Spin textures in Quantum Hall ferromagnets

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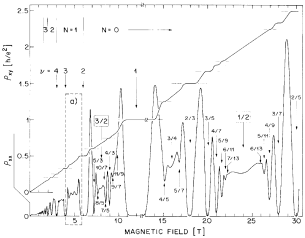
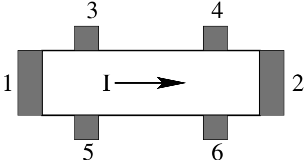
Quantum Hall effect





$$
\begin{aligned}
R_{xx} &= \left(V(3) - V(4)\right)/I \\
R_{xy} &= \left(V(3) - V(5)\right)/I
\end{aligned}
$$

Quantum nature of Hall resistance plateaus

Plateaus observed for (ν integer):

$$\rho_{xy} = \frac{B}{ne} = \frac{h}{\nu e^2}$$

$\rightarrow$ Quantized electronic densities:

$$n = \nu \frac{eB}{h}$$

In terms of $\Phi_0 = \frac{h}{e}$: "Flux quantum"

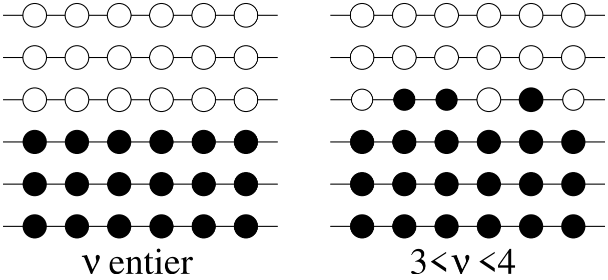
$$N_{\mathrm{electrons}} = \nu \frac{\text{Total magnetic flux}}{\Phi_0}$$

Landau levels are degenerate

Intuitively, each state occupies the same area as a flux quantum Φ_0, so that the number of states per Landau level =

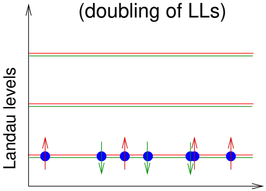
$$\frac{\text{Total magnetic flux}}{\Phi_0}$$

ν is interpreted as the number of occupied Landau levels

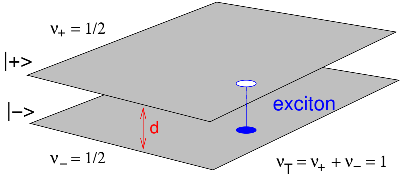


Multi-Component Systems (Internal Degrees of Freedom)

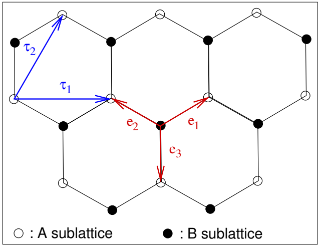
(A) physical spin: SU(2)



(B) bilayer: SU(2) isospin



(C) graphene (2D graphite)



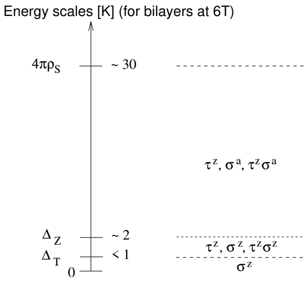
two−fold valley degeneracy
$\longrightarrow$ SU(2) isospin

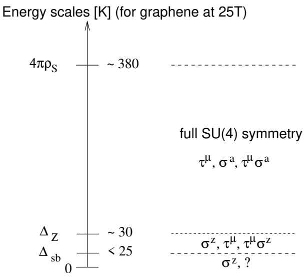
spin + isospin : SU(4)

Realistic anisotropies

Hamiltonian can approximately have high $SU(4)$ symmetry

- Zeeman anisotropy: $SU(2) \to U(1)$
- Graphene: valley weakly split, $O(a/l_B)$
- Bilayers: charging energy: $SU(2) \to U(1)$; neglect tunnelling





Quantum Hall ferromagnets

N internal states (spin, valley, layer indices, e. g. $N = 4$ for graphene).
Integer filling factor M with $1 \leq M \leq N - 1$.
Large magnetic field $\rightarrow$ Projection onto the lowest Landau level (LLL). Assume that largest sub-leading term is given by Coulomb interactions (small g factor). This selects a ferromagnetic state

Main question: What happens when $\nu = M + \delta\nu$, $\delta\nu << 1$?

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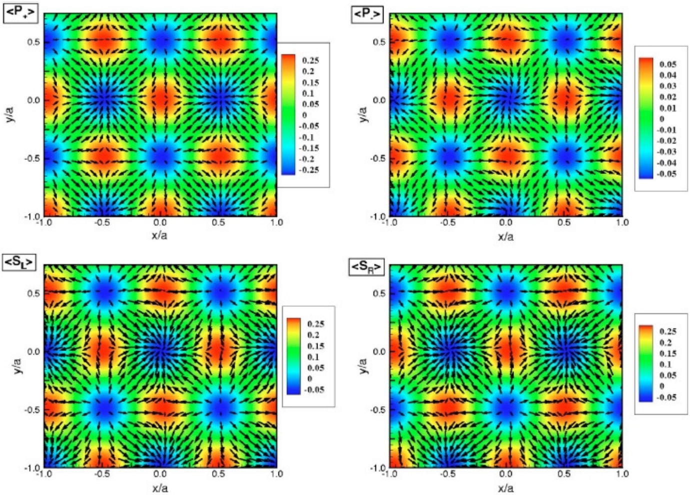
Main question: What happens when $\nu = M + \delta\nu$, $\delta\nu << 1$?

Ferromagnetic state is replaced by slowly varying textures (e. g. Skyrmions lattices for $M = 1$).

Sondhi, Karlhede, Kivelson, Rezayi, PRB **47**, 16419, (1993), Brey, Fertig, Côté and MacDonald, PRL **75**, 2562 (1995)

Example of entangled textures ($N = 4$, $M = 1$)



Bourassa et al, Phys. Rev. B **74**, 195320 (2006)

Description of uniform states

Work in lowest Landau level with $\nu = M$, $1 \leq M \leq N-1$. We choose an M-dimensional subspace in $\mathbb{C}^N$, which corresponds to the M occupied internal states. Explicitely, this subspace is generated by the columns of an $N \times M$ matrix V.

Consider now a complete basis $\chi^{(\alpha)}(\mathbf{r})$ in the LLL (orbital degree of freedom). A ferromagnetic state is obtained by taking the Slater determinant $|\mathcal{S}_V\rangle$ built from single particle states of the form $|\Psi^{(i\alpha)}\rangle$, $(1 \leq i \leq M)$, given by:

$$\Psi_a^{(i\alpha)}(\mathbf{r}) = V_{ai}\, \chi^{(\alpha)}(\mathbf{r}), \quad 1 \leq a \leq N$$

Terminology: The continuous set of M-dimensional subspaces in $\mathbb{C}^N$ is a smooth complex manifold of dimension $(N-M)M$, called the Grassmannian $\mathrm{Gr}(M, N)$.

Slater determinants in the LLL associated to smooth textures (I)

Physical space manifold: $\Sigma = \mathbb{R}^2$
Textures: Smooth maps $\Sigma \to \mathcal{M} = \mathrm{Gr}(M, N)$
Explicitly: Pick an $N \times M$ matrix $V_{ij}(\mathbf{r})$ of maps.
This defines a local projector in internal (generalized spin space)
$P_V(\mathbf{r}) = V(\mathbf{r})(V^\dagger(\mathbf{r})V(\mathbf{r}))^{-1}V^\dagger(\mathbf{r})$.

Auxiliary single-particle Hamiltonian:

$$H_{\mathrm{aux},V} = -\mathcal{P}_{LLL}\left(\int d^2\mathbf{r} \sum_{a,b} P_V(\mathbf{r})_{ab}\Psi_a^\dagger(\mathbf{r})\Psi_b(\mathbf{r})\right)\mathcal{P}_{LLL}$$

The ground-state of $H_{\mathrm{aux},V}$ is a Slater determinant $|\mathcal{S}_V\rangle$.

Slater determinants in the LLL associated to smooth textures (II)

Main effect of $\mathcal{P}_{LLL}$: (Moon et al. (1995), Pasquier (2000),...)

$$\begin{aligned} n_{\rm el}(\mathbf{r}) &= \frac{M}{2\pi l^2} - Q(\mathbf{r}) + O(l^2) \\ N_{\rm el} &= M N_\Phi - Q_{\rm top} \quad \rightarrow \quad \text{CONSTRAINT} \end{aligned}$$

Energy functional:

$$E_{\rm tot} = E_{\rm loc} + E_{\rm non-loc}$$

$E_{\rm loc}$: exchange energy (generalized ferromagnet), given by a non-linear σ model energy functional (next slides).
$E_{\rm non-loc} = \frac{e^2}{8\pi\epsilon} \int d^2\mathbf{r} \int d^2\mathbf{r}' \frac{Q(\mathbf{r})Q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$.

Slater determinants in the LLL associated to smooth textures (III)

$$E_{\text{tot}} = E_{\text{loc}} + E_{\text{non-loc}}$$

If filling factor is close to M, $E_{\text{non-loc}} << E_{\text{loc}}$. To find optimal textures, we can therefore:

1. Minimize E_{loc} in the presence of the $N_{\text{el}} = MN_{\Phi} - Q_{\text{top}}$ constraint. This leads to a continuous family of degenerate configurations (next slides).
2. Lift this degeneracy by minimizing $E_{\text{non-loc}}$ within this degenerate family. Physically, this favors textures in which the topological charge density is as uniform as possible.

Kähler manifolds

$\mathcal{M}$ complex manifold with local complex coordinates w_i.
$\mathcal{M}$ is equipped with an Hermitian metric

$$ds^2 = \sum_{ij} h_{ij}\, dw_i d\bar{w}_j$$

such that the corresponding associated (1,1) form

$$\omega = \frac{i}{2} \sum_{ij} h_{ij}\, dw_i \wedge d\bar{w}_j$$

is closed.
This implies that, locally, the metric derives from a Kähler potential Φ, i.e. that:

$$h_{ij} = \frac{\partial^2 \Phi}{\partial w_i \partial \bar{w}_j}$$

Grassmannians are Kähler manifolds

Choice of local coordinates on $\mathrm{Gr}(M, N)$: Pick a rank M, $N \times M$ matrix V. Then it has at least one non-zero $M \times M$ minor determinant. Assuming this is the first one, we get a dense open subset of $\mathrm{Gr}(M, N)$. $V = \begin{pmatrix} V_u \\ V_d \end{pmatrix}$. Multiplying V on the right by V_u^{-1} leads to the same M-dimensional subspace. This changes V into

$$\begin{pmatrix} I_m \\ W \end{pmatrix}$$

where $W = V_d V_u^{-1}$ is an arbitrary $(N - M) \times M$ matrix.

Kähler potential:

$$\Phi(W, W^\dagger) = \frac{1}{\pi} \log \det(I + W^\dagger W)$$

Energy functionals for maps to Kähler manifolds

Classical energy functional for a map $(x, y) \to (w_i)$:

$$E = \frac{g}{2} \int d^2\mathbf{r} \; h_{ij}(w(\mathbf{r}), \bar{w}(\mathbf{r})) \nabla w_i . \nabla \bar{w}_j$$
$$E = g \int d^2\mathbf{r} \; h_{ij}(\partial_z w_i \partial_{\bar{z}} \bar{w}_j + \partial_{\bar{z}} w_i \partial_z \bar{w}_j)$$

The topological charge density is defined by:

$$Q = \int d^2\mathbf{r} \; f^* \omega$$

Explicitely:

$$Q = \int d^2\mathbf{r} \; h_{ij}(\partial_z w_i \partial_{\bar{z}} \bar{w}_j - \partial_{\bar{z}} w_i \partial_z \bar{w}_j)$$

$d\omega = 0$ implies that Q does not change to first order under any infinitesimal variation of the map f, so Q depends only on the homotopy class of f. In many interesting situations, Q takes only integer values.

Bogomolnyi inequality and its consequences

$$\begin{aligned} E &= g(A+B) \\ Q &= A-B \end{aligned}$$

$$A = \int d^2\mathbf{r}\; h_{ij}\partial_z w_i \partial_{\bar{z}} \bar{w}_j \quad B = \int d^2\mathbf{r}\; h_{ij}\partial_{\bar{z}} w_i \partial_z \bar{w}_j$$

Since $h_{ij} = \bar{h}_{ji}$ is positive definite, A and B are both real and non-negative. Then $A + B \geq |A - B|$, so:

$$E \geq g|Q|$$

Minimal energy configurations with fixed Q:
If $Q > 0$, $B = 0$, so $\partial_{\bar{z}} w_i = 0$: minimal configurations are holomorphic.
If $Q < 0$, $A = 0$, so $\partial_z w_i = 0$: minimal configurations are anti-holomorphic.

A list of questions

Consider physical space to be a two-dimensional manifold Σ

1. How to construct (and parametrize) the whole family of holomorphic maps $\Sigma \longrightarrow \mathcal{M}$?
2. How to minimize spatial variations of the topological charge density (Coulomb energy) ?
3. Space of maps $\Sigma \longrightarrow \mathcal{M}$ as classical phase-space. How to study the quantum Hamiltonian associated to the local energy functional E ?

Holomorphic maps from the sphere to $\mathbb{CP}(N-1)$ (I)

$S^2 \cong \mathbb{CP}(1) \cong \mathbb{C} \cup \{\infty\}$ so we use one coordinate $z \in \mathbb{C}$.
Kähler potential on the sphere: $\Phi = \frac{1}{\pi}\log(1+|z|^2)$
Volume element: $\omega = \frac{dx \wedge dy}{\pi(1+|z|^2)^2}$
Holomorphic maps $f : S^2 \to \mathbb{CP}(N-1)$: collections of N polynomials $P_1(z), ..., P_N(z)$.
Topological charge: number of intersection points of $f(S^2)$ with an arbitrary hyperplane in $\mathbb{CP}(N-1)$ = maximal degree d of $P_1(z), ..., P_N(z)$.
Topological charge density:

$$Q(z,\bar{z}) = (1+|z|^2)^2 \partial_z \partial_{\bar{z}} \log(\sum_{i=1}^{N} |P_i(z)|^2)$$

$Q(z,\bar{z})$ is constant when:

$$\sum_{i=1}^{N} |P_i(z)|^2 = (1+|z|^2)^d$$

Holomorphic maps from the sphere to $\mathbb{C}\mathrm{P}(N-1)$ (II)

Hermitian scalar product on degree d polynomials:

$$(P,Q)_d = \frac{d+1}{\pi}\int d^2\mathbf{r}\, \frac{\overline{P(z)}Q(z)}{(1+|z|^2)^{d+2}}$$

Orthonormal basis: $e_p(z) = \left(\begin{array}{c} d \\ p \end{array}\right)^{1/2} z^p$

General texture of degree d: $P_i(z) = \sum_{i=0}^{d} A_{ij} e_j(z)$
$Q(z,\bar{z})$ is constant when: $A^\dagger A = I_{d+1}$
If $d \geq N$: No solution
If $d \leq N-2$: many solutions, but not all components of the maps are linearly independent.
If $d = N-1$: $AA^\dagger = I_N = A^\dagger A$, so $(P_i, P_j)_d = \delta_{ij}$.
Textures with uniform topological charge density $\Leftrightarrow$ Components form an orthonormal basis.

Holomorphic maps from the torus to $\mathbb{CP}(N-1)$ (I)

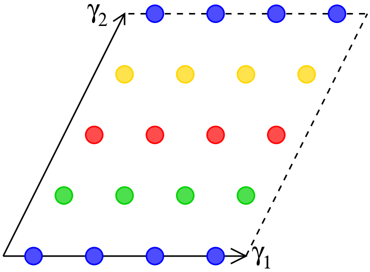
$$\theta(z+\gamma) = e^{a_\gamma z + b_\gamma}\theta(z)$$
$$(\theta,\theta')_d = \int d^2\mathbf{r}\ \exp(-\frac{\pi d|z|^2}{|\gamma_1 \wedge \gamma_2|})\overline{\theta(z)}\theta'(z)$$

Optimal textures $(d = N)$

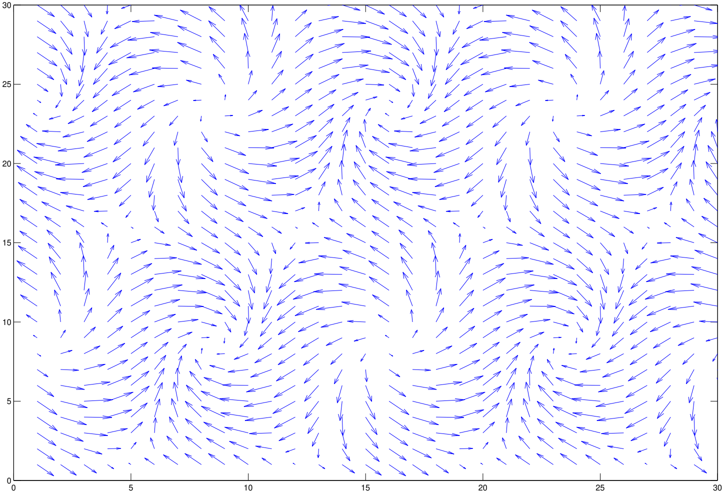
$$|\Psi(z)\rangle = \begin{pmatrix} \theta_0(z) \\ \theta_1(z) \\ . \\ . \\ . \\ \theta_{d-1}(z) \end{pmatrix}$$

$$(\theta_i, \theta_j)_d = \delta_{ij}$$

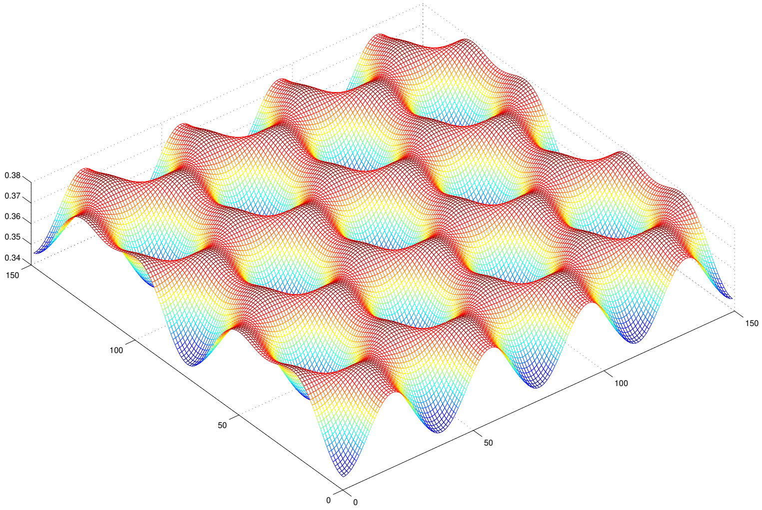
Pattern of zeros $(d{=}4)$



$d = N = 2$



$d = N = 4$



Spatial variations of topological charge: $Q(r)$ is always γ_1/d and γ_2/d periodic. Unlike on the sphere, $Q(r)$ is not exactly constant.

At large d the modulation contains mostly the lowest harmonic, and its amplitude decays exponentially with d.

Large d behavior for a square lattice:

$$Q(x,y) \simeq \frac{2}{\pi} - 4de^{-\pi d/2}[\cos(2\sqrt{d}x) - 2e^{-\pi d/2}\cos^2(4\sqrt{d}x) + (x \leftrightarrow y)] + \ldots$$

Only the triangular lattice seems to yield a true local energy minimum. This has been evidenced by computing eigenfrequencies of small deformation modes.

B. Douçot, D. Kovrizhin, R. Moessner, PRL 110, 186802 (2013)

Holomorphic maps from Σ to $\mathbb{CP}(N-1)$ (I)

Components of a map $f : \Sigma \to \mathbb{CP}(N-1)$ were polynomials on the sphere and θ functions on the torus. Note that polynomials have poles at $z \to \infty$, and θ functions are multivalued.

More general construction: Pick a line bundle L over Σ, and choose the components of the maps $s_j(z)$ as global holomorphic sections of L, for $1 \leq j \leq N$.

Recipe for optimal textures: $N =$ dimension of the space of global holomorphic sections of L. Choose components forming an orthonormal basis for a well chosen hermitian product.

Holomorphic maps from Σ to $\mathbb{CP}(N-1)$ (II)

Geometric quantization recipe for the hermitian product

ω : volume form associated to constant curvature metric on Σ

h^d : hermitian metric on fibers of L^d whose curvature form equals $-d(2\pi i)\omega$

$$(s, s')_{L,d} = \int_\Sigma h^d(s(x), s'(x))\omega(x)$$

Topological charge form: $\omega_{\mathrm{top}} - \omega = \frac{1}{\pi}\partial_z \partial_{\bar{z}} \log B(z, \bar{z})$.

$B(z, \bar{z})_{L,d} = \sum_{j=1}^{N} h^d(s_j(z), s_j(z))$

For an orthonormal basis $B(z, \bar{z})$ is the Bergman kernel, whose large d asymptotics has been studied a lot in the 90's.

Holomorphic maps from Σ to $\mathbb{CP}(N-1)$ (III)

Bergman kernel asymptotics (Tian, Yau, Zelditch, Catlin, Lu,...(1990 to 2000)):
$B(z,\bar{z}) = d + a_0(z,\bar{z}) + a_{-1}(z,\bar{z})d^{-1} + a_{-2}(z,\bar{z})d^{-2} + \ldots$, such that $a_j(z,\bar{z})$ is a polynomial in the curvature and its covariant derivatives at $(z,\bar{z})$.

Interesting consequence: If ω is associated to the constant curvature metric on Σ, the previous family of textures have uniform topological charge, up to corrections which are smaller than any power of $1/d$.

"Practical" questions: How to effectively construct such orthonormal bases of sections, when Σ has genus ≥ 2 ?
Optimization of the exponentially small corrections in d with respect to the line bundle L ?

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Key idea: View classical textures as coherent states.

Basic idea of geometric quantization

Kostant (1970), Souriau (1970), Berezin (1974), Rawnsley (1977): to each compact symplectic (Kähler) manifold $(\mathcal{M}, \omega)$, that admits an ample complex line bundle L, we associate:

- A quantum Hilbert space $\mathcal{H}_n$: the space of holomorphic sections of the $\otimes^n L$ bundle over $\mathcal{M}$.
- Pick a hermitian metric h on the fibers of $\otimes^n L$, such that the curvature form of the associated connection is proportional to ω. Then $\langle s_1 | s_2 \rangle = \int_{\mathcal{M}} h(s_1, s_2) \omega^{\mathrm{dim} \mathcal{M}}$.
- The coherent state at x is defined by: $s(x) = \langle \Phi_x | s \rangle s_0(x)$, s_0 being a reference section.

$$\mathrm{Symb}(\hat{A})(x) = \langle \Phi_x | \hat{A} | \Phi_x \rangle / \langle \Phi_x | \Phi_x \rangle$$

$$\mathrm{Symb}([\hat{A}, \hat{B}]) \xrightarrow[n \to \infty]{} \frac{i}{n} \{ \mathrm{Symb}(\hat{A}), \mathrm{Symb}(\hat{B}) \}$$

Geometric quantization on $\mathbb{C}P(N-1)$ (I)

Consider N bosonic modes with $[a_i, a_j^\dagger] = \delta_{ij}$ for $0 \leq i, j \leq N-1$. Take m positive integer, and consider the finite dimensional subspace of bosonic Fock space, defined by the constraint: $\sum_{i=0}^{N-1} a_i^+ a_i = m$.

Orthonormal basis: $|\vec{n}\rangle = \frac{(\hat{a}_0^+)^{n_0} \cdots (\hat{a}_{N-1}^+)^{n_{N-1}}}{\sqrt{n_0! \cdots n_{N-1}!}} |0\rangle, \quad \sum_{i=0}^{N-1} n_i = m,$ $n_i \geq 0$.

Coherent states: $|e_{\bar{v}}\rangle = \sum \frac{\bar{v}_1^{n_1} \cdots \bar{v}_{N-1}^{n_{N-1}}}{\sqrt{n_0! \cdots n_{N-1}!}} |\vec{n}\rangle$

Overlaps: $\langle e_{\bar{v}'} | e_{\bar{v}} \rangle = \frac{(1+\langle v|v'\rangle)^m}{m!}$

Reproducing kernel: $\mathbf{I} = \frac{(m+N-1)!}{\pi^{(N-1)} m!} \int \frac{\prod_{j=1}^{N-1} dv_j d\bar{v}_j}{(1+\langle v|v\rangle)^N} \frac{|e_{\bar{v}}\rangle\langle e_{\bar{v}}|}{\langle e_{\bar{v}}|e_{\bar{v}}\rangle}$

Geometric quantization on $\mathbb{C}P(N-1)$ (II)

Covariant symbols: $\frac{\langle e_{\bar{v}}| \prod_{j=0}^{N-1} (a_j^+)^{m_j} a_j^{n_j} |e_{\bar{v}}\rangle}{\langle e_{\bar{v}}|e_{\bar{v}}\rangle} = \frac{m!}{(m-n)!} \frac{\prod_{j=1}^{N-1} v_j^{m_j} \bar{v}_j^{n_j}}{(1+\langle v|v\rangle)^n}$

Consider $\hat{H}$ an operator which can be written as a power series in bosonic mode operators a_i, $a_j^\dagger$, and whose covariant symbol $H(v,\bar{v}) = \frac{\langle e_{\bar{v}}|\hat{H}|e_{\bar{v}}\rangle}{\langle e_{\bar{v}}|e_{\bar{v}}\rangle}$ is such that $H(v,\bar{v})$ has a minimum at $v=0$ and its Taylor expansion around $v=0$ doesn't contain any monomial composed only of v_j's nor only of $\bar{v}_j$'s.

Then $|e_0\rangle$ is an exact eigenstate of $\hat{H}$, with eigenvalue $H(0,0)$.

Key property of the Hessian

$\mathcal{M}$ complex Kähler manifold with local complex coordinates w_i,
$ds^2 = \sum_{ij} h_{ij} dw_i d\bar{w}_j$, $\omega = \frac{i}{2}\sum_{ij} h_{ij} dw_i \wedge d\bar{w}_j$, $d\omega = 0$, so locally:
$h_{ij} = \frac{\partial^2 \Phi}{\partial w_i \partial \bar{w}_j}$. We consider maps $\mathbb{R}^2 \to \mathcal{M}$.

$$E = g\int d^2\mathbf{r}\, \frac{\partial^2 \Phi}{\partial w_i \partial \bar{w}_j}(\partial_z w_i \partial_{\bar{z}} \bar{w}_j + \partial_{\bar{z}} w_i \partial_z \bar{w}_j)$$

$$Q = \int d^2\mathbf{r}\, f^*\omega = \int d^2\mathbf{r}\, \frac{\partial^2 \Phi}{\partial w_i \partial \bar{w}_j}(\partial_z w_i \partial_{\bar{z}} \bar{w}_j - \partial_{\bar{z}} w_i \partial_z \bar{w}_j)$$

$$E = gQ + 2g\int d^2\mathbf{r}\, \frac{\partial^2 \Phi}{\partial w_i \partial \bar{w}_j}\partial_{\bar{z}} w_i \partial_z \bar{w}_j$$

Pick a holomorphic texture $w_{\mathrm{cl},i}(\mathbf{r})$ and write
$w_i(\mathbf{r}) = w_{\mathrm{cl},i}(\mathbf{r}) + \delta w_i(\mathbf{r})$. Using the fact that $\partial_{\bar{z}} w_{\mathrm{cl},i} = 0$, we see that the Taylor expansion of E does not contain any term involving only $\delta w_i(\mathbf{r})$'s nor any term involving only $\delta \bar{w}_i(\mathbf{r})$'s.

B. Douçot, D. Kovrizhin, R. Moessner, PRB 93, 094426 (2016)

Towards continuum limit: lattice regularization

Take a 2D lattice, and associate to each site the quantized Hilbert space obtained from the classical $\mathbb{C}P(N-1)$ manifold, with the same m at each site. The classical limit is obtained as $m \to \infty$. Consider the Hamiltonian:

$$\hat{H} = -\sum_{\langle \mathbf{r}\mathbf{r}' \rangle} \sum_{ij} a_i^\dagger(\mathbf{r}) a_j(\mathbf{r}) a_j^\dagger(\mathbf{r}') a_i(\mathbf{r}')$$

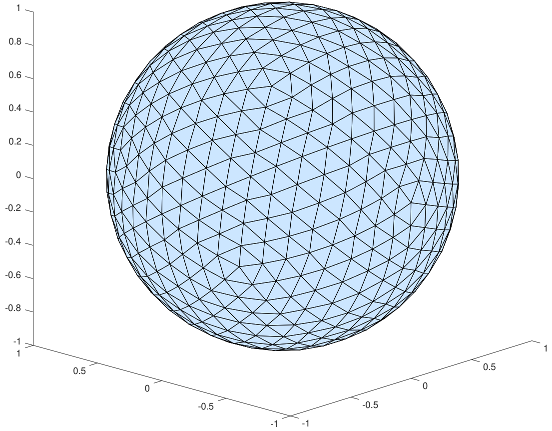
Its covariant symbol is:

$$H(v,\bar{v}) = -m^2 \frac{(1+\langle v(\mathbf{r})|v(\mathbf{r}')\rangle)(1+\langle v(\mathbf{r}')|v(\mathbf{r})\rangle)}{(1+\langle v(\mathbf{r})|v(\mathbf{r})\rangle)(1+\langle v(\mathbf{r}')|v(\mathbf{r}')\rangle)}$$

This provides a lattice discretization of the classical $\mathbb{C}P(N-1)$ energy functional, together with a well defined quantization associated to it.

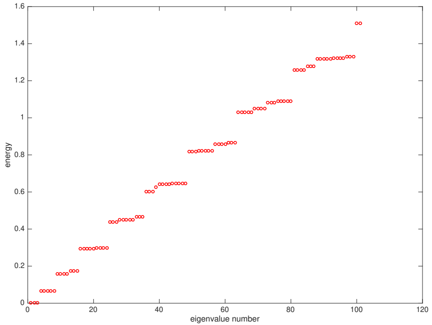
Numerical experiments (D. Kovrizhin)

Triangulation on the sphere (642 sites)



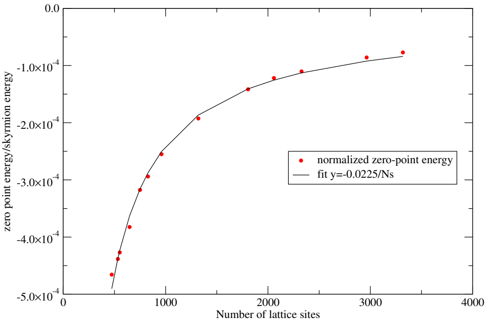
Numerical experiments (D. Kovrizhin)

Harmonic mode spectrum around a single Skyrmion classical configuration: compatible with the magnetic Laplacian on the sphere with a charge 2 magnetic monopole: manifestation of the spin Berry phase associated to a slow twist of the spin background.



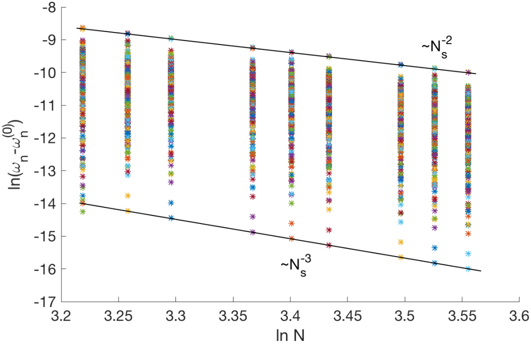
Numerical experiments (D. Kovrizhin)

Harmonic mode spectrum around a single Skyrmion classical configuration: quantum zero point correction normalized to classical Skyrmion energy



Remark on lattice effects (I)

Absence of magnon-type excitations, due to holomorphic nature of the texture, holds only when magnon wave-length » lattice spacing.



Magnon frequency shift due to magnon non-conserving terms in the quadratic approximation around a classical Skyrmion solution

Remark on lattice effects (II)

It turns out that the total quantum correction to the ground-state energy of a Skyrmion configuration goes to zero as $1/N_{\mathrm{sites}}$ when $N_{\mathrm{sites}} \to \infty$, even in the presence of small residual quantum fluctuations induced by the lattice discretization.

B. Douçot, D. Kovrizhin, R. Moessner, arXiv:1808.06783

Final remarks

Ubiquity of geometric quantization:

- Derivation of energy functionals and physical effects due to projection onto lowest Landau level.
- "Re-quantization" around classical textures and analysis of quantum zero point motion correction to total energy.
- More surprinsingly, provides a geometrical description of optimal textures, i.e. those with most uniform topological charge density.

Main open challenge: To identify optimal Grassmannian textures. For them, we are no longer dealing with line bundles, but with vector bundles over the space manifold Σ.

Maps $\Sigma \to \mathrm{Gr}(M, N)$ and rank M vector bundles (I)

Basic fact: there exists a 1 to 1 correspondence between:

- Maps $f : \Sigma \to \mathrm{Gr}(M, N)$
- Rank M vector bundles $\mathcal{V}$ over Σ, together with a choice of N sections of $\mathcal{V}$, which generate the fiber $\mathcal{V}_x$ at each $x \in \Sigma$, modulo automorphisms of $\mathcal{V}$.

$$\begin{array}{ccc} \mathcal{V} \cong f^*\mathcal{T}^* & \xrightarrow{\bar{f}} & \mathcal{T}^* \\ \uparrow s_i & 1 \leq i \leq N & \uparrow t_i \\ \Sigma & \xrightarrow{f} & \mathrm{Gr}(M, N) \end{array}$$

$f^*\mathcal{T}^*$: dual of tautological rank M vector bundle over $\mathrm{Gr}(M, N)$.
For $V \in \mathrm{Gr}(M, N)$, $t_i(V)$ is the linear form on V defined by the i-th component in $\mathbb{C}^N$ ($V \subset \mathbb{C}^N$).

Maps $\Sigma \to \mathrm{Gr}(M, N)$ and rank M vector bundles (II)

Conversely, we start from a rank M vector bundle over Σ, and a choice of N sections $s_i(x)$, $1 \leq i \leq N$ of $\mathcal{V}$, which generate the fiber $\mathcal{V}_x$ at each $x \in \Sigma$.

Using local frames in open subsets U_α covering Σ, each section $s_i(x)$ may be seen as an M-component row-vector. These N rows form an $N \times M$ matrix $V^{(\alpha)}(x)$, and if $x \in U_\alpha \cap U_\beta$:

$$V^{(\alpha)}(x) = V^{(\beta)}(x) t^{(\beta\alpha)}(x)$$

where $t^{(\beta\alpha)}(x)$ are the transition functions of $\mathcal{V}$.

The linear span in $\mathbb{C}^N$ of the columns of $V^{(\alpha)}(x)$ form a well defined $f(x) \in \mathrm{Gr}(M, N)$.

Elements of $\mathcal{V}_x \longleftrightarrow$ M-component row-vectors

Elements of $f(x) \longleftrightarrow$ M-component column-vectors

$$\mathcal{V}_x \cong f(x)^*$$

Using the Plücker embedding of $\mathrm{Gr}(M, N)$ into $\mathbb{C}P(\tilde{N}-1)$

$$\tilde{N} = \binom{N}{M}$$

$$\begin{array}{ccccc}
\mathrm{Det}\ \mathcal{V} & \xrightarrow{\ \overline{f}\ } & \mathrm{Det}\ \mathcal{T}^* & \xrightarrow{\ \overline{i_{\mathcal{P}}}\ } & \mathcal{O}(1) \\
\uparrow {\scriptstyle s_{i_1} \wedge \ldots \wedge s_{i_M}} & & \uparrow {\scriptstyle t_{i_1} \wedge \ldots \wedge t_{i_M}} & & \uparrow {\scriptstyle X_{i_1,\ldots,i_M}} \\
\Sigma & \xrightarrow{\ f\ } & \mathrm{Gr}(M, N) & \xrightarrow{\ i_{\mathcal{P}}\ } & \mathbb{C}P(\tilde{N}-1)
\end{array}$$

Suggests to consider $i_{\mathcal{P}}\, f$, which is generated by the $\tilde{N}$ sections $s_{i_1} \wedge \ldots \wedge s_{i_M}$ of Det $\mathcal{V}$.