Analytic continuation of Chern-Simons theory

Davide Polini

LisMath Seminar

29th April 2016

Summary

- General context
- Simplified example
- Adaptation to Chern-Simons

Quantum Field Theory: aims to describe the fundamental forces of nature. Formally all the physical information can be extracted from the generating functional:

$$Z = \int D\varphi \exp\left(i \int d^n x \mathcal{L}[\varphi]\right)$$

Problems:

- $D\varphi = \prod_x \delta\varphi(x)$
- Oscillatory integrand

A way to treat the oscillatory behaviour is to perform an analytical continuation of the integral.

Chern-Simons action with gauge group $SL(2,\mathbb{C})$:

$$I = \frac{t}{8\pi}\int_M Tr(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) + \frac{\tilde{t}}{8\pi}\int_M Tr(\bar{\mathcal{A}}\wedge d\bar{\mathcal{A}} + \frac{2}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}})$$

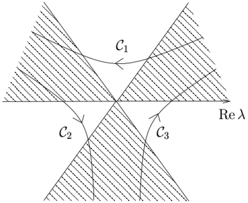
$$Z = \int_{\mathcal{Y}} D\mathcal{A}D\bar{\mathcal{A}}e^{iI}$$

The analytic continuation of this theory is of particular relevance for 3d Quantum Gravity and knot theory.

Simplified model: Airy function

$$Z_\lambda = \int_{-\infty}^{\infty} dx \exp\left(i\lambda(x^3/3 - x)\right)$$

Analytic continuation consists in taking $\lambda \in \mathbb{C}, x \in \mathbb{C}$ and integrate over a suitable cycle $\mathcal{C}$.
Suitable cycle means a cycle that connects two regions such that: $Re(i\lambda(x^3/3 - x)) < 0$



How to generalize

We want to get this result by means of a general theory.

This will be achieved through homology.

We define the set:

$$X_{-T} = \{x \in X | Re(I) < -T\}$$

We want the integration cycles to have their ends lying in X_{-T}.

In terms of homology this means that they should be element of the first relative homology group $H_1(X, X_{-T}, \mathbb{Z})$.

Morse theory establishes a relation between the homology of a space X and the critical points of a function $h : X \to \mathbb{R}$ defined on the space.

Given a metric g on the space this is obtained by means of the downward flow equation:

$$\frac{\partial x^i}{\partial t} = -g^{ij} \frac{\partial h}{\partial x^j}$$

In particular the cycles are given by the union of all flow lines that start out at the critical point.

It is worth to notice that the result is independent of the choice of the metric and of the function.

The Airy function case

In our case the Morse function is $h = ReI$.

Since h is unbounded the cycles will flow to X_{-T}, so the generated Homology is the relative one that we are looking for.

Every good integration cycle is a linear combination of the critical point cycles with integer coefficients.

So if $\mathcal{C} = \sum_\alpha \mathfrak{n}_\alpha \mathcal{C}_\alpha$ we can write:

$$Z_\lambda = \sum_\alpha \mathfrak{n}_\alpha \int_{\mathcal{C}_\alpha} exp(i\lambda(x^3/3 - x))$$

It is possible to evaluate the asymptotic behavior of the integral over the cycles $\mathcal{C}_\alpha$ for $\lambda \to \infty$:

$$Z_{\alpha,\lambda} \sim \exp(I_\alpha)(-i\lambda)^{-1/2} \sum_{t=0}^{\infty} b_t \lambda^{-t}$$

Therefore for a generic cycle $\mathcal{C} = \sum_\alpha \mathfrak{n}_\alpha \mathcal{C}_\alpha$, we have:

$$Z_{\mathcal{C},\lambda} = \sum_\alpha \mathfrak{n}_\alpha Z_{\alpha,\lambda}$$

So given a generic cycle $\mathcal{C} = \sum_\alpha \mathfrak{n}_\alpha \mathcal{C}_\alpha$ we want to be able to determine the coefficients $\mathfrak{n}_\alpha$.

A natural dual to the downward flowing cycles are the upward flowing cycles:

$$\frac{\partial x^i}{\partial t} = g^{ij} \frac{\partial h}{\partial x^j}.$$

The only point where an upward flowing cycle $\mathcal{K}_\sigma$ and a downward flowing cycle $\mathcal{C}_\tau$ can meet is the critical point to which they are attached, so we have:

$$\langle \mathcal{C}_\tau , \mathcal{K}_\sigma \rangle = \delta_{\sigma\tau}$$

Therefore:

$$\mathfrak{n}_\alpha = \langle \mathcal{C}, \mathcal{K}_\alpha \rangle.$$

Let's try to see how to express the real cycle $\mathcal{C}_{\mathbb{R}}$ in terms of Morse cycles.

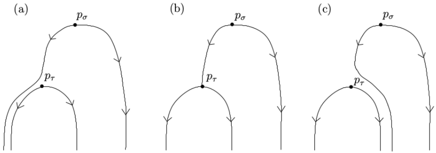
We have two critical points in $x = \pm 1$ that both lie on $\mathcal{C}_{\mathbb{R}}$.

A non-trivial upward flow starting at a critical point will end at a point with $h > 0$ and won't lie on $\mathcal{C}_{\mathbb{R}}$ since h vanishes identically on it.

This implies that the coefficients are 1 and then:

$$\mathcal{C}_{\mathbb{R}} = \mathcal{C}_{+} + \mathcal{C}_{-}$$

It can happen to have flows between critical points, not flowing to $-\infty$. Then the solution of the downward flow equation is not in the relative homology.



This happens under certain conditions for λ that we can control.

$$I = \frac{t}{8\pi}\int_M Tr(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) + \frac{\tilde{t}}{8\pi}\int_M Tr(\tilde{\mathcal{A}}\wedge d\tilde{\mathcal{A}} + \frac{2}{3}\tilde{\mathcal{A}}\wedge\tilde{\mathcal{A}}\wedge\tilde{\mathcal{A}})$$

We consider $A, \tilde{A}$ to be independent and $t, \tilde{t} \in \mathbb{C}$.

What Chern-Simons has in common with what we showed?

- Critical points
- Downward flow equations

What is needed to adapt this method?

- Manifold
- Non-isolated critical points
- $\mathbb{Z} \to \mathbb{C}$

1. Witten, Edward. "Analytic continuation of Chern-Simons theory." Chern–Simons gauge theory 20 (2011): 347-446.
2. Hutchings, Michael. "Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves)." available at math. berkeley. edu/ hutching/teach/276-2010/mfp. ps (2002).
3. Berry, M. V., and C. J. Howls. "Hyperasymptotics for integrals with saddles." Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences. Vol. 434. No. 1892. The Royal Society, 1991.