Invariants and Hochschild cohomology of rings of differential operators in one variable

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The A_h **family**

Heisenberg's Uncertainty Principle

$$YX - XY = 1$$

This is the defining relation for the Weyl algebra A_1.

A_1 can also be seen as the algebra of first order differential operators on $\mathbb{F}[t]$ with polynomial coefficients:

$X =$ multiplication by t
$Y = \frac{d}{dt}$

$$\begin{aligned} YX.p(t) &= \frac{d}{dt} t\, p(t) = p(t) + t\, \frac{d}{dt} p(t) = XY.p(t) + p(t) \\ &= (XY + 1).p(t) \end{aligned}$$

The Meromorphic Weyl Algebra *aka* The Jordan Plane

Now replace differentiation by integration:

$$X.p(t) = \int_0^t p(z)\,dz$$
$$Y.p(t) = t\,p(t)$$

$$\begin{aligned} XY.p(t) &= \int_0^t z\,p(z)\,dz \qquad \text{(integrate by parts)} \\ &= t\int_0^t p(z)\,dz - \int_0^t \int_0^z p(w)\,dw\,dz \\ &= YX.p(t) - X^2.p(t). \end{aligned}$$

$YX - XY = X^2$ defines the Jordan Plane

The A_h family

More generally, we can consider operators x, y satisfying:

$$yx - xy = h(x)$$

for a fixed $h \in \mathbb{F}[x]$.

Define the algebra thus generated by A_h:

$$A_h = \mathbb{F}\langle x, y\rangle / \left([y, x] = h(x)\right)$$

We can view $A_h \subseteq A_1$ as the subalgebra generated by:

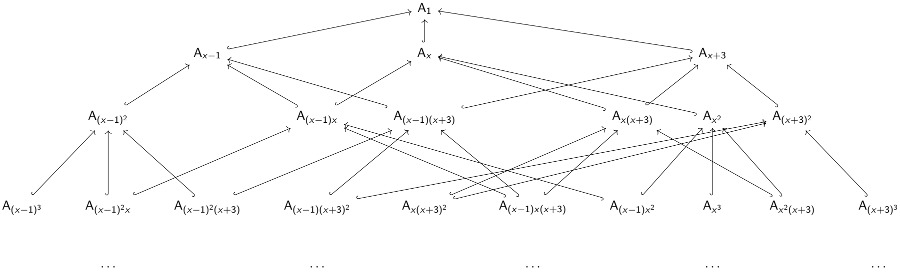
$$x \quad \text{and} \quad y = h(x)\frac{d}{dx}.$$

Examples

- $A_0 = \mathbb{F}[x, y]$ (commutative) polynomial algebra
- $A_1 = \mathbb{A}_1(\mathbb{F})$ Weyl algebra
- $A_x = U(\mathfrak{L})$ env. algebra of the 2-dim'l non-abelian Lie algebra
  $\mathfrak{L} = \mathbb{F}x \oplus \mathbb{F}y$, $[y, x] = x$
- A_{x^2} Jordan plane – Artin-Schelter regular of dimension 2
- $A_{x(x+1)}$, etc.

Purpose: To describe this family as a whole.

Divisibility poset on $\mathbb{F}[X]$ – inclusion poset on A_h



Classification of Ore extensions over $\mathbb{F}[x]$

Thm. (cf. Awami, Van den Bergh, Van Oystaeyen '88)

If A is an Ore extension over $\mathbb{F}[x]$, then A is isomorphic to:

- **quantum plane:** $yx = qxy$ for some $q \in \mathbb{F}^*$;
- **quantum Weyl algebra:** $yx - qxy = 1$ for some $q \in \mathbb{F}^*$;
- **one of the algebras** A_h for some $h = h(x) \in \mathbb{F}[x]$.

Remark: Quantum planes and quantum Weyl algebras are examples of generalized Weyl algebras and have been extensively studied. From the homological point of view see the recent paper:

M. Gerstenhaber, A. Giaquinto
On the cohomology of the Weyl algebra, the quantum plane, and the q-Weyl algebra
Journal of Pure and Applied Algebra **(2014)**

Examples of Generalized Weyl Algebras

GWAs were introduced by Bavula in '92.

Prototypical examples are:

- **(quantum) plane**
- **(quantum) Weyl algebra**
- $\mathbf{U}(\mathfrak{sl}_2), \ \mathbf{U_q}(\mathfrak{sl}_2)$
- **Noetherian (generalized) down-up algebras**

Thm. (BLO)

$$\left\{ \mathsf{A}_h \ \mid \ h \in \mathbb{F}[x] \right\} \bigcap \left\{ \text{GWAs over } \mathbb{F}[x] \right\} \overset{\text{iso}}{=} \left\{ \mathsf{A}_0, \mathsf{A}_1 \right\}$$

Isomorphism problem

Thm. (BLO) Isomorphism problem

$$\mathsf{A}_h \cong \mathsf{A}_g \iff g(x) = \nu h(\alpha x + \beta),$$

for some $\alpha, \beta, \nu \in \mathbb{F}$, $\alpha\nu \neq 0$.

Automorphisms and invariants of A_h

Automorphisms of A_1

For $f \in \mathbb{F}[x] \subseteq A_h$ there is $\phi_f \in \mathrm{Aut}_{\mathbb{F}}(A_h)$ defined by

$$\phi_f(x) = x, \qquad \phi_f(y) = y + f(x)$$

Furthermore,

$$\{\phi_f \mid f \in \mathbb{F}[x]\} \cong (\mathbb{F}[x], +) \quad \text{is a subgroup of } \mathrm{Aut}_{\mathbb{F}}(A_h).$$

In case $h \in \mathbb{F}^*$ (Weyl algebra):

$\sigma : A_1 \to A_1, \quad x \mapsto -y, \quad y \mapsto x$ gives an automorphism of A_1 of order 4, and

Dixmier '68 ($\mathrm{char}(\mathbb{F}) = 0$), Makar-Limanov '84 ($\mathrm{char}(\mathbb{F}) > 0$):

$\mathrm{Aut}_{\mathbb{F}}(A_1)$ is generated by $\mathbb{F}[x]$ and σ.

Remark: A similar result holds for $\mathrm{Aut}_{\mathbb{F}}(A_0)$, by Jung ($\mathrm{char}(\mathbb{F}) = 0$) and Van der Kulk ($\mathrm{char}(\mathbb{F}) > 0$).

Structure of $\mathrm{Aut}_{\mathbb{F}}(\mathsf{A}_h)$, for $\deg h \geq 1$

Consider the automorphisms:

$$\tau_{\alpha,\beta}(x) = \alpha x + \beta, \qquad \tau_{\alpha,\beta}(y) = \alpha^{\deg h - 1} y.$$

for $(\alpha, \beta) \in \mathbb{P} = \{(\alpha, \beta) \in \mathbb{F}^* \times \mathbb{F} \mid h(\alpha x + \beta) = \alpha^{\deg h} h(x)\}$.
The following are subgroups of $\mathrm{Aut}_{\mathbb{F}}(\mathsf{A}_h)$:

$$\tau_{\mathbb{P}} := \{\tau_{\alpha,\beta} \mid (\alpha, \beta) \in \mathbb{P}\} \quad \text{and} \quad \tau_{1,\mathbb{G}} = \{\tau_{1,\nu} \mid (1, \nu) \in \mathbb{P}\}.$$

Thm. (BLO). Assume $\deg h \geq 1$.

- $\mathbb{F}[x]$ is a normal subgroup of $\mathrm{Aut}_{\mathbb{F}}(\mathsf{A}_h)$, and
  $$\mathrm{Aut}_{\mathbb{F}}(\mathsf{A}_h) = \mathbb{F}[x] \rtimes \tau_{\mathbb{P}}$$
- $\tau_{1,\mathbb{G}}$ is isomorphic to a finite subgroup of $(\mathbb{F}, +)$, which is trivial when $\mathrm{char}(\mathbb{F}) = 0$
- $\mathbb{F}[x] \rtimes \tau_{1,\mathbb{G}}$ is a normal subgroup of $\mathrm{Aut}_{\mathbb{F}}(\mathsf{A}_h)$

If the subgroup $\mathbb{F}[x] \rtimes \tau_{1,\mathbb{G}}$ is proper, there is some $(\alpha, \beta) \in \mathbb{P}$ with $\alpha \neq 1$. The next result draws conclusions in that case.

Thm. (BLO). Assume h has k distinct roots in $\overline{\mathbb{F}}$ for $k \geq 1$.

Case $k = 1$ Let λ be the unique root of h in $\overline{\mathbb{F}}$.

- If $\lambda \notin \mathbb{F}$, then $\mathrm{Aut}_{\mathbb{F}}(\mathsf{A}_h) = \mathbb{F}[x]$.
- If $\lambda \in \mathbb{F}$, then $\tau_{\mathbb{P}} \cong \mathbb{F}^*$, and $\mathrm{Aut}_{\mathbb{F}}(\mathsf{A}_h) = \mathbb{F}[x] \rtimes \mathbb{F}^*$.

Case $k \geq 2$ $\tau_{\mathbb{P}} = \tau_{1,\mathbb{G}} \rtimes \langle \tau_{\alpha,\beta} \rangle$, where $\langle \tau_{\alpha,\beta} \rangle$ is a finite cyclic group and

$$\mathrm{Aut}_{\mathbb{F}}(\mathsf{A}_h) \cong (\mathbb{F}[x] \rtimes \tau_{1,\mathbb{G}}) \rtimes \langle \tau_{\alpha,\beta} \rangle.$$

Invariants of A_h

Let $\mathfrak{A} = \mathrm{Aut}_{\mathbb{F}}(\mathsf{A}_h)$. We determine the invariants under $\mathfrak{A}$ in A_h:

$$\mathsf{A}_h^{\mathfrak{A}} = \{a \in \mathsf{A}_h \mid \omega(a) = a \quad \forall\, \omega \in \mathfrak{A}\}.$$

Thm. (BLO). Suppose $\mathfrak{A} = \mathrm{Aut}_{\mathbb{F}}(\mathsf{A}_h)$. Then

(i) $\mathsf{A}_h^{\mathfrak{A}} = \mathbb{F}[x]$ if $\mathfrak{A} = \mathbb{F}[x]$.

(ii) $\mathsf{A}_h^{\mathfrak{A}} = \mathbb{F}$ if $\mathfrak{A} = \mathbb{F}[x] \rtimes \mathbb{F}^*$ and $|\mathbb{F}| = \infty$.

(iii) $\mathsf{A}_h^{\mathfrak{A}} = \mathbb{F}[t]$, where $t \in \mathbb{F}[x]$ can be taken as follows:

(a) If $\tau_{\mathbb{P}} = \tau_{1,\mathbb{G}}$, then $t(x) = \prod_{\nu \in \mathbb{G}} (x + \nu)$.

(b) If $\tau_{\mathbb{P}} = \tau_{1,\mathbb{G}} \rtimes \langle \tau_{\alpha,\beta} \rangle$, where α is a primitive ℓth root of unity, then $t(x) = \prod_{\nu \in \mathbb{G}} \left(x + \frac{\beta}{\alpha - 1} + \nu \right)^{\ell}$.

Remark: This could be summarized as follows

$$\dim \left(\mathbb{F}[x]/\mathsf{A}_h^{\mathfrak{A}} \right) = [\mathfrak{A} : \mathbb{F}[x]]$$

Properties $\mathcal{P}_1$ and $\mathcal{P}_2$

L. Ben Yakoub and M.P. Malliavin '96 studied the following properties relative to an algebra R and an automorphism (resp. derivation) ϕ of R:

$\mathcal{P}_1$ ϕ extends to any algebra $S \supseteq R$ (& extensions are compatible).

$\mathcal{P}_2$ For any ideal I of R, ϕ induces a map $\overline{\phi}$ on R/I and $\overline{\phi}$ satisfies $\mathcal{P}_1$.

Note: $\mathcal{P}_2 \implies \mathcal{P}_1$

Any inner automorphism (resp. inner derivation) satisfies $\mathcal{P}_2$, but there are algebras with non-inner automorphisms satisfying $\mathcal{P}_2$.

Thm. (BLO)

$$\phi \in \mathrm{Aut}_{\mathbb{F}}(\mathsf{A}_h) \quad \text{satisfies } \mathcal{P}_1 \quad \iff \quad \phi = \mathrm{id}_{\mathsf{A}_h}$$

$$D \in \mathrm{Der}_{\mathbb{F}}(\mathsf{A}_h) \quad \text{satisfies } \mathcal{P}_2 \quad \overset{\mathrm{char}(\mathbb{F})=0}{\iff} \quad D \in \mathrm{Inder}_{\mathbb{F}}(\mathsf{A}_h)$$

Derivations of A_h

Derivations of the Weyl algebra

Theorem (Dixmier '66) Derivations of A_1, $\mathrm{char}(\mathbb{F}) = 0$

In characteristic 0, all derivations of the Weyl algebra are inner.

However, over fields of characteristic $p > 0$, A_1 has two special derivations E_x and E_y, which are specified by

$$E_x(x) = y^{p-1}, \quad E_x(y) = 0, \quad \text{and} \quad E_y(x) = 0, \quad E_y(y) = x^{p-1}.$$

Theorem (BLO) Derivations of A_1, $\mathrm{char}(\mathbb{F}) = p > 0$

- Let $Z(A_1) = \mathbb{F}[x^p, y^p]$ be the center of A_1. Then

$$\mathrm{Der}_{\mathbb{F}}(A_1) = Z(A_1)E_x \oplus Z(A_1)E_y \oplus \mathrm{Inder}_{\mathbb{F}}(A_1).$$

- The restriction map $\mathrm{Res} : \mathrm{Der}_{\mathbb{F}}(A_1) \to \mathrm{Der}_{\mathbb{F}}(Z(A_1))$ induces a Lie algebra isomorphism

$$\mathrm{HH}^1(A_1) = \mathrm{Der}_{\mathbb{F}}(A_1)/\mathrm{Inder}_{\mathbb{F}}(A_1) \cong \mathrm{Der}_{\mathbb{F}}(\mathbb{F}[t_1, t_2]).$$

Locally nilpotent derivations

For $g \in \mathbb{F}[x]$, consider the derivation D_g of A_h with

$$D_g(x) = 0 \quad \text{and} \quad D_g(y) = g.$$

Prop. (BLO)

- $[D_f, D_g] = 0$ for all $f, g \in \mathrm{R}$.
- $\mathcal{G} = \{D_g \mid g \in \mathrm{R}\}$ is an abelian Lie subalgebra of $\mathrm{Der}_{\mathbb{F}}(\mathrm{A}_h)$.
- D_g is locally nilpotent.
- Assume $\mathrm{char}(\mathbb{F}) = 0$. Then

$$\phi_g = \exp(D_g) = \sum_{n=0}^{\infty} \frac{(D_g)^n}{n!}.$$

Remark: In characteristic 0, any locally nilpotent derivation of A_1 is conjugate under $\mathrm{Aut}_{\mathbb{F}}(\mathrm{A}_1)$ to one in $\mathcal{G}$ (Dixmier '68). Similarly for A_0 (Rentschler '68). If $\deg h \geq 1$ then $\mathcal{G} = \mathrm{LND}(\mathrm{A}_h)$.

Lie structure of $\mathrm{HH}^1(\mathrm{A}_h)$ for $\mathrm{char}(\mathbb{F}) = 0$

Let $\pi_h = \dfrac{h}{\gcd(h, h')}$.

Thm. (BLO)

The Lie algebra $\mathrm{HH}^1(\mathrm{A}_h) = \mathrm{Der}_{\mathbb{F}}(\mathrm{A}_h)/\mathrm{Inder}_{\mathbb{F}}(\mathrm{A}_h)$ decomposes as

$$\mathrm{HH}^1(\mathrm{A}_h) = \mathrm{Z}(\mathrm{HH}^1(\mathrm{A}_h)) \oplus [\mathrm{HH}^1(\mathrm{A}_h), \mathrm{HH}^1(\mathrm{A}_h)],$$

where

- $\mathrm{Z}(\mathrm{HH}^1(\mathrm{A}_h)) = \left\{ D_{rh/\pi_h} \mid \deg r < \deg \pi_h \right\}$,
- $\dim_{\mathbb{F}} \mathrm{Z}\left(\mathrm{HH}^1(\mathrm{A}_h)\right) = \deg \pi_h$.

Lie structure of $[\mathrm{HH}^1(\mathrm{A}_h), \mathrm{HH}^1(\mathrm{A}_h)]$ for $\mathrm{char}(\mathbb{F}) = 0$

The *Witt algebra* W of vector fields on the unit circle (centerless Virasoro algebra) is defined as

$$\mathrm{W} := \mathrm{span}_{\mathbb{F}}\{w_n \mid n \geq -1\},$$

where $[w_m, w_n] = (n-m)w_{m+n}$ for all $m, n \geq -1$.

Let $q_1, \ldots, q_k$, $k \geq 0$, be the prime factors of h with multiplicity ≥ 2.

Consider the ideal of $[\mathrm{HH}^1(\mathrm{A}_h), \mathrm{HH}^1(\mathrm{A}_h)]$:

$$\mathcal{J} = \mathrm{span}_{\mathbb{F}}\{\mathrm{ad}_{ra_n} \mid r \in q_1 \cdots q_k \mathbb{F}[x],\ n \geq 0\}$$

Lie structure of $[\mathrm{HH}^1(\mathrm{A}_h), \mathrm{HH}^1(\mathrm{A}_h)]$ for $\mathrm{char}(\mathbb{F}) = 0$

Thm. (BLO)

(i) $\mathcal{J}$ is the largest nilpotent ideal of $[\mathrm{HH}^1(\mathrm{A}_h), \mathrm{HH}^1(\mathrm{A}_h)]$ and

$$[\mathrm{HH}^1(\mathrm{A}_h), \mathrm{HH}^1(\mathrm{A}_h)]/\mathcal{J} \cong \mathrm{W}_1 \oplus \cdots \oplus \mathrm{W}_\mathrm{k},$$

sum of k simple Lie algebras $\mathrm{W}_\mathrm{i} = (\mathbb{F}[x]/\mathbb{F}[x]q_i) \otimes \mathrm{W}$.

(ii) Assume $h \mid \pi_h^2$ (h is cube-free). Then $\mathcal{J} = 0$ and:
- if h is square-free, then $[\mathrm{HH}^1(\mathrm{A}_h), \mathrm{HH}^1(\mathrm{A}_h)] = 0$;
- otherwise, $[\mathrm{HH}^1(\mathrm{A}_h), \mathrm{HH}^1(\mathrm{A}_h)]$ is semisimple (cf. above).

(iii) Assume $h \nmid \pi_h^2$. Then $\mathcal{J} \neq 0$, and $[\mathrm{HH}^1(\mathrm{A}_h), \mathrm{HH}^1(\mathrm{A}_h)]$ is neither nilpotent nor semisimple.

Special cases, $\mathrm{char}(\mathbb{F}) = 0$

Cor. (BLO)

Let $D_1 \in \mathrm{Der}_{\mathbb{F}}(\mathsf{A}_h)$ be defined by $D_1(x) = 0$ and $D_1(y) = 1$.

A_1: $\mathrm{Der}_{\mathbb{F}}(\mathsf{A}_1) = \mathrm{Inder}_{\mathbb{F}}(\mathsf{A}_1)$, so $\mathrm{HH}^1(\mathsf{A}_1) = (0)$.

A_x: $\mathrm{Der}_{\mathbb{F}}(\mathsf{A}_x) = \mathbb{F}D_1 \oplus \mathrm{Inder}_{\mathbb{F}}(\mathsf{A}_x)$, so $\mathrm{HH}^1(\mathsf{A}_x) = \mathbb{F}D_1$.

A_{x^n}, $n \geq 2$: $\mathrm{HH}^1(\mathsf{A}_{x^n})/\mathcal{J} = \mathbb{F}D_{x^{n-1}} \oplus \mathsf{W}$,

where $\mathsf{W} := \mathrm{span}_{\mathbb{F}}\{w_i \mid i \geq -1\}$ is the Witt algebra.
The ideal $\mathcal{J}$ is nilpotent of index $\leq n-1$. In particular, $\mathcal{J} = 0$ when $n = 2$.

Derivations in prime characteristic

This case is more intricate and we will reveal only a few structural properties. Let

$$\mathsf{Res} : \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$$

be the restriction map and $\overline{\mathsf{Res}} : \mathsf{HH}^1(\mathsf{A}_h) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$ be the induced map.

Thm. (BLO)

- $\operatorname{im} \mathsf{Res} = \operatorname{im} \overline{\mathsf{Res}}$ is a free $\mathsf{Z}(\mathsf{A}_h)$-submodule of $\mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$ of rank 2, and $\mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$ is the Witt algebra in 2 variables;
- $\mathsf{HH}^1(\mathsf{A}_h)$ is a free $\mathsf{Z}(\mathsf{A}_h)$-module $\iff \gcd(h, h') = 1$. In this case, $\overline{\mathsf{Res}}$ is an isomorphism onto the image.

Higher cohomology and (twisted) Calabi-Yau property

Hochdchild cohomology groups

$$\begin{array}{rcl} HH^0(A) & = & Z(A) \\ HH^1(A) & = & \mathrm{Der}_{\mathbb{F}}(A)/\mathrm{Inder}_{\mathbb{F}}(A) \\ HH^2(A) & = & \text{formal deformations of A} \\ HH^3(A) & = & \text{obstructions to formal deformations of A} \end{array}$$

A formal deformation of an associative algebra A is a $\mathbb{F}[[\hbar]]$-algebra structure A_μ on $A[[\hbar]]$:

$$a \star b = ab + \mu_1(a,b)\hbar + \mu_2(a,b)\hbar^2 + \mu_3(a,b)\hbar^3 + \cdots$$

for all $a, b \in A$, where ab is the product in A.

Thus, we retrieve A on setting $\hbar = 0$.
The idea of algebraic deformation parallels the theory of deformations of complex analytic structures, initiated by Kodaira and Spencer in

K. Kodaira and D. Spencer, On deformations of complex analytic structures I & II, Ann. of Math. 67 (1958) 328-466.

Examples

Let $\mathrm{A} = \mathbb{F}[x, y]/(y^2 - x^3)$. In algebraic terms, this could be interpreted as setting

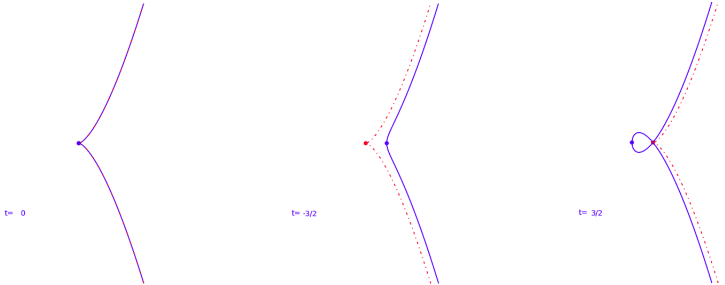
$$y \cdot y = x^3$$

in A.

Now consider $\mathrm{A}_t = \mathbb{F}[x, y, t]/(y^2 - x^3 - x^2 t)$, seen as a deformation of A; so

$$y \star y = x^3 + (x^2)t$$

in A_t. It is in fact a formal deformation of A.



Deformations and Hochdchild cohomology

Theorem

- If $HH^2(A) = 0$ then A is rigid (all deformations are trivial).
- If $HH^3(A) = 0$ then any 2-cocycle of A can be lifted to an associative deformation of A.

Examples

- $\mathbb{F}G$ is rigid for G a finite group with $|G| \neq 0$ in $\mathbb{F}$ (Maschke's theorem).
- $U(\mathfrak{g})$ is rigid for $\mathfrak{g}$ a finite-dimensional complex semisimple Lie algebra (Weyl's theorem).
- The Weyl algebra A_1 is rigid if $\mathrm{char}(\mathbb{F}) = 0$ (Sridharan).

Calabi-Yau algebras

V. Ginzburg, Calabi-Yau algebras, `arXiv:math/0612139`.
Motivation: Geometry of Calabi-Yau manifolds and mirror symmetry.

Definition We say A is a ν-twisted Calabi-Yau algebra of dimension $d \geq 0$ if

- A is homologically smooth, i.e., it admits a f.g. projective resolution of finite length, as a bimodule over itself;
- $$\mathrm{Ext}_{A^e}(A, A^e) = \begin{cases} 0, & \text{if } i \neq d \\ A^{\nu} & \text{if } i = d, \end{cases}$$

  where $A^e = A \otimes A^{\mathrm{op}}$ and $\nu \in \mathrm{Aut}_{\mathbb{F}}(A)$.
  The automorphism ν is unique up to inner automorphisms and is called the Nakayama automorphism of A.

Twisted Poincaré duality – Van den Bergh duality

By

M. Van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proceedings AMS (1998); Erratum, (2002).

if A is a ν-twisted Calabi-Yau algebra of dimension *d* then there is a twisted Poincaré duality between homology and cohomology:

$$\mathrm{HH}^{d-i}(A) \xrightarrow{\cong} \mathrm{HH}_i(A, A^{\nu})$$

In particular:

$$\mathrm{HH}^{d}(A) \xrightarrow{\cong} \mathrm{HH}_0(A, A^{\nu}) = A/\left[A, A^{\nu}\right].$$

The Nakayama automorphism for A_h

Can we determine the full cohomology ring $HH^*(A_h)$? Yes.

Thm.

- The algebras A_h are twisted Calabi-Yau of dimension 2.

  Follows from
  L.-Y. Liu, S.-Q. Wang and Q.-S. Wu, *Twisted Calabi-Yau property of Ore extensions*, Journal of Noncommutative Geometry (2014).

- The Nakayama automorhism ν of A_h is given by

$$\nu(x) = x, \qquad \nu(y) = y + h'(x).$$

  Follows from private communication with Q.-S. Wu.

The cohomology ring $\mathrm{HH}^*(\mathrm{A}_h)$

Cor.

- $$\mathrm{HH}^0(\mathrm{A}_h) = \mathrm{Z}(\mathrm{A}_h) = \begin{cases} \mathbb{F}, & \text{if } \mathrm{char}(\mathbb{F}) = 0; \\ \mathbb{F}[x^p, z_p], & \text{if } \mathrm{char}(\mathbb{F}) = p > 0. \end{cases}$$
- $\mathrm{HH}^1(\mathrm{A}_h) = \mathrm{Der}_{\mathbb{F}}(\mathrm{A}_h)/\mathrm{Inder}_{\mathbb{F}}(\mathrm{A}_h)$ has been determined in the previous slides.
- $\mathrm{HH}^2(\mathrm{A}_h) \cong \mathrm{A}_h / [\mathrm{A}_h, \mathrm{A}_h^{\nu}] = \mathrm{A}_h / \{ab - b\nu(a) \mid a, b \in \mathrm{A}_h\}$.
- $\mathrm{HH}^{\mathrm{n}}(\mathrm{A}_h) = 0$ for all $n \geq 3$; in particular, all Hochschild 2-cocycles can be integrated to full deformations of A_h.

In particular, we retrieve Sridharan's result and further that:

$$\text{the Weyl algebra } \mathrm{A}_1 \text{ is rigid } \iff \mathrm{char}(\mathbb{F}) = 0.$$

Deformations related to A_h

Thus it remains to determine $\mathsf{A}_h / [\mathsf{A}_h, \mathsf{A}_h^{\nu}]$ and describe all formal deformations of A_h.

Two other points of view:

- Use the ideas of J. Goodman and U. Krhmer (cf. also M. Suárez-Alvarez) to obtain an *untwisted* (i.e., Calabi-Yau) extension of A_h.
- Can the deformations of A_h be seen as deformations of a polynomial algebra?

Generalization of Moyal product

Example: $A = \mathbb{F}[x, y]$ commutative polynomial ring, $\mathrm{char}(\mathbb{F}) = 0$.

Let $\phi = \partial_y$, $\psi = h(x)\partial_x$. Then

$$a \star b = \sum_{n \geq 0} \frac{\phi^n(a)\psi^n(b)}{n!} \hbar^n$$

defines an associative product on $A[[\hbar]]$ with:

$$x \star x = x^2, \qquad y \star y = y^2,$$
$$y \star x = yx + h(x)\hbar, \qquad x \star y = xy.$$

So

$$y \star x - x \star y = h(x)\hbar.$$

Setting $\hbar = 1$ we retrieve all members of the family A_h as deformations of the commutative polynomial algebra $A = \mathbb{F}[x, y]$.