Anderson Localization in the presence of topologically protected channels

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Harvard University

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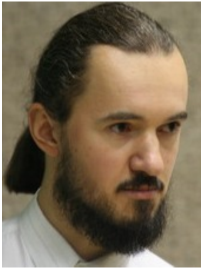
Lisbon

Acknowledgment

Pavel Ostrovksy
Max Planck Institute for Solid State Research



Mikhail Skvortsov
Landau Institute for theoretical physics

- E. Khalaf, M. A. Skvortsov, and P. M. Ostrovsky, PRB 2016
- E. Khalaf, P. M. Ostorvsky, PRL 2017
- E. Khalaf, P. M. Ostorvsky, PRB 2017
- E. Khalaf, M. A. Skvortsov, and P. M. Ostrovsky, in preparation.
- E. Khalaf, PhD Thesis

Anderson Localization

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- Absence of diffusion in disorder one or two dimensional systems for any disorder strength and in three dimensional systems for sufficiently strong disorder (Anderson 58, Abraham *et al.* 80)

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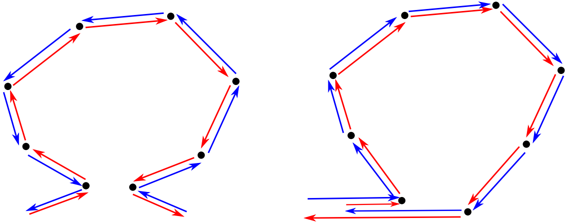

P' is the same as P $\qquad$ P' is the time-reverse of P

Topological protection

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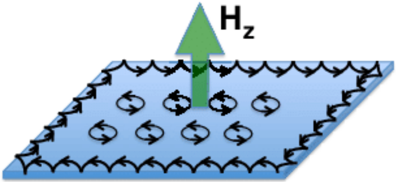
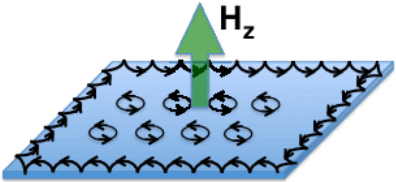


Quantum Hall effect
von Klitzing 81

Topological protection



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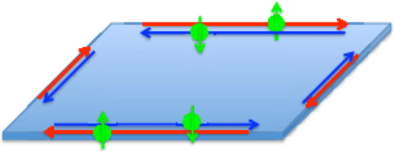

Quantum Spin Hall effect
Kane Mele 05, Bernevig *et al.* 06, K$\acute{}$onig *et al* 07

Localization + protected channels

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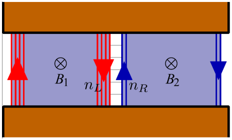
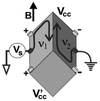
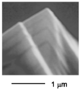


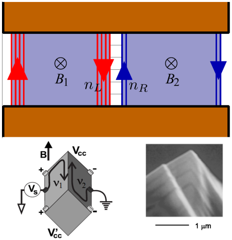




Interface between two quantum Hall systems

(Grayson *et. al.* 07,08)

Localization + protected channels



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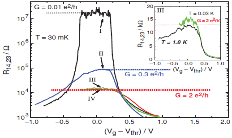

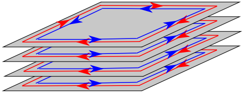

Stack of quantum spin-Hall systems (Koenig *et. al.* 07)

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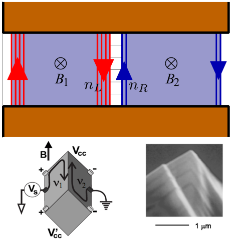

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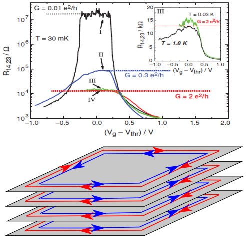

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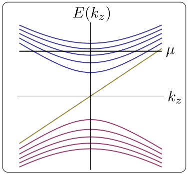
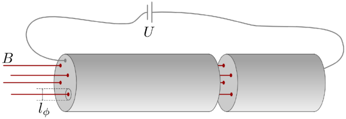


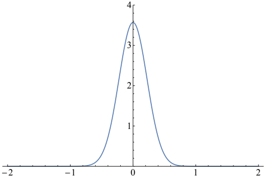


Weyl semimetal in a magnetic field (Wan *et. al.* '11, Zyuzin & Burkov '12, Altland & Bagrets '15, Huang *et. al.* '15, Shekhar *et. al.* '15, . . .)

Diffusion + Drift

Classical Diffusion

$$W(x,t) = \frac{e^{-\frac{x^2}{2Dt}}}{2\sqrt{\pi D t}}$$



Chiral channels →

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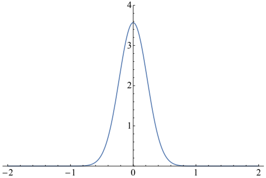

↓ Quantum interference

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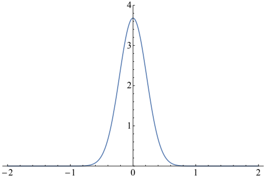
Anderson localization

$$W(x,t) = ?$$



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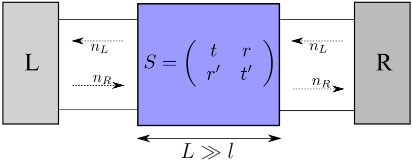

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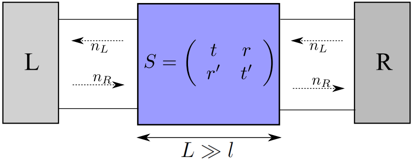

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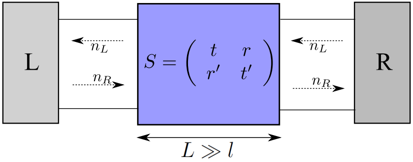

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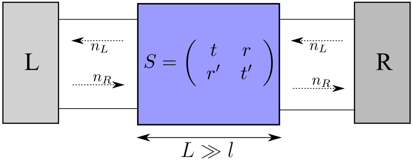

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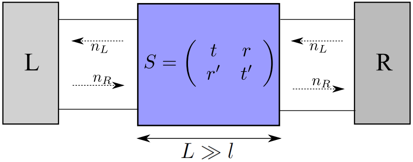

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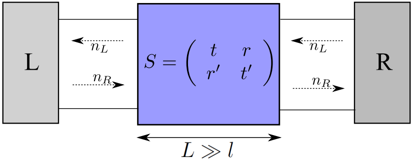

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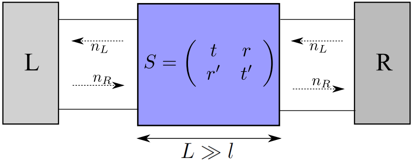

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- Lyapunov exponent $\lambda \geq 0$: $T = 1/\cosh^2 \lambda$
  - $\lambda = 0$: perfect transmission $(T = 1)$
  - $\lambda = \infty$: zero transmission $(T = 0)$

Effective field theory of disorder

Efetov '83

Effective field theory of disorder

- Moments in terms of Green's functions: Kubo formula

$$\mathrm{tr}(t^\dagger t)^n = \mathrm{tr}[\hat{v}(x_L) G^R_\epsilon(x_L, x_R) \hat{v}(x_R) G^A_\epsilon(x_R, x_L)]^n$$

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$$S = -\int_0^L dx \, \mathrm{str} \left[\frac{\xi}{8} (\partial_x Q)^2 + S_{\mathrm{top}} \right],$$
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The tenfold way

Symmetry class Cartan label	Symmetry $\mathcal{T}$	$\mathcal{C}$	$\mathcal{S}$	NLSM compact sector	Spatial dimension 1	2
A	0	0	0	$U(2n)/U(n) \times U(n)$	0	$\mathbb{Z}$
AIII	0	0	1	$U(n) \times U(n)/U(n)$	$\mathbb{Z}$	0
AI	1	0	1	$Sp(4n)/Sp(2n) \times Sp(2n)$	0	0
BDI	1	1	1	$U(2n)/Sp(2n)$	$\mathbb{Z}$	0
D	0	1	0	$O(2n)/U(n)$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	-1	1	1	$O(n) \times O(n)/O(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII	-1	0	0	$O(2n)/O(n) \times O(n)$	0	$\mathbb{Z}_2$
CII	-1	-1	1	$U(n)/O(n)$	$\mathbb{Z}$	0
C	0	-1	0	$Sp(2n)/U(n)$	0	$\mathbb{Z}$
CI	1	-1	1	$Sp(2n) \times Sp(2n)/Sp(2n)$	0	0

Altland, Zirnbauer 97, Kitaev 09, Schnyder *et al.* 09

The tenfold way

Symmetry class Cartan label	Symmetry $\mathcal{T}$	$\mathcal{C}$	$\mathcal{S}$	NLSM compact sector	Spatial dimension 1	2
A	0	0	0	$U(2n)/U(n) \times U(n)$	0	$\mathbb{Z}$
AIII	0	0	1	$U(n) \times U(n)/U(n)$	$\mathbb{Z}$	0
AI	1	0	1	$Sp(4n)/Sp(2n) \times Sp(2n)$	0	0
BDI	1	1	1	$U(2n)/Sp(2n)$	$\mathbb{Z}$	0
D	0	1	0	$O(2n)/U(n)$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	-1	1	1	$O(n) \times O(n)/O(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII	-1	0	0	$O(2n)/O(n) \times O(n)$	0	$\mathbb{Z}_2$
CII	-1	-1	1	$U(n)/O(n)$	$\mathbb{Z}$	0
C	0	-1	0	$Sp(2n)/U(n)$	0	$\mathbb{Z}$
CI	1	-1	1	$Sp(2n) \times Sp(2n)/Sp(2n)$	0	0

- A, C, D: arbitrary number of protected (chiral) channels ($\mathbb{Z}$)

Altland, Zirnbauer 97, Kitaev 09, Schnyder *et al.* 09

The tenfold way

Symmetry class Cartan label	Symmetry $\mathcal{T}$	$\mathcal{C}$	$\mathcal{S}$	NLSM compact sector	Spatial dimension 1	2
A	0	0	0	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathrm{U}(n)\times\mathrm{U}(n)/\mathrm{U}(n)$	$\mathbb{Z}$	0
AI	1	0	1	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$	0	0
BDI	1	1	1	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$	$\mathbb{Z}$	0
D	0	1	0	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	-1	1	1	$\mathrm{O}(n)\times\mathrm{O}(n)/\mathrm{O}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII	-1	0	0	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$	0	$\mathbb{Z}_2$
CII	-1	-1	1	$\mathrm{U}(n)/\mathrm{O}(n)$	$\mathbb{Z}$	0
C	0	-1	0	$\mathrm{Sp}(2n)/\mathrm{U}(n)$	0	$\mathbb{Z}$
CI	1	-1	1	$\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)/\mathrm{Sp}(2n)$	0	0

- A, C, D: arbitrary number of protected (chiral) channels ($\mathbb{Z}$)
- AII, DIII: 0 or 1 protected (helical) channels ($\mathbb{Z}_2$).

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The tenfold way

Symmetry class Cartan label	Symmetry $\mathcal{T}$	 $\mathcal{C}$	 $\mathcal{S}$	NLSM compact sector	Spatial dimension 1	 2
A	0	0	0	$\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathrm{U}(n) \times \mathrm{U}(n)/\mathrm{U}(n)$	$\mathbb{Z}$	0
AI	1	0	1	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$	0	0
BDI	1	1	1	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$	$\mathbb{Z}$	0
D	0	1	0	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	-1	1	1	$\mathrm{O}(n) \times \mathrm{O}(n)/\mathrm{O}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII	-1	0	0	$\mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n)$	0	$\mathbb{Z}_2$
CII	-1	-1	1	$\mathrm{U}(n)/\mathrm{O}(n)$	$\mathbb{Z}$	0
C	0	-1	0	$\mathrm{Sp}(2n)/\mathrm{U}(n)$	0	$\mathbb{Z}$
CI	1	-1	1	$\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)/\mathrm{Sp}(2n)$	0	0

- A, C, D: arbitrary number of protected (chiral) channels ($\mathbb{Z}$)
- AII, DIII: 0 or 1 protected (helical) channels ($\mathbb{Z}_2$).
- Quantum Hall edge, Weyl semimetal: class A

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The tenfold way

Symmetry class Cartan label	Symmetry $\mathcal{T}$	$\mathcal{C}$	$\mathcal{S}$	NLSM compact sector	Spatial dimension 1	2
A	0	0	0	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathrm{U}(n)\times\mathrm{U}(n)/\mathrm{U}(n)$	$\mathbb{Z}$	0
AI	1	0	1	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$	0	0
BDI	1	1	1	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$	$\mathbb{Z}$	0
D	0	1	0	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	-1	1	1	$\mathrm{O}(n)\times\mathrm{O}(n)/\mathrm{O}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII	-1	0	0	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$	0	$\mathbb{Z}_2$
CII	-1	-1	1	$\mathrm{U}(n)/\mathrm{O}(n)$	$\mathbb{Z}$	0
C	0	-1	0	$\mathrm{Sp}(2n)/\mathrm{U}(n)$	0	$\mathbb{Z}$
CI	1	-1	1	$\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)/\mathrm{Sp}(2n)$	0	0

- A, C, D: arbitrary number of protected (chiral) channels ($\mathbb{Z}$)
- AII, DIII: 0 or 1 protected (helical) channels ($\mathbb{Z}_2$).
- Quantum Hall edge, Weyl semimetal: class A
- Quantum spin Hall edge: class AII

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The tenfold way

Symmetry class Cartan label	Symmetry $\mathcal{T}$	 $\mathcal{C}$	 $\mathcal{S}$	NLSM compact sector	Spatial dimension 1	 2
A	0	0	0	$\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathrm{U}(n) \times \mathrm{U}(n)/\mathrm{U}(n)$	$\mathbb{Z}$	0
AI	1	0	1	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$	0	0
BDI	1	1	1	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$	$\mathbb{Z}$	0
D	0	1	0	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	-1	1	1	$\mathrm{O}(n) \times \mathrm{O}(n)/\mathrm{O}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII	-1	0	0	$\mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n)$	0	$\mathbb{Z}_2$
CII	-1	-1	1	$\mathrm{U}(n)/\mathrm{O}(n)$	$\mathbb{Z}$	0
C	0	-1	0	$\mathrm{Sp}(2n)/\mathrm{U}(n)$	0	$\mathbb{Z}$
CI	1	-1	1	$\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)/\mathrm{Sp}(2n)$	0	0

- A, C, D: arbitrary number of protected (chiral) channels ($\mathbb{Z}$)
- AII, DIII: 0 or 1 protected (helical) channels ($\mathbb{Z}_2$).
- Quantum Hall edge, Weyl semimetal: class A
- Quantum spin Hall edge: class AII

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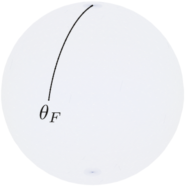
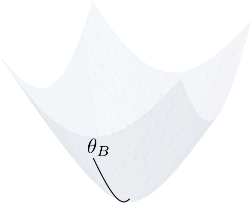
Unitary class

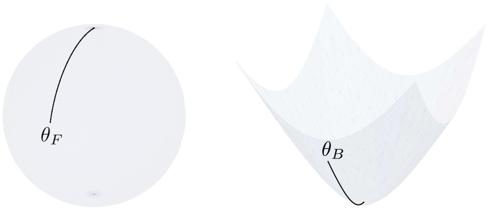
- Sigma model manifold: compact sector (Sphere) + non-compact sector (Hyperboloid)





Unitary class

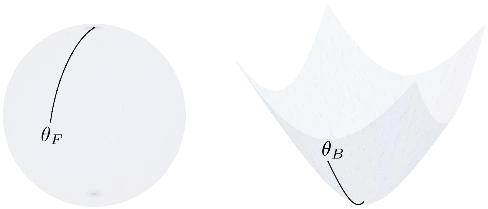
- Sigma model manifold: compact sector (Sphere) + non-compact sector (Hyperboloid)



- Boundary conditions at $x = 0$: $\theta_F = \theta_B = 0$

Unitary class

- Sigma model manifold: compact sector (Sphere) + non-compact sector (Hyperboloid)



- Boundary conditions at $x = 0$: $\theta_F = \theta_B = 0$
- Transmission distribution functions

$$\rho\left(\lambda\right) = \frac{2}{\pi} \operatorname{Re} \frac{\partial Z[\theta_F, \theta_B]}{\partial \theta_F}\Big|_{\theta_F = i\theta_B = \pi - \epsilon - 2i\lambda}$$

Unitary class

- Sigma model manifold: compact sector (Sphere) + non-compact sector (Hyperboloid)

- Boundary conditions at $x = 0$: $\theta_F = \theta_B = 0$
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$$\rho\left(\lambda\right) = \frac{2}{\pi} \operatorname{Re} \frac{\partial Z[\theta_F, \theta_B]}{\partial \theta_F}\Big|_{\theta_F = i\theta_B = \pi - \epsilon - 2i\lambda}$$

- Topological term: field of a magnetic monopole

$$S_{\text{top}} = \frac{im}{2} \int dx (1 - \cos\theta)\dot{\phi}, \qquad m = n_L - n_R$$

Saddle-point approximation ($L \ll \xi$)

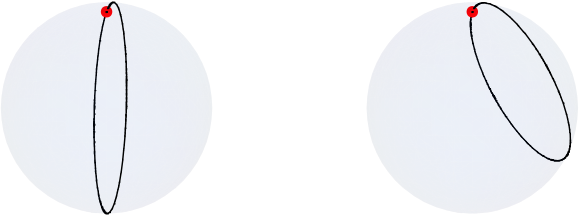
Action in the compact sector

$$S_F = \int_0^L dx \left[\frac{\xi}{4}(\dot\theta^2 + \sin^2\theta\dot\phi^2) + i\frac{m}{2}(1-\cos\theta)\dot\phi \right], \quad \xi = (n_R+n_L)l = 2\pi\nu D$$

Saddle-point approximation ($L \ll \xi$)

Action in the compact sector

$$S_F = \int_0^L dx \left[\frac{\xi}{4}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + i\frac{m}{2}(1-\cos\theta)\dot{\phi} \right], \quad \xi = (n_R + n_L)l = 2\pi\nu D$$



$m = 0$ $\qquad$ $m \neq 0$

$\rightarrow \theta_F$ close to π is classically unreachable

Saddle-point approximation: distribution function

- Distribution function

$$\rho\,(\lambda) = \frac{2}{\pi}\,\mathrm{Re}\,\frac{\partial Z[\theta_F,\theta_B]}{\partial \theta_F}\Big|_{\theta_F = i\theta_B = \pi - \epsilon - 2i\lambda} = m\delta(\lambda) + \rho_{\rm ns}(\lambda),$$

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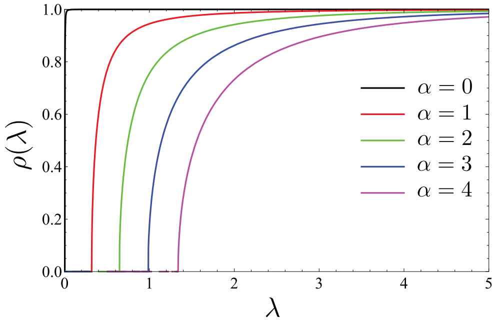
- Distribution function

$$\rho(\lambda) = \frac{2}{\pi} \mathrm{Re} \frac{\partial Z[\theta_F, \theta_B]}{\partial \theta_F}\Big|_{\theta_F = i\theta_B = \pi - \epsilon - 2i\lambda} = m\delta(\lambda) + \rho_{\rm ns}(\lambda),$$

$$\alpha = \frac{mL}{\xi}$$



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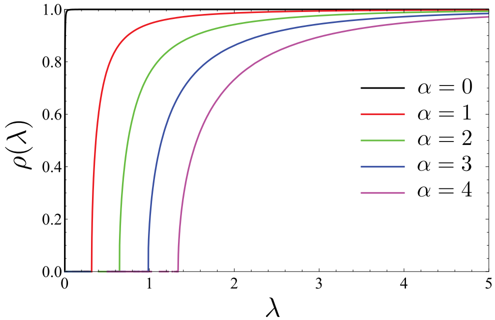
Saddle-point approximation: distribution function

- Distribution function

$$\rho\left(\lambda\right) = \frac{2}{\pi}\,\mathrm{Re}\,\frac{\partial Z[\theta_F,\theta_B]}{\partial\theta_F}\bigg|_{\theta_F=i\theta_B=\pi-\epsilon-2i\lambda} = m\delta(\lambda)+\rho_{\mathrm{ns}}(\lambda),$$

$$\alpha = \frac{mL}{\xi}$$



- Gap close to unit transmission $\implies$ transport from the unprotected channels gets *suppressed*

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Soft modes

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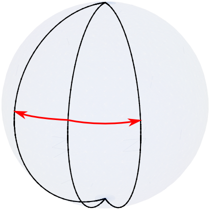
Soft modes



- $\theta_F \approx \pi \rightarrow$ many trajectories joining north pole to south pole $\rightarrow$ ground state degeneracy $\rightarrow$ some modes become very soft for $\pi - \theta \ll \sqrt{\frac{L}{\xi}}$

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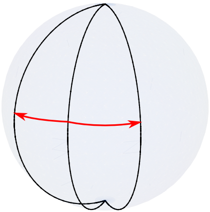
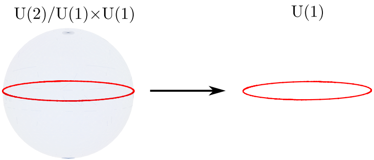
Soft modes



- $\theta_F \approx \pi \rightarrow$ many trajectories joining north pole to south pole $\rightarrow$ ground state degeneracy $\rightarrow$ some modes become very soft for $\pi - \theta \ll \sqrt{\frac{L}{\xi}}$
- Projection onto the manifold of soft modes $\rightarrow$ trajectories labelled by polar angle ϕ "equator".



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Mapping to 0D

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Mapping to 0D

H	"E"	Q_{FF}	d=1	d=2
A	AIII	$\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$	0	$\mathbb{Z}$
AIII	A	$\mathrm{U}(n) \times \mathrm{U}(n)/\mathrm{U}(n)$	$\mathbb{Z}$	0
AI	CI	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$	0	0
BDI	AI	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$	$\mathbb{Z}$	0
D	BDI	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	D	$\mathrm{O}(n) \times \mathrm{O}(n)/\mathrm{O}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII	DIII	$\mathrm{O}(2n)/\mathrm{O}(n) \times \mathrm{O}(n)$	0	$\mathbb{Z}_2$
CII	AII	$\mathrm{U}(n)/\mathrm{O}(n)$	$\mathbb{Z}$	0
C	CII	$\mathrm{Sp}(2n)/\mathrm{U}(n)$	0	$\mathbb{Z}$
CI	C	$\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)/\mathrm{Sp}(2n)$	0	0

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Mapping to 0D

H	"E"	Q_{FF}	d=1	d=2
A	AIII	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$	0	$\mathbb{Z}$
AIII	A	$\mathrm{U}(n)\times\mathrm{U}(n)/\mathrm{U}(n)$	$\mathbb{Z}$	0
AI	CI	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$	0	0
BDI	AI	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$	$\mathbb{Z}$	0
D	BDI	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	D	$\mathrm{O}(n)\times\mathrm{O}(n)/\mathrm{O}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII	DIII	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$	0	$\mathbb{Z}_2$
CII	AII	$\mathrm{U}(n)/\mathrm{O}(n)$	$\mathbb{Z}$	0
C	CII	$\mathrm{Sp}(2n)/\mathrm{U}(n)$	0	$\mathbb{Z}$
CI	C	$\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)/\mathrm{Sp}(2n)$	0	0

1D edge with protected channels $\leftrightarrow$ random matrix with zero eigenvalues

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Mapping to 0D

H	"E"	Q_{FF}	d=1	d=2
A	AIII	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$	0	$\mathbb{Z}$
AIII	A	$\mathrm{U}(n)\times\mathrm{U}(n)/\mathrm{U}(n)$	$\mathbb{Z}$	0
AI	CI	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$	0	0
BDI	AI	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$	$\mathbb{Z}$	0
D	BDI	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	D	$\mathrm{O}(n)\times\mathrm{O}(n)/\mathrm{O}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII	DIII	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$	0	$\mathbb{Z}_2$
CII	AII	$\mathrm{U}(n)/\mathrm{O}(n)$	$\mathbb{Z}$	0
C	CII	$\mathrm{Sp}(2n)/\mathrm{U}(n)$	0	$\mathbb{Z}$
CI	C	$\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)/\mathrm{Sp}(2n)$	0	0

1D edge with protected channels $\leftrightarrow$ random matrix with zero eigenvalues

Quantum Hall classes A, C and D $\leftrightarrow$ chiral random matices AIII, CII, BDI

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Mapping to 0D

H	"E"	Q_{FF}	d=1	d=2
A	AIII	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$	0	$\mathbb{Z}$
AIII	A	$\mathrm{U}(n)\times\mathrm{U}(n)/\mathrm{U}(n)$	$\mathbb{Z}$	0
AI	CI	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$	0	0
BDI	AI	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$	$\mathbb{Z}$	0
D	BDI	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	D	$\mathrm{O}(n)\times\mathrm{O}(n)/\mathrm{O}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII	DIII	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$	0	$\mathbb{Z}_2$
CII	AII	$\mathrm{U}(n)/\mathrm{O}(n)$	$\mathbb{Z}$	0
C	CII	$\mathrm{Sp}(2n)/\mathrm{U}(n)$	0	$\mathbb{Z}$
CI	C	$\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)/\mathrm{Sp}(2n)$	0	0

1D edge with protected channels	$\leftrightarrow$	random matrix with zero eigenvalues
Quantum Hall classes A, C and D	$\leftrightarrow$	chiral random matices AIII, CII, BDI
$\mathbb{Z}_2$ classes AII and DIII	$\leftrightarrow$	Random matrix classes DIII and D

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Mapping to 0D

H	"E"	Q_{FF}	d=1	d=2
A	AIII	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$	0	$\mathbb{Z}$
AIII	A	$\mathrm{U}(n)\times\mathrm{U}(n)/\mathrm{U}(n)$	$\mathbb{Z}$	0
AI	CI	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$	0	0
BDI	AI	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$	$\mathbb{Z}$	0
D	BDI	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	D	$\mathrm{O}(n)\times\mathrm{O}(n)/\mathrm{O}(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AII	DIII	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$	0	$\mathbb{Z}_2$
CII	AII	$\mathrm{U}(n)/\mathrm{O}(n)$	$\mathbb{Z}$	0
C	CII	$\mathrm{Sp}(2n)/\mathrm{U}(n)$	0	$\mathbb{Z}$
CI	C	$\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)/\mathrm{Sp}(2n)$	0	0

1D edge with protected channels	$\leftrightarrow$	random matrix with zero eigenvalues
Quantum Hall classes A, C and D	$\leftrightarrow$	chiral random matices AIII, CII, BDI
$\mathbb{Z}_2$ classes AII and DIII	$\leftrightarrow$	Random matrix classes DIII and D
Transmission distribution function	$\leftrightarrow$	Spectral density

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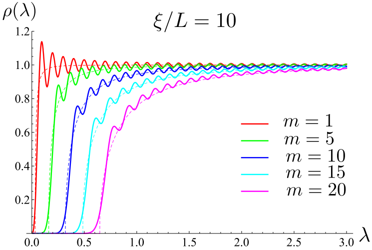
Results

$$\rho_A(\lambda, m) = \frac{\pi\xi u}{2L}\left[J_m^2(u) - J_{m+1}(u)J_{m-1}(u)\right] + m\delta(\lambda), \qquad u = \frac{\pi\xi\lambda}{L}$$



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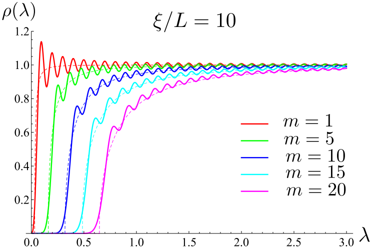
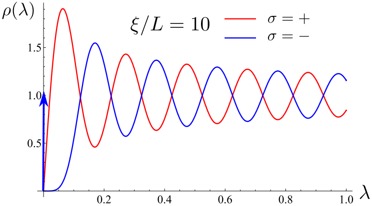
Results

$$\rho_{\mathsf{A}}(\lambda, m) = \frac{\pi\xi u}{2L}\left[J_m^2(u) - J_{m+1}(u)J_{m-1}(u)\right] + m\delta(\lambda), \qquad u = \frac{\pi\xi\lambda}{L}$$



$$\rho_{\mathsf{AII}}(\lambda, \sigma = \pm) = \frac{\pi\xi u}{2L}[J_1^2(u) + J_0(u)J_1'(u)] \pm \frac{\pi\xi}{2L}J_1(u) + 2\delta_{\sigma,-}\delta(\lambda)$$



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Exact solution: transfer matrix method

- Transfer matrix method: 1D path integral → Schrödinger equation

$$\partial_t \psi(Q,t) = -\mathcal{H}\psi(Q,t), \qquad t = x/\xi, \qquad \psi(Q,0) = \delta(Q,\Lambda)$$

Rejaei 96, EK, Ostrovsky PRL 17

Exact solution: transfer matrix method

- Transfer matrix method: 1D path integral → Schrödinger equation

$$\partial_t \psi(Q,t) = -\mathcal{H}\psi(Q,t), \qquad t = x/\xi, \qquad \psi(Q,0) = \delta(Q,\Lambda)$$

- NLSM action: particle motion on a curved (super)manifold with a vector potential

$$S = \int dt \left[\frac{1}{4}\dot{y}^\alpha \dot{y}^\beta g_{\beta\alpha} + \dot{y}^\alpha A_\alpha\right]$$

$$dy^\alpha dy^\beta g_{\beta\alpha} = -\frac{1}{2}\,\mathrm{str}(dQ)^2, \quad dy^\alpha A_\alpha = -\frac{m}{2}\,\mathrm{str}\, T^{-1}\Lambda dT$$

Rejaei 96, EK, Ostrovsky PRL 17

Exact solution: transfer matrix method

- Transfer matrix method: 1D path integral → Schrödinger equation

$$\partial_t \psi(Q,t) = -\mathcal{H}\psi(Q,t), \qquad t = x/\xi, \qquad \psi(Q,0) = \delta(Q,\Lambda)$$

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Rejaei 96, EK, Ostrovsky PRL 17

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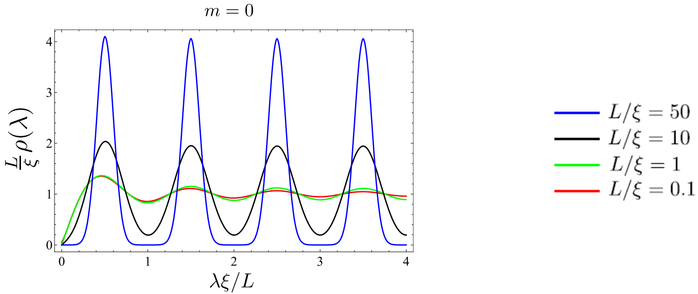
- Vector potential $\to$ Hamiltonian only simplifies in a special gauge
- Spherical eigenfunctions: gauge transformation + integration (EK, Skvortsov and Ostrovsky (to appear), EK (PhD thesis))

$$\phi_\nu(\mathbf{h}) = \int_{\mathcal{K}} dK \, e^{i\mathbf{p}_\nu \cdot \mathbf{a}(\mathbf{h}, K)} [\text{sdet} \, K_I(\mathbf{h}, K)]^m$$

valid also without supersymmetry (compact or non-compact nlsm)

Effect of topology: Class A (unitary)



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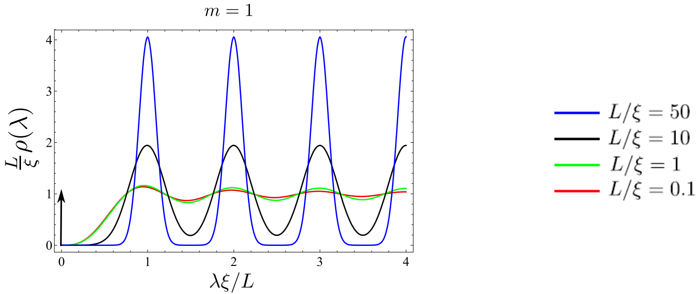

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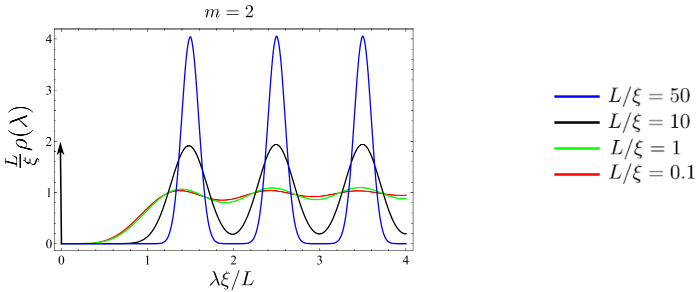

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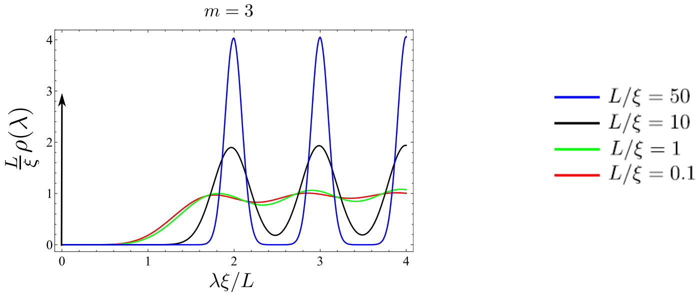

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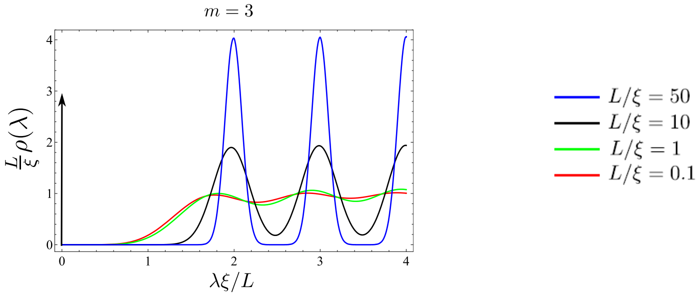

$$G = \int_0^\infty d\lambda \frac{\rho(\lambda)}{\cosh^2 \lambda} \sim \int_0^\infty d\lambda e^{-2\lambda} \rho(\lambda)$$

- Localization length for the unprotected channels $\xi_m = \xi/(m+1)$

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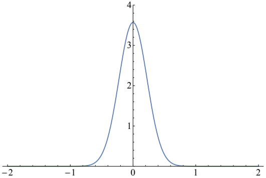
Diffusion + Drift

Classical Diffusion

$$W(x,t) = \frac{e^{-\frac{x^2}{2Dt}}}{2\sqrt{\pi D t}}$$



Chiral channels →

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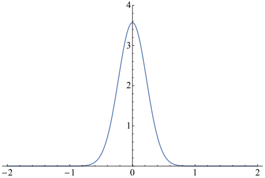

Quantum interference ↓

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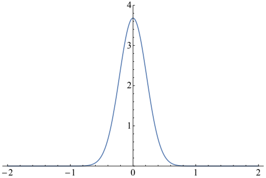
Anderson localization

$$W(x,t) = ?$$



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Dynamical correlations: Theory

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- Sigma model at finite frequency $\omega + i0 \mapsto i\Omega$

$$S[Q] = -\int dx \; \mathrm{str}\left[\frac{\xi}{8}(\partial_x Q)^2 - \frac{\kappa^2}{16\xi}\Lambda Q + S_{\mathrm{top}}\right], \quad \kappa = 2N\sqrt{\Omega\tau_e}$$

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  - Eigenfunction: Coulomb spherical functions (no integral representation, no generalization to more variables)

Zero mode and local dynamical correlations

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- Zero mode($Q = T^{-1} \Lambda T$)

$$|\Psi_0\rangle = \int_{\mathcal{K}} dK \, (\mathrm{sdet}\, K_{R,A})^{\pm m} \exp\left[-\frac{\kappa}{2} \,\mathrm{str}\, P_{\pm} \left(KT + T^{-1} K^{-1} \right) \right]$$

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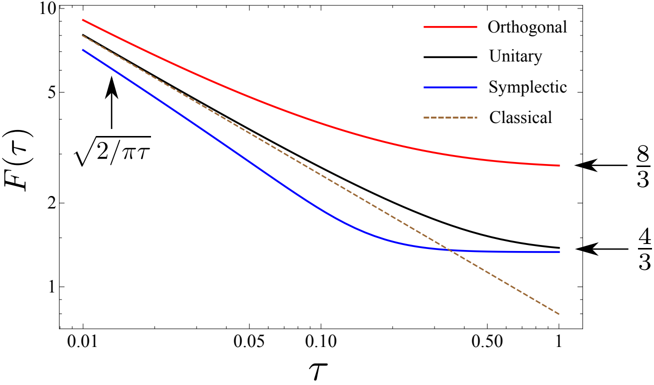

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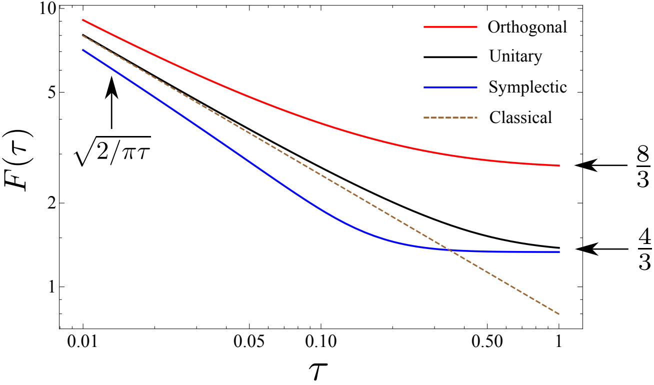

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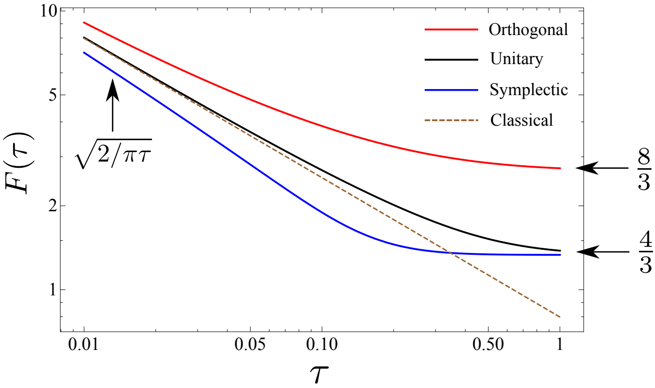

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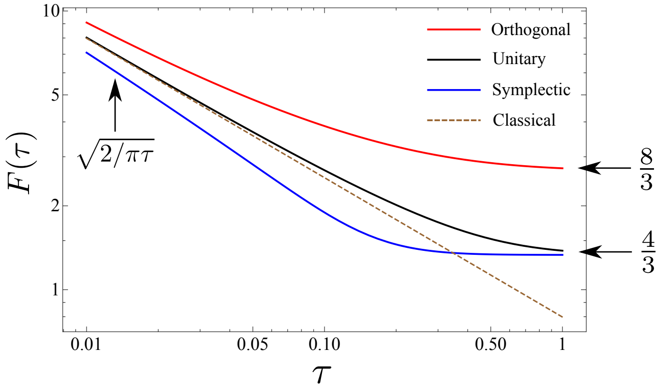

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- The saturation value is approached as a power-law: unitary $\sim 1/\tau^3$, orthogonal $\sim 1/\tau^2$ and symplectic $\sim 1/\tau^5$.

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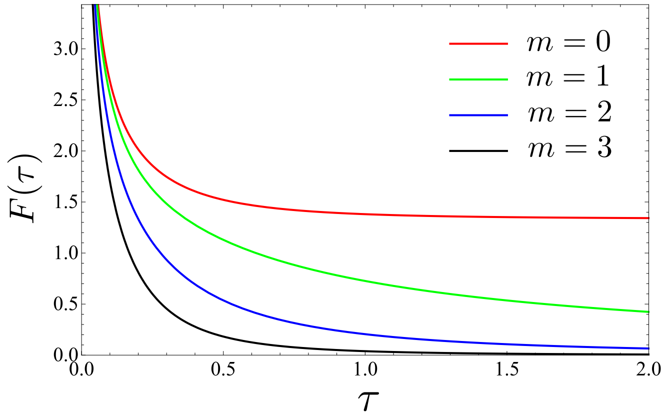

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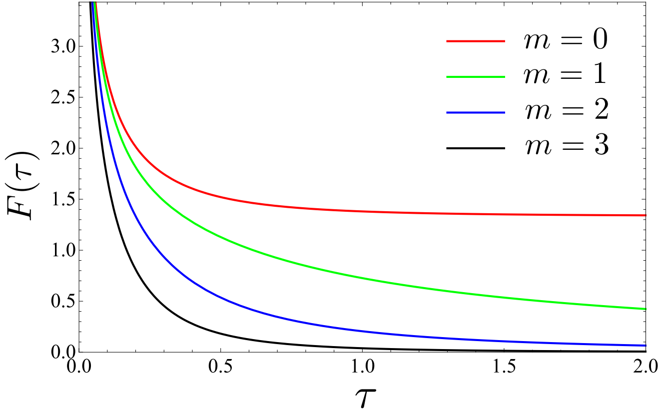

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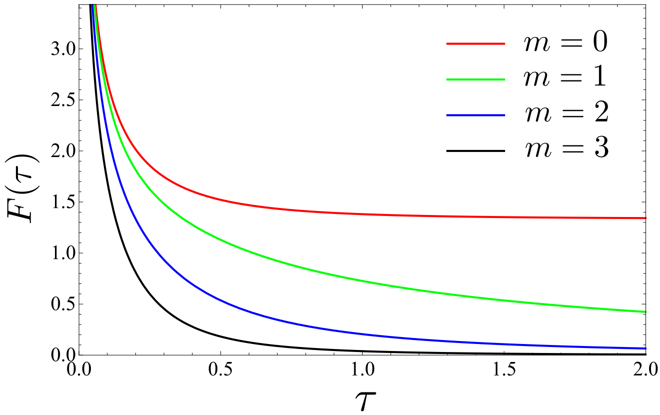

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- Level statistics: connection to non-Hermitian systems? (Lee *et al.* PRL 2020)

Open questions

- Mathematical reason for the integral representation of the zero mode?
- How to understand the long time asymptotics and wave-packet dynamics in the presence of topological channels?
- Level statistics: connection to non-Hermitian systems? (Lee *et al.* PRL 2020)
- Wavefunction statistics: how does the wavefunctions look like in this system?