Absence of positive eigenvalues of magnetic Schrödinger operators

Hynek Kovařík (Brescia)

joint work with

Silvana Avramska-Lukarska and Dirk Hundertmark (Karlsruhe)

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Basic set up

The main object of our interest: magnetic Schrödinger operators of the form

$$H_{A,V} = (P - A)^2 + V, \qquad P = -i\nabla, \qquad \text{in} \ \ L^2(\mathbb{R}^n),$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ generates a magnetic field B via the identity **rot** $A = B$.
$V : \mathbb{R}^n \to \mathbb{R}$.

Throughout we assume that $B(x) \to 0$ and $V(x) \to 0$ as $|x| \to \infty$. Then

$$\sigma_{\rm es}(H_{A,V}) = [0, \infty).$$

The question:

Under which conditions on B and V can we exclude a presence of **positive eigenvalues of** $H_{A,V}$?

Or, more in general, above certain energy level $\mathbf{\Lambda > 0}$?

History

-**[Kato 1959]**: $H_{0,V} = -\Delta + V$ in $L^2(\mathbb{R}^n)$ has no positive eigenvalues if V is continuous and if

$$V(x) = o(|x|^{-1}) \qquad |x| \to \infty.$$

-**[Simon 1967]** if $V = V_1 + V_2$ outside a compact set and if $V_1 = o(|x|^{-1}),\ V_2(x) = o(1)$, then $H_{0,V}$ in $L^2(\mathbb{R}^3)$ has **no eigenvalues above**

$$\omega_0 = \limsup_{|x|\to\infty} x \cdot \nabla V_2(x)$$

-**[Agmon 1970]** $H_{0,V}$ in $L^2(\mathbb{R}^n)$ has **no eigenvalues above** $\omega_0/2$.

- By **[Wigner-von Neumann 1929]** the operator $H_{0,V}$ in $L^2(\mathbb{R}^3)$ with

$$V(r) = -\frac{32 \sin r \Big(g(r)^3 \cos r - 3g^2(r) \sin^3 r + g(r) \cos r + \sin^3(r)\Big)}{(1+g(r)^2)^2},$$

$g(r) = 2r - \sin(2r)$, has eigenvalue 1. Here $V(r) = \mathcal{O}(r^{-1})$, and $\omega_0 = 16$.

History

-**[Froese, Herbst, Hoffmann-Ostenhof,Hoffmann-Ostenhof 1982]**: absence of **all positive eigenvalues** of $H_{0,V}$ in $L^2(\mathbb{R}^n)$ under much weaker regularity assumptions on V. Extension to many-body operators.

Further generalisatios, by relaxing regularity conditions on V, by **[Jerison, Kenig, 1985]**, **[Ionescu, Jerison, 2003]** and **[Koch, Tataru, 2006]**.

Less is known about the case $B \neq 0$, in particular for $n = 2$.

-**[Fanelli, Krejčiřík, Vega, 2018]** :

sufficient conditions on B, V for the absence of **all eigenvalues** of $H_{A,V}$ in $L^2(\mathbb{R}^2)$.

Assumptions: pointwise version

We identify a magnetic field B with an **antisymmetric matrix–valued** function

$$B_{j,k}(x), \quad j,k \in \{1,\dots,n\}, \quad x \in \mathbb{R}^n.$$

Let $\widetilde{B} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field with components

$$\widetilde{B}_j(x) = \sum_{k=1}^{n} B_{j,k}(x)\, x_k\,, \quad j = 1,\dots,n\,.$$

- If $n = 3$, then $B = (B_{3,2},\, B_{1,3},\, B_{2,1})$ is a vector field, and $\widetilde{B} = B \wedge x$.

Assumptions:

1. $|\widetilde{B}| \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ with $p > n$, and $|\widetilde{B}(x)| = \mathcal{O}(1)$ as $|x| \to \infty$.
2. $V \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ with $q > n$, and $V = V_1 + V_2$ on $\mathbb{R}^n \setminus K$ (K compact)

$$V_1 \in C^0(\mathbb{R}^n \setminus K), \;\; V_2 \in C^1(\mathbb{R}^n \setminus K).$$

3. $V_j(x) = o(1)$ as $|x| \to \infty$.
4. $xV_1(x) \in L^\infty(\mathbb{R}^n \setminus K)$, and $x \cdot \nabla V_2(x) \in L^\infty(\mathbb{R}^n \setminus K)$.

The gauge

We fix the vector potential A in the **Poincaré gauge** :

$$A(x) = \int_0^1 \widetilde{B}(tx)\, dt$$

Then under the above assumptions $A \in L^2_{\rm loc}(\mathbb{R}^n)$. This allows to define $H_{A,V}$ via the closed quadratic form

$$q_{A,V}(\varphi,\varphi) = \|(P-A)\varphi\|_2^2 + \big\langle \varphi, V\varphi \big\rangle_{L^2(\mathbb{R}^n)}$$

with the form domain

$$D(q_{A,V}) = D(P-A) := \Big\{ u \in L^2(\mathbb{R}^n) \ : \ (P-A)\, u \in L^2(\mathbb{R}^n) \Big\}$$

Note also that

$$x \cdot A(x) = 0 \qquad \forall\, x \in \mathbb{R}^n$$

Main result

Theorem 1: Under the stated assumptions $H_{A,V}$ has **no eigenvalues** above

$$\Lambda(B,V) = \Lambda := \frac{1}{4}\Big(\beta + \omega_1 + \sqrt{(\beta+\omega_1)^2 + 2\omega_0}\Big)^2,$$

where

$$\beta = \limsup_{|x|\to\infty} |\widetilde{B}(x)|, \quad \omega_1 = \limsup_{|x|\to\infty} |x|\,|V_1(x)|, \quad \omega_0 = \limsup_{|x|\to\infty} x\cdot\nabla V_2(x)$$

- Λ depends only on the behavior of B and V at infinity!

- If $B(x) = o(|x|^{-1})$ as $|x|\to\infty$, then $\beta = 0$. The splitting $V_1 = V$ and $V_2 = 0$ thus gives $\Lambda = \omega_1^2$, which extends the result of Kato to magnetic Schrödinger operators.

On the other hand, the splitting $V_1 = 0$ and $V_2 = V$ gives $\Lambda = \omega_0/2$. This extends the result of Agmon and Simon.

- **Is the decay condition $B(x) = \mathcal{O}(|x|^{-1})$ optimal** ?

Example: Miller-Simon

[Miller-Simon, 1980] considered the case $V = 0$ and radial magnetic field $B(x) = b(r)$, $r = |x|$, in dimension two. They proved that

1. If $b(r) = \mathcal{O}(r^{-\alpha})$ with $\alpha > 1$ then the spectrum of $H_{A,0}$ is **purely absolutely continuous** in $(0, \infty)$.

2. If $b(r) = r^{-\alpha} + \mathcal{O}(r^{-1-\varepsilon})$ with $0 < \alpha < 1$ and $\varepsilon > 0$ then the spectrum of $H_{A,0}$ is **dense pure point** in $(0, \infty)$.

3. If $b(r) = b_0\, r^{-1} + \mathcal{O}(r^{-1-\varepsilon})$, then the spectrum of $H_{A,0}$ is **dense pure point** in $\left[0, b_0^2\right]$ and **absolutely continuous** in $\left(b_0^2, \infty\right)$. Mobility edge at b_0^2.

- Note that in this case $\Lambda = (\limsup_{|x|\to\infty} |x|\,|B(x)|)^2$.

- Hence $\Lambda = 0$ in case 1. and $\Lambda = b_0^2$ in case 3. This shows that the decay conditoin $B(x) = \mathcal{O}(|x|^{-1})$ is optimal, and that the value of Λ is **sharp**!

Example: Wigner-von Neumann

Recall that $n = 3,\ B = 0,\ V(|x|) = V(r)$, and that $H_{0,V}$ has eigenvalue 1. Note also that for large r

$$V(r) = -\frac{8\sin(2r)}{r} + \mathcal{O}(r^{-2}).$$

Theorem 1: given a splitting $V = V_1 + V_2$, $H_{0,V}$ has no eigenvalues above

$$\Lambda = \frac{1}{4}\Big(\omega_1 + \sqrt{\omega_1^2 + 2\omega_0}\ \Big)^2.$$

Optimise the splitting as to **minimise** Λ: the optimal choice is $V_1 = 0$ and $V_2 = V$. Then

$$\Lambda = \frac{\omega_0}{2} = 8.$$

This coincides with the result of Agmon.

Main ingredients of the proof

Assume, for simplicity, that $\Lambda = 0$. Then $\beta = \omega_0 = \omega_1 = 0$.

Weighted commutator. Recall that $P = -i\nabla$ and let

$$D = \tfrac{1}{2}(P \cdot x + x \cdot P) = \tfrac{1}{2}((P-A) \cdot x + x \cdot (P-A))$$

be the generator of unitary dilations on $L^2(\mathbb{R}^n)$.

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth and bounded radial function, such that

$$\nabla F(x) = x g(x), \qquad g > 0,$$

and assume that $\nabla(|\nabla F|^2)$, $(1+|\cdot|^2)g$, $x \cdot \nabla g$ and $(x \cdot \nabla)^2 g$ are bounded.

Let $\psi \in D(P-A)$ be a weak eigenfunction of the magnetic Schrödinger operator $H_{A,V}$, i.e.,

$$E\langle \varphi, \psi \rangle = q_{A,V}(\varphi, \psi)$$

for some $E \in \mathbb{R}$ and all $\varphi \in D(P-A)$.

Main ingredients of the proof

Weighted commutator. Let $\psi_F = e^F \psi$. Then

$$
\begin{aligned}
\big\langle \psi_F, i\,[H,D]\,\psi_F \big\rangle = \big\langle \psi_F, \big(E + |\nabla F|^2\big)\,\psi_F \big\rangle + \|(P-A)\psi_F\|_2^2 \\
+ 2\,\mathrm{Re}\,\big\langle (P-A)\,\psi_F, \widetilde{B}\,\psi_F \big\rangle - 2\,\mathrm{Im}\,\big\langle (P-A)\,\psi_F, xV_1\,\psi_F \big\rangle \\
+ \big\langle \psi_F, (n\,V_1 - V)\psi_F \big\rangle - \big\langle \psi_F, x \cdot \nabla V_2\,\psi_F \big\rangle,
\end{aligned}
$$

and at the same time

$$
\big\langle \psi_F, i\,[H,D]\,\psi_F \big\rangle = -4\,\|\sqrt{g}\,D\,\psi_F\|_2^2 + \big\langle \psi_F, \big((x\cdot\nabla)^2 g - x\cdot\nabla|\nabla F|^2\big)\psi_F \big\rangle.
$$

Main ingredients of the proof

Fast decay of the eigenfunctions. Set set $\langle x\rangle_\lambda = \sqrt{\lambda + |x|^2}$.

Lemma: Assume that $H_{A,V}\,\psi = E\psi$. If $E > \Lambda$, then

$$x \mapsto e^{\mu\langle x\rangle_\lambda}\,\psi(x) \in L^2(\mathbb{R}^n) \qquad \forall\,\mu > 0, \quad \forall\,\lambda \geq 0.$$

Idea of the proof: by contradiction (for $\lambda = 1$). If

$$\sup\left\{\mu \geq 0\,:\, e^{\mu\langle x\rangle_1}\,\psi \in L^2(\mathbb{R}^n)\right\} < \infty,$$

and one can construct a sequence of bounded smooth functions F_k such that

$$\|\mathrm{e}^{F_k}\,\psi\| \to \infty \qquad k \to \infty.$$

Then the sequence

$$\varphi_k(x) = \frac{e^{F_k}\,\psi}{\|e^{F_k}\,\psi\|} \quad \text{satisfies} \quad \varphi_k \to 0 \quad \text{in} \quad H^1_{\rm loc}(\mathbb{R}^n)$$

Main ingredients of the proof

Hence $\|\varphi_k\| = 1$, and from the weighted commutator identity for φ_k:

$$\begin{aligned}\big\langle \varphi_k, i\,[H,D]\,\varphi_k\big\rangle = \big\langle \varphi_k, \big(E + |\nabla F_k|^2\big)\,\varphi_k\big\rangle + \|(P-A)\varphi_k\|_2^2 \\ + 2\,\mathrm{Re}\,\big\langle (P-A)\,\varphi_k, \widetilde{B}\,\varphi_k\big\rangle - 2\,\mathrm{Im}\,\big\langle (P-A)\,\varphi_k, xV_1\,\varphi_k\big\rangle \\ + \big\langle \varphi_k, (n\,V_1 - V)\varphi_k\big\rangle - \big\langle \varphi_k, x\cdot\nabla V_2\,\varphi_k\big\rangle,\end{aligned}$$

$$\big\langle \varphi_k, i\,[H,D]\,\varphi_k\big\rangle = -4\,\|\sqrt{g_k}\,D\,\varphi_k\|_2^2 + \big\langle \varphi_k, \big((x\cdot\nabla)^2 g_k - x\cdot\nabla|\nabla F_k|^2\big)\varphi_k\big\rangle.$$

where $\nabla F_k = x g_k$, and from the assumption

$$\limsup_{|x|\to\infty} |\widetilde{B}(x)| = \limsup_{|x|\to\infty} |x|\,|V_1(x)|, = \limsup_{|x|\to\infty} x\cdot\nabla V_2(x) = 0,$$

it follows that

Main ingredients of the proof

$$\big\langle \varphi_k, i\,[H,D]\,\varphi_k \big\rangle \geq E + o_{k\to\infty}(1)$$

and at the same time

$$\big\langle \varphi_k, i\,[H,D]\,\varphi_k \big\rangle = -4\,\|\sqrt{g_k}\,D\,\varphi_k\|_2^2 + o_{k\to\infty}(1)$$

These equations are obviously in contradiction if $E > 0$.

Main ingredients of the proof

Now we define

$$F(x) = F_{\mu,\varepsilon,\lambda}(x) = \frac{\mu}{\varepsilon}\Big(1 - e^{-\varepsilon\,\langle x\rangle_\lambda}\Big) \qquad \Big(\text{recall} : \langle x\rangle_\lambda = \sqrt{\lambda + |x|^2}\Big)$$

Note that $\lim_{\varepsilon\to 0} F_{\mu,\varepsilon,\lambda}(x) = \mu\langle x\rangle_\lambda$.

Inserting this into the weighted commutator identity and letting $\varepsilon \to 0$ gives

$$\mu^2 \Big\langle e^{\mu\langle x\rangle_\lambda}\,\psi, \frac{|x|^2}{\lambda + |x|^2}\, e^{\mu\langle x\rangle_\lambda}\,\psi\Big\rangle \le C\,\|e^{\mu\langle x\rangle_\lambda}\,\psi\|^2 \qquad \forall\,\mu, \lambda > 0,$$

for some $C > 0$. Passing to the limit $\lambda \to 0$ we thus get

$$\mu^2\,\|e^{\mu\langle x\rangle_\lambda}\,\psi\|^2 \le C\|e^{\mu\langle x\rangle_\lambda}\,\psi\|^2 \qquad \forall\,\mu > 0,$$

This is a contradiction unless $\psi = 0$.

Remark

The possibility of $H_{A,V}$ having eigenvalue $E = 0$ cannot be excluded! Indeed, if $B : \mathbb{R}^2 \to \mathbb{R}$ is continuous and compactly supported with

$$\int_{\mathbb{R}^2} B < -2\pi,$$

then by the Aharonov-Casher Theorem the operator

$$H_{A,B} = (P - A)^2 + B$$

has eigenvalue 0. Note that here $\Lambda = 0$.

References

[1] S. Agmon, *Lower bounds for solutions of Schrödinger equations.* J. Anal. Math., **23** (1970) 1–25.

[2] S. Avramska-Lukarska, D. Hundertmark, H. Kovařík: *Absence of positive eigenvalues for magnetic Schrödinger operators*. Submitted. arXiv: 2003.07294.

[3] R. Froese, I. Herbst, *Exponential Bounds and Absence of Positive Eigenvalues for Many-Body Schrödinger Operators.* Comm. Math. Phys., **87** (1982), 429–447.

[4] R. Froese, I. Herbst, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, *On the absence of positive eigenvalues for one-body Schrödinger operators.* J. Anal. Math. **41** (1982), 272–284.

[5] A.D. Ionescu, D. Jerison, *On the absence of positive eigenvalues of Schrödinger operators with rough potentials.* Geom. Funct. Anal., **13** (2003), 1029 –1081.

[6] D. Jerison, C.E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operator.* Annals Math. **121** (1985), 463–494.

[7] T. Kato, *Growth Properties of Solutions of the Reduced Wave Equation With a Variable Coefficient.* Commun. Pure Appl. Math., **12** (1959), 403–425.

[8] H. Koch, D. Tataru, *Carleman estimates and absence of embedded eigenvalues.* Comm. Math. Phys., **267** (2006), 419–449.

[9] K. Miller, B. Simon, *Quantum Magnetic Hamiltonians with Remarkable Spectral Properties.* Phys. Rev. Lett. **44** (1980), 1706–1707.

[10] B. Simon, *On positive eigenvalues of one body Schrödinger operators.* Commun. Pure Appl. Math. **22** (1967), 531–538.