NuHAG Numerical Harmonic Analysis Group

Banach Frames and Banach Gelfand Triples (and some applications)

Hans G. Feichtinger
hans.feichtinger@univie.ac.at
.
WEBPAGE: www.nuhag.eu

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Goals of my presentation

- introduce and motivate two not-so-standard functional analytic concepts;
- show how to formulate the basic setting and how it can be used;
- provide motivation by mixing usual linear background with the situation encountered in Gabor Analysis;
- suggests a particular viewpoint which emphasizes *families of Banach function spaces* (over topological vector spaces).

Relevant tools for my work

- Banach modules (Factorization theorem)
- homogeneous Banach spaces, solid BF-spaces
- **Wiener amalgam spaces** (and function spaces)
- **Banach Gelfand Triples !!**
- **Coorbit Theory** (including metric approximation prop)
- the role of BUPUs and BAPUS
- decomposition spaces, clusters, coverings, etc.
- irregular sampling and reconstruction
- spline-type spaces (and sampling)
- **Gabor analysis** (foundations and applications)
- Gabor multipliers (theory and applications)
- from linear algebra to distribution theory
- generalized stochastic process from a FA viewpoint

The role of MATLAB/OCTAVE (math.softw.)

It is one of my side-goals to emphasize the possibility of *teaching Fourier Analysis* starting from Linear Algebra (I plan to give a short presentation at the MATLAB EXPO in Muenich on this topic, coming up: May 10th 2016).

MATLAB (by def.: A MATrix LABoratory) FOR ME: **linear algebra in a box** (a software covering more or less all the aspects of linear algebra, and certainly more than most people can teach in a two semester course). Using MATLAB allows to make things more concrete and even clarify concepts (like dual spaces) in a very practical sense (can be discussed separately with those interested in this subject).

At NuHAG we have plenty of software (and a PhD thesis) on this subject, exercises, demonstrations, tutorials.

The role of MATLAB/OCTAVE II

The different roles of MATLAB in my work:

1. Experimental Mathematics, checking for numbers
2. Building finite models which are correctly implementing Fourier and TF-analysis over finitegroups;
3. Visualize facts
4. Verify/estimate condition numbers, eigenvalues
5. more to come...

There will be no place here to discuss how the finite models approximate the continuous (infinite dimensional) one!

Changes in the World, Changes in Mathematics?

Some GENERAL THOUGHTS:
We are obviously living in a world of changes.

`And sometimes mathematicians are proud of the fact that so little has changed in MATHEMATICS!`

Very much in the same way as **Classical Musicians** are (and should be) proud of the fact that they are performing the classical pieces in an almost perfect way, that we have master recordings of all the important ones. But on the *Creative Side* of the Music Business we see al kinds of Popular Music, Musicals, Rock, and for example **Jazz**, which lives from "breaking strict rules", but it has its own rules of improvisation. It's not going without rules.

Changes in the World, Changes in Mathematics? (2)

Taking a comparison from another domain of real life: **Mobility**

For a century (since the time of Lebesgue and Ford) this meant: How should one build CARS, how should one prepare the road infra-structure for these vehicles, from highways to parking houses. The *postmodern question* is not anymore: how to build a faster or more shiny car, but it is: how can we (in an ecological and economical responsible way) satisfy the NEEDS OF MOBILITY.

The young generation is not anymore working in order to buy cars. They use car-sharing pools, TESLA is just announcing a revolution concerning electric cars, so the ability of building powerful Diesel engines is not securing a bright future in car industry.

So were do we start, what are the current tools?

- We certainly have to understand linear algebra, vector spaces, linear mappings, bases, coordinate systems, matrix analysis;
- We cannot avoid NOT-FINITE-dimensional signal spaces on a continuous world (signals on $\mathbb{R}^d$, like images), so we have to resort to functional analysis, which could be seen as the STUDY of such vector spaces through their FINITE-dimensional subspaces (cf. weak convergence!)
- We have to work with norms (perhaps families of seminorms)
- We have to allow measures and distributions (Dirac!) ...

But the concrete setting that we choose to formulate our results is not necessarily optimal if we just make use of the tools which our ancestors have developed and provided to us. It is legitimate to ask, if they are useful nowadays.

The producers versus the costumers viewpoint

As mathematicians we are more like the car-producers, the pharmaceutical industry, the technic freaks, or the extreme climbers in alpinism. The beauty and difficulty of our products/ achievements is giving us reputation and academic careers. But it is healthy to look from time to time into the value of our achievments for a broader group of partners or costumers: e.g. the applied scientist of all kinds (not only physics & computer science)! Are the benefitting from our achievements. Is the level of refinement of individual tools (e.g. function spaces) in a good equilibrium with the demonstration of their usefulness:
Does anyone provide consumer reports, telling the users, which function space to use for which task? And I think we should start thinking about an analogue for science!

What are good generating systems in $\mathbb{R}^n$?

A set of generators is good, if *all vectors in* $\mathbb{R}^n$ can be well represented with not too much effort. It could be considered as *well balanced* if the effort of representing the vector are *well comparable* for all vectors of length one.
So we have a **cost or representation** (which is the ℓ^2-norm of the coefficients required to represent the given unit length vector) and the *uniformity of costs* over the unit ball.
Formally: Given a sequence of vectors $\boldsymbol{A} = (\vec{a}_k)_{k=1}^n$

$$A\|\vec{x}\|^2 \leq \sum_{k=1}^{n} |\langle \vec{x}, \vec{a}_k \rangle|^2 \leq B\|\vec{x}\|^2 \qquad (1)$$

with optimal parameters $A, B > 0$ *iff* $\mathrm{Col}(\boldsymbol{A}) = \mathbb{C}^m$.

What are good generating systems in $\mathbb{R}^n - II$?

Obviously an orthonormal system is a perfect generating system, because one can choose $A = B = 1$ in equation (1).
Is there a converse to this equation (A sequence of vectors in $\mathbb{C}^m$ or $\mathbb{R}^m$ satisfying (1) with $A = B$ is called a *tight frame*) valid. Or equivalently (and maybe more surprising), is any system of vectors which allows to represent arbitrary vectors as

$$\vec{x} = \sum_{k=1}^{n} \langle \vec{x}, \vec{a}_k \rangle \vec{a}_k$$

necessarily an ONB for $\mathbb{C}^m$?

What are good generating systems in $\mathbb{R}^n$ – *II*?

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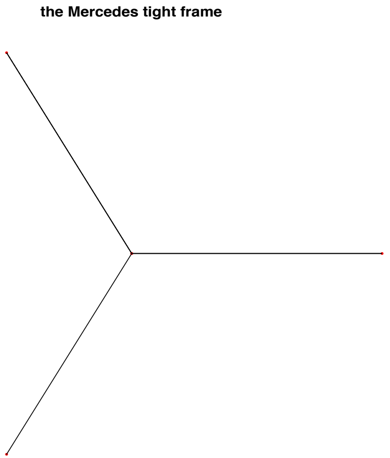
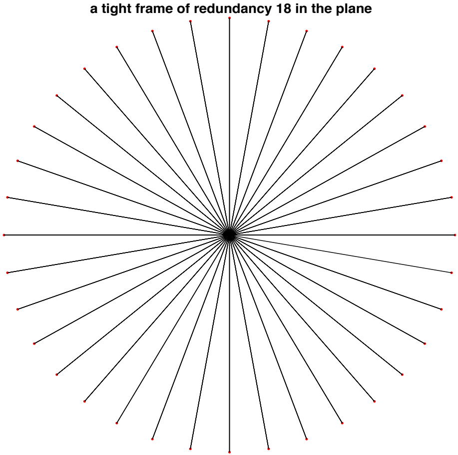
necessarily an ONB for $\mathbb{C}^m$?
OF COURSE NOT! (for $m < n$ no hope, for $m = n$: OK! but for $m > n$?) *There are many easy examples!*

The Mercedes frame!



Tight frames of high redundancy



Frames in Hilbert Spaces

The above inequality (which on the basis of compactness arguments is valid if and only if the sequence $(\vec{a}_k)$ is a set of generators for $\mathbb{C}^m$) can be taken as a definition.
Replace the sequence by an indexed family of vectors $(h_i)_{i \in I}$ and require that for some $A, B > 0$ one has for any $h \in \mathcal{H}$ ones has:

$$A\|h\|^2 \leq \sum_{i \in I} |\langle h, h_i \rangle|^2 \leq B\|h\|^2 \qquad (2)$$

Note that such a set has to be *total* (the closed linear span is all of $\mathcal{H}$), but the converse is *not valid!*. Costs may depend in an unbounded way on the direction of the vector $h \in \mathcal{H}$.

Riesz Basic Sequences

Frames are the *correct version* for the concept of generating systems in a Hilbert space (incorporating stability assumptions). There is also a natural concept of *stable linear independence*. Instead of saying, that zero cannot be represented as non-trivial vector one is quantifying the minimal (positive) length of a linear combination of a given set of vectors, using coefficients of length 1 in $\mathbb{C}^n$.

The formal definition of a *Riesz basic sequence* reads as follows.

Definition

A family $(b_j)_{j\in J}$ is called a RBS (or a *Riesz basis for its closed linear span*) if and only if there exist constants $C, D > 0$ such that

$$C\|\vec{c}\|^2 \leq \|\sum_{j\in J} c_j b_j\|_2^2 \leq D\|\vec{c}\|^2, \forall \vec{c} \in \ell^2(J). \qquad (3)$$

Gabor Analysis is a perfect place to study frames

We have to learn about the following expressions:

- translation operators T_x;
- frequency shifts or modulation operators M_s;
- time-frequency shift operators $\pi(\lambda) = M_s \circ T_t$, for $\lambda = (t, s)$;
- the Short-time Fourier Transform of $f \in \mathcal{S}'(\mathbb{R}^d)$, with window $g \in \mathcal{S}(\mathbb{R}^d)$:

$$STFT_g f(t, s) = \langle f, \pi(\lambda(g)) \rangle, \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

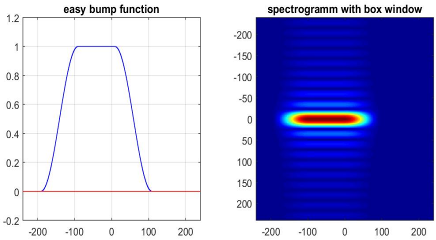
- *Gabor families* are families of the form $(\pi(\lambda_i)g)_{i \in I}$;
- We are mostly interested in Gaborian frames (for signal representations) Gaborian RBS (mobile communication!), and ??? Gaborian Riesz basis! (D.!Gabor, 1946 paper!).

Is there an orthonormal Gaborian Basis for $(L^2(\mathbb{R}), \|\cdot\|_2)$?

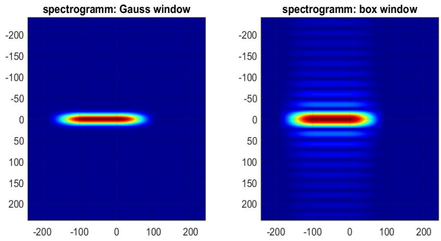
Yes of course. One can take the indicator function $\mathbf{1}_{[0,a]}$ for any $a > 0$, take all of its translates along $a\mathbb{Z}$ and then do a Fourier expansions (in the sense of a-periodic functions) of the pieces.

But how does the spectrogram (STFT) of a nice function are rather broad (spread out in the frequency direction, no integrable, even if the signal f is a nice bump function!). This is due to the bad decay properties of $\mathcal{F}(box) = SINC$ in this case.
Consequently the box-car function is *not a good window* for local Fourier analysis! Let us illustrate this by some MATLAB pictures.

Bad spread of spectrogram in frequency direction

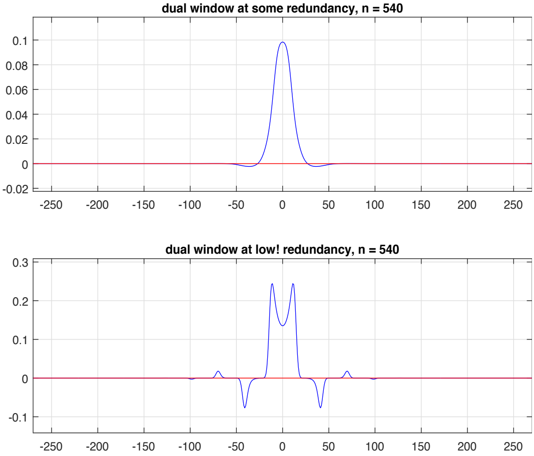


Ideal concentration for Gaussian Window



TF-concentration of Dual of Gaussian Window

Two dual windows, for different lattice parameters:



Are there good orthonormal Gaborian bases for $L^2(\mathbb{R}^d)$?

Here, in *contrast to the situation in wavelet theory* one has to say:

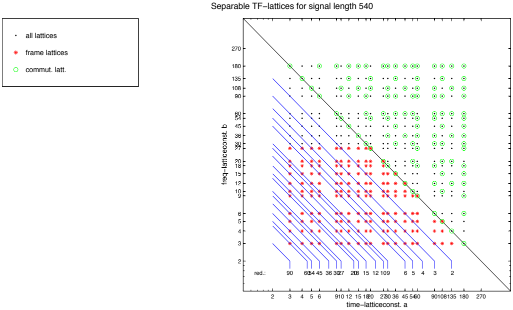
Unfortunately the answer is *negative*!

One can construct suitable ONBs, like *local Fourier basis* of so-called Wilson bases which are rather close to Gaborian bases, but there is NO Gaborian Riesz basis with e.g. a Schwartz window g (and not even with $g \in \boldsymbol{S}_0(\mathbb{R}^d)$).

Consequently the work in Gabor Analysis has forced us to think more carefully about "good atomic representations" through frames, including *Banach frames for families of function spaces*, in fact for families of modulation spaces.

The better *symmetry* between the time and the frequency variables (compared to time and scale in wavelet theory) also is at the foundation of a particular duality principle (can be derived from Poisson's formula for the symplectic Fourier transform).

The Ron-Shen Duality Theory



lower part: small lattice parameters $>$ frames, red > 1;
upper part: large frame constants $>$ Gaborian Riesz bases,
In the middle: **critical line, redundancy** $= 1$.

The Ron=Shen duality explained

The Ron-Shen duality (later formulated for general lattices by Feichtinger-Kozek) states:
If there is a lattice such that the regular Gabor system generated from (g, Λ), where g is the Gabor atom, is a Gabor frame, then (and only then) is the adjoint Gabor system (g, Λ°) a Gaborian Riesz basis. Moreover the dual Gabor atom (resp. the generator of the biorthogonal Gaborian RBS) are the same (up to normalization), and furthermore the condition number of the frame operator and the Gram matrix (of the RBS) are the same.
Let us mention that for $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ one has $\Lambda^\circ = 1/b \cdot \mathbb{Z}^d \times 1/a \cdot \mathbb{Z}^d$.

So what are we looking for?

Within our landscape of Gabor lattices we look out for lattices of *not too high redundancy* which allows us to build good Gabor frames (with well TF-concentrated dual window, or even better well TF-concentrated tight Gabor frames) which do not have too high redundancy, i.e. corresponding to lattices near the critical line. For mobile communication we search for Gaborian Riesz basis of "high spectral efficiency" (so high redundancy, coming close to the crictial line from above, while again still having good biorthogonal generators).

Of course in each case one can also consider (the rich family) of non-separable lattices, i.e. general lattices within $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, not just those of the form $a\mathbb{Z}^d \times b\mathbb{Z}^d$.

What is the problem with Gabor's suggestion?

Formally the technical problem with Gabor's idea of using a maximally TF-localized window (namely the Gauss function g_0, with $g_0 = e^{-\pi|t|^2}$, which is a minimizer to the Heisenberg uncertainty relation) is the Balian- Low theorem. In fact, while *most likely*, formulated in a modern terminology, D. Gabor was *hoping* to suggest a Riesz basis obtained from a family of TF-shifts of the Gauss-function along the integer lattice $\mathbb{Z}^2$, i.e. with $a = 1 = b$, the analysis in the 80th showed that it is neither a frame nor a Riesz basic sequence, so of course *not a Riesz basis*. What has been overlooked by D. Gabor (at least there is no indication that he was aware of this problem) that the more one comes to the critical lattice (e.g. by letting $a = b$ tend to the critical value $a = 1$) the more delocalized (in the TF-sense) the dual window is, i.e. the optimal localization of the Gabor atoms is in sharp contrast with the significant unsharpness of the overall system (Gabor and dual Gabor frame!).

The frame-condition in a diagram

Both the frame condition and the Riesz basis sequence can be characterized by a commutative diagram involving Hilbert spaces. The differ only by the direction of the arrows (typically between ℓ^2 and some Hilbert space $\mathcal{H}$).

For FRAMES we have an injective embedding of $\mathcal{H}$ into $\ell^2(I)$, via $h \mapsto (\langle h, h_i \rangle)_{i \in I}$, which established an isomorphism between $\mathcal{H}$ and the (!closed) range of this coefficient mapping.

For RIESZ BASIC SEQUENCES we look at the synthesis mapping in a similar way: $\vec{c} \mapsto \sum_{j \in J} c_j b_j$.

Let us recall: we are dealing with Hilbert spaces, and therefore in this case the isomorphic identification of one Hilbert space with the a closed subspace as *the same as* the identification with a closed and (orthogonally) complemented subspace.

REFERENCE

For a continuation of the talk and the corresponding slides look-up the talk held in Trondheim, at the link

http://www.univie.ac.at/nuhag-php/program/talks_details.php?id=3013

A diagram

Definition

A linear mapping $\mathcal{C}$ is defining a retract from $\boldsymbol{X}$ into $\boldsymbol{Y}$ if there exists a left inverse to it, i.e. a mapping $\mathcal{R}$ from $\boldsymbol{Y} into \boldsymbol{X}$ such that $\mathcal{R} \circ \mathcal{C} = Id_{\boldsymbol{X}}$.

The method of retracts is often used to push results known for vector-valued $\boldsymbol{L}^p$-spaces to the setting of Besov- or modulation spaces.

A DIAGRAM

Frames and Riesz Bases: Commutative Diagrams

Think of $\boldsymbol{X}$ as something like $\boldsymbol{L}^p(\mathbb{R}^d)$, and $\boldsymbol{Y} = \ell^p$:

Frame case: $\mathcal{C}$ is injective, but not surjective, and $\mathcal{R}$ is a left inverse of $\mathcal{R}$. This implies that $P = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range Y_0 of $\mathcal{C}$ in $\boldsymbol{Y}$:

$$
\begin{tikzcd}
 & \boldsymbol{Y} \arrow[dl, "\mathcal{R}"'] \arrow[d, "P"] \\
\boldsymbol{X} \arrow[r, shift right, "\mathcal{C}"'] & Y_0 \arrow[l, shift right, "\mathcal{R}"']
\end{tikzcd}
$$

Riesz Basis case: E.g. $\boldsymbol{X}_0 \subset \boldsymbol{X} = L^p$, and $\boldsymbol{Y} = \ell^p$:

$$
\begin{tikzcd}
\boldsymbol{X} \arrow[d, "P"'] \arrow[dr, "\mathcal{C}"] & \\
\boldsymbol{X}_0 \arrow[r, shift left, "\mathcal{C}"] & \boldsymbol{Y} \arrow[l, shift left, "\mathcal{R}"]
\end{tikzcd}
$$