The infinite unitriangular group

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The talk

1. Asymptotic Representation theory
   - Locally finite groups
   - Characters
   - Bratelli diagram, harmonic functions and measures
   - Asymptotic characters
2. Supercharacters for finite groups
   - Brief description of a supercharacter theory for algebra groups
3. Supercharacters for the Infinite unitriangular group
   - Supercharacters
   - Super-Bratelli diagram, super-harmonic functions and measures
   - Asymptotic supercharacters

Motivation

Motivation

- Locally finite groups are discrete infinite groups. Most of such groups are wild, in the sense that the study of it irreducible representations is not a reasonable problem;
- The main object of study of such groups are characters; they are the traces of representations of finite Von Neumann type;

Motivation

- In the 80's, S. Kerov and A. Vershick, developed a method of constructing extreme characters by considering weak limits of irreducible characters of finite subgroups, the so called *ergodic method*;
- such approach implies the knowledge of the Bratelli diagram, hence all irreducible representations of the finite subgroups; This may be "impossible" in some cases;

Motivation

- For finite groups Supercharacter Theory have been shown an alternative approach to approximate irreducible characters;
- We intend to develop an ergodic method for supercharacters in order to obtain the analogue of supercharacters for locally finite groups;

Locally finite groups

Locally finite groups

- A locally finite group is defined as a direct limit of directed system of finite groups.
- Consider a family of finite groups, sharing the same identity, such tat $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n \subseteq \ldots$ and consider the group $G_\infty = \bigcup_{n \in \mathbb{N}} G_n$.
- G_∞ is in fact a locally finite group, and for simplicity, all groups will be considered of this form.

Two important examples:

- Let S_n denote the symmetric group of order n, one can consider the Locally finite group $S_\infty = \bigcup_{n \in \mathbb{N}} S_n$.
- Fix q a power of a prime, let $U_n = U_n(q)$ the group of $n \times n$ upper triangular matrices with entries in the finite field $\mathbb{F}_q$ and 1's on the diagonal. Such group is called the $n \times n$ unitriangular group (over q).
- There is a natural inclusion of U_n in U_{n+1}, hence one can consider $U_\infty = \bigcup_{n \in \mathbb{N}} U_n$, the infinite unitriangular group (over q).

In what follows we will fix a group $G = \bigcup_{n \in \mathbb{N}} G_n$. $\chi : G \to \mathbb{C}$ is a character if:

- positive definite : $\sum_{i,j=1}^{n} \chi(g_i g_j^{-1}) z_i \overline{z_j} \geq 0$
- central: $\chi(g^{-1} h g) = \chi(h)$
- normalized: $\chi(1) = 1$

- Denote $char(G)$ the set of all characters of G;
- With the pointwise topology $char(G)$ is a convex and compact set; let $\mathcal{E}$ denote the set of extreme characters;
- By Choquet's theorem, for every character $\chi \in char(G)$ there is a measure μ supported on $\mathcal{E}$ such that:

$$\chi(g) = \int_{\mathcal{E}} \chi_\delta(g) d\mu(\delta)$$

Bratelli diagram

- Let $Irr(G_n) = \{\chi_\lambda \;:\; \lambda \in \Gamma_n\}$;
- For $\Lambda \in \Gamma_{n+1}$ and $\lambda \in \Gamma_n$ denote $m(\lambda, \Lambda)$ the multiplicity of χ_λ in the restriction of χ_Λ to G_n;
- The Bratelli diagram of G is a graded graph, where its vertices are $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ and there is a edge $\lambda \nearrow \Lambda$ iff $m(\lambda, \Lambda) \neq 0$;
- The edges are labelled precisely by $m(\lambda, \Lambda)$;

Bratelli diagram

- We note that $\forall \Lambda \in \Gamma_{n+1}$ there is at least one $\lambda \in \Gamma_n$ such that $\lambda \nearrow \Lambda$;
- Vertices of Γ are connected only by edges to the adjacent floors;
- Each floor is a finite set;

Bratelli diagram

- A graded graph with the previous properties is called a branching graph;
- For each branching graph there is only one (up to isomorphism) C^*-algebra such that the branching graph coincides with its Bratelli diagram;
- the corresponding algebra of a Bratelli diagram is $C^*(G)$;

- In a branching graph we consider T to be set of paths;
- We consider the topology in T which has the basis

$$F(u) = \{t \in T : P_n(t) = u\} \quad u \in T_n$$

Harmonic functions

- Consider $\chi \in char(G)$, denote χ_n its restriction to G_n;
- $\chi_n = \sum_{\lambda \in \Gamma_n} \phi(\lambda)\chi_\lambda$;
- $\phi(\lambda) = \sum_{\Lambda \in \Gamma_{n+1}} m(\lambda, \Lambda)\phi(\Lambda)$
- a function on a branching graph with such properties is called *harmonic*;

- A non-negative harmonic function ϕ defines a measure M on T:

$$M(F(u)) = \phi(\lambda)d(\lambda) \ \ u \in T_n, \ u_n = \lambda$$

  where $d(\lambda) = m(\emptyset, \lambda)$;

- A measure M on T is called central (or ergodic) if $M(F(u)) = M(F(v))$, for $u, v \in T_n, u_n = v_n$;

- A central measure defines a non-negative harmonic function in the obvious way;

Theorem

Given a non negative harmonic function ϕ there exists a unique character ψ such that $\forall g \in G_n$

$$\psi(g) = \sum_{\lambda \in \Gamma_n} \phi(\lambda)\chi_\lambda(g)$$

- Having The two bijections Characters $\leftrightarrow$ non-negative Harmonic functions $\leftrightarrow$ central measures, one can prove:

Theorem

A character is extreme iff the corresponding measure is extreme.

- By describing extreme central measures, one describes the extreme characters; Let M be a central measure and ϕ its corresponding harmonic function;
- Adapting Birkhoff's ergodic theorem one can prove that

$$\phi(\lambda) = \lim_{n\to\infty} \frac{m(\lambda, \mu_n)}{d(\mu_n)}$$

for M-almost $\mu = (\mu_0, \mu_1, ...) \in T$

Theorem

Given a character ψ of G, for each G_n there is a χ_{λ_n} such that

$$\psi(g) = \lim_{n\to\infty} \frac{\chi_{\lambda_n}(g)}{\chi_{\lambda_n}(1)}$$

Some remarks:

- The description of limits can be a very difficult task;
- The set $char(G)$ may not be closed, that is, the limit may exist but it may not be a character;
- In some cases one can take limits of reducible characters and obtain an extreme character;
- Nevertheless the knowledge of extreme characters heavily depends on the knowledge of the Bratelli diagram;

Supercharacters

- Let G be a fixed finite group;
- Consider $\mathcal{K}$ to be a partition of G and $\mathcal{X}$ a partition of $Irr(G)$;
- $\forall X \in \mathcal{X}$ define $\xi_X = \sum_{\chi \in X} \chi(1)\chi$;
- We say that $\mathcal{K}$ and $\mathcal{X}$ defines a supercharacter theory if
  1. $\{1\} \in \mathcal{K}$;
  2. $|\mathcal{K}| = |\mathcal{X}|$;
  3. ξ_X is constant in the elements of $\mathcal{K}$, for all $X \in \mathcal{X}$;

Supercharacters

- Elements of $\mathcal{K}$ are called superclasses;
- The characters ξ_X are called Supecharacters;

- Supercharacters replace the irreducible characters; Superclasses replace conjugacy classes;
- The irreducible characters form a supercharacter theory, such theory is called the trivial one;

Supercharacters

The main properties of supercharacters can be summarized:

- Supercharacters form an orthogonal basis of superclass functions;
- Their sum is the regular character;
- For each irreducible character χ of G there exists one and only one supercharacter ξ such that χ occurs in ξ;

Supercharacters

- A group G is said to be a group algebra if $G = 1 + A$, where A is a nilpotent algebra; U_n is a group algebra;
- For groups algebra there is a canonical way to construct a supercharacter theory; In particular superclasses are considered to have the form $1 + GaG$;
- Considering the canonical SCT of $1 + B \leq 1 + A$, the restriction of supercharacters is a non-negative integer linear combination of supercharacters;

- To determine $Irr(U_n)$ is considered to be a "wild" problem;
- Supercharacters have been proved to be a useful alternative to irreducible characters in order to study the representation theory of U_n;

Supercharacters of U_n

- Let $\chi_\lambda \in Irr(G)$, with $\lambda \in \Gamma_n$;
- We shall denote by $\xi_{[\lambda]}$ the only supercharacter in which χ_λ occurs;
- Supercharacters of U_n have a nice combinatorial description in terms of basic pairs;
- Supercharacters of U_n can be decomposed in a product of elementary supercharacters;

Supercharacters of U_∞

Supercharacters of U_∞

- A supercharacter of U_∞ is a character constant on the superclasses $1 + U_\infty a U_\infty$;
- Denote by $Schar(U_\infty)$ the set of supercharacters of U_∞;
- Such supercharacters are intended to approximate characters;

- Let $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ be the Bratelli diagram of U_∞;
- Denote $\tilde{\Gamma}_n := \{[\lambda] : \lambda \in \Gamma_n\}$;
- Set $\tilde{m}([\lambda], [\Lambda])$ to be the multiplicity of $\xi_{[\lambda]}$ in the restriction of $\xi_{[\Lambda]}$;
- We define the super-bratelli diagram as $\tilde{\Gamma} = \bigcup_{n\in\mathbb{N}} \tilde{\Gamma}_n$ with $\tilde{m}$ as labelling function;

Super-Bratelli diagram

- As a branching graph, the super-Bratelli diagram gives rise to a C^*-algebra $\mathcal{A}$;
- Such branching graph determines the set $char(\mathcal{A})$;
- The sets $Schar(U_\infty)$ and $Char(\mathcal{A})$ are homeomorfic;
- $Schar(U_\infty)$ is a compact convex set; Denote by $\mathcal{E}Sch(U_\infty)$ the extreme points of $Schar(U_\infty)$;

Super-Bratelli diagram

In complete analogy:

- Non-negative super-harmonic functions $\leftrightarrow$ $Schar(U_\infty)$;
- Non-negative super-harmonic functions $\leftrightarrow$ super-central measures

Proposition

Given $\psi \in \mathcal{E}Sch(U_\infty)$, for each U_n, there is $\xi_{[\lambda_n]}$ such that

$$\psi(g) = \lim_{n \to \infty} \frac{\xi_{[\lambda_n]}(g)}{\xi_{[\lambda_n]}(1)}$$

Some remarks:

- In the same way that Supercharactrs of U_n admits a characterization via basic pairs, it is expected that the same happens in U_∞;
- There are several combinatoric tools that can be used, namely: set partitions; Hopf Algebras;

The Agenda

The agenda

- Full description of supercharacters;
- Construct explicit superrepresentations;
- Factorization properties;
- Irreducible Supercharacters;

Bibliography

- A. M. Vershik, S. V. Kerov, *Asymptotic theory of characters of the symmetric group*, Funct. Anal. and its Appl., v.15 (1981) 246-255.
- A. M. Vershik, S. V. Kerov, *Locally semisimple algebras. Combinatorial theory and the K_0-functor*, Ser. Sovrem. Probl. Mat. Nov. Dostizh., 26, VINITI, Moscow, 1985, 3-56
- P. Diaconis, I.M Issacs, *Superchedgeters and Superclasses for Algebra groups*, Trans. Amer. Soc. 360 (5) (2008) 2359-2392.
- C, André, *Basic characters of the unitriangular group (for arbitrary primes)*, Proceedings of AMS 130(7): 1943-1954 - Jan- 2002