

Weighted Euclidean Billiard Partitions

G. Andrews, V. Dragović, M. Radnović, 2019

Combinatorics of periodic ellipsoidal billiards, soon to appear.

Additionally, we consider weighting the partitions in $\mathcal{D}$ in accordance with the Audin Alternative. Suppose $\pi \in \mathcal{D}$ and that π has d parts with largest part n and s odd parts. The weight $\phi(n, d, \pi)$ is given by:

$$\phi(2m, d, \pi) = 2^{d-1-2s};$$
$$\phi(2m+1, d, \pi) = 2^{d-2s}.$$

Examples:

$$\pi_1 = (5, 4, 2),\ \phi(5, 3, \pi_1) = 2^{3-2\cdot 1} = 2;$$
$$\pi_2 = (6, 5, 2),\ \phi(6, 3, \pi_2) = 2^{3-1-2\cdot 1} = 1;$$
$$\pi_3 = (6, 4, 2),\ \phi(6, 3, \pi_3) = 2^{3-1-2\cdot 0} = 4.$$

G. Andrews, V. D., M. Radnović, 2019

Let $p_{\mathcal{D}}(m, n)$ denote the number of weighted partitions of n in $\mathcal{D}$ with weight m.

Theorem

The generating function for the weighted Euclidean billiard partitions has the following formula:

$$1 + \sum_{n \geq 1, m \geq 0} p_{\mathcal{D}}(m, n) q^n = 1 + \sum_{d=1}^{\infty} \sum_{n=0}^{\infty} \frac{s(d, n)}{(q^2; q^2)_d},$$

where

$$s(d, 2n) = x^{2n-d-1} q^{2n^2-2dn-n+d^2+2d} \begin{bmatrix} n-1 \\ 2n-d-1 \end{bmatrix}_{q^2};$$

$$s(d, 2n+1) = x^{2n-d} q^{2n^2-2dn-n+d^2+3n} \begin{bmatrix} n-1 \\ 2n-d \end{bmatrix}_{q^2}.$$

G. Andrews, V. D., M. Radnović, 2019

$s(d,n)$: the generating function for the partitions in $\bar{\mathcal{D}}$ that have exactly d (dimension of the space) parts and largest part equal n (the period). $\hat{s}(d,n)$: the generating function for the partitions in $\mathcal{D}$ that have exactly d (dimension of the space) parts and largest part equal n (the period).

$$\hat{s}(d,n) = s(d,n)\frac{1}{(1-q^2)(1-q^4)\cdots(1-q^{2d})} = \frac{s(d,n)}{(q^2;q^2)_d},$$

where the product

$$\frac{1}{(q^2;q^2)_d}$$

generates the general partition into at most d even parts. The Gaussian polynomials or q-binomial coefficients:

$$\begin{bmatrix} A \\ B \end{bmatrix}_q = \begin{cases} 0, & \text{if } B<0 \text{ or } B>A \\ \frac{(q;q)_A}{(q;q)_B(q;q)_{A-B}}, & 0 \le B \le A \end{cases}$$

and $(x;q)_N = (1-x)(1-xq)\cdots(1-xq^{N-1})$.

Further results:

- V. Dragović, M. Radnović, *Periodic ellipsoidal billiard trajectories and extremal polynomials*, arXiv:1804.02515, Comm. Math. Phys. 372 (2019), no. 1, 183–211.
- V. Dragović, M. Radnović, *Caustics of Poncelet polygons and classical extremal polynomials* Regular Chaotic Dynamics, Vol. 24, No. 1, 2019, p. 1-35.
- A. Adabrah, V. Dragović, M. Radnović, *Periodic billiards within conics in the Minkowski plane and Akhiezer polynomials*, 2019, arXiv:1906.04911, Regular and Chaotic Dynamics, No. 5, Vol. 24, 2019, p. 464-501.
- G. Andrews, V. Dragović, M. Radnović, *Combinatorics of periodic billiards within quadrics*, arXiv: 1908.01026, The Ramanujan Journal, DOI: 10.1007/s11139-020-00346-y.

Although this may seem a paradox, all exact science is dominated by the idea of approximation. When a man tells you that he knows the exact truth about anything, you are safe in infering that he is an inexact man.

Bertrand Russell