Mixing times for exclusion processes with open boundaries

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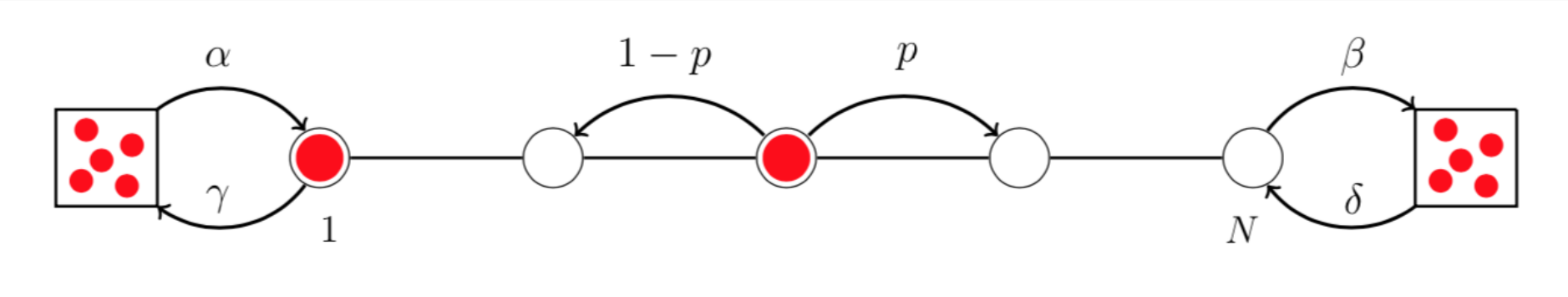
The model: exclusion process with open boundaries

Consider an exclusion process $(\eta_t)_{t\geq 0}$ on the segment $[1, 2, \ldots, N]$. Particles jump to the right at rate p, to the left at rate $1-p$, and they can only jump when the target site is empty. In addition, particles can enter at the left side with rate α and exit at the left side at rate γ, and enter at the right side at rate δ and exit at the right side at rate β.



Parameters: $p \in [\frac{1}{2}, 1]$ and $\alpha, \beta, \gamma, \delta \geq 0$.

More formal description

The process $(\eta_t)_{t\geq 0}$ is a Feller process with state space $\Omega_N := \{0,1\}^N$ generated by

$$\begin{aligned}\mathcal{L}f(\eta) = \mathcal{L}_{\text{ex}}f(\eta) + \alpha(1-\eta(1))\ \left[f(\eta^1) - f(\eta)\right] + \gamma\eta(1)\ \left[f(\eta^1) - f(\eta)\right] \\ + \delta(1-\eta(N))\left[f(\eta^N) - f(\eta)\right] + \beta\eta(N)\left[f(\eta^N) - f(\eta)\right]\end{aligned}$$

where $\eta^x \in \Omega_N$ denotes the configuration in which we flip the values at position x in $\eta \in \Omega_N$. Here

$$\begin{aligned}\mathcal{L}_{\text{ex}}f(\eta) = &\sum_{x=1}^{N-1} p\ \eta(x)(1-\eta(x+1))\left[f(\eta^{x,x+1}) - f(\eta)\right] \\ &+ \sum_{x=2}^{N} q\ \eta(x)(1-\eta(x-1))\left[f(\eta^{x,x-1}) - f(\eta)\right]\end{aligned}$$

where $\eta^{x,y} \in \Omega_N$ denotes the configuration in which we exchange the values at positions x and y in $\eta \in \Omega_N$.

Why study this?

- Basic model of an interacting particle system
- Toy model for non-equilibrium behaviour
- A lot of recent interest in fluctuations in equilibrium, often for $p = p_N \to \frac{1}{2}$ (called weakly asymmetric case)
- Interest from statistical physics as well as combinatorics

For each N, $(\eta_t)_{t\geq 0}$ is a Markov chain in continuous time with state space $\Omega_N := \{0,1\}^N$. Assume that $p, \alpha, \beta, \delta, \gamma$ are such that the process has a unique invariant distribution μ. Then, the law of $(\eta_t)_{t\geq 0}$ converges to μ for any starting configuration.

How fast do the laws of $(\eta_t)_{t\geq 0}$ converge towards μ as $N \to \infty$?

Goal: study the order of $t^N_{\text{mix}}(\varepsilon)$ when N goes to infinity.

Mixing time

For a sequence of Markov chains $(\eta_t^N)_{t\geq 0}$ with invariant distributions $\mu = \mu_N$ define

- Total variation distance

$$\|\mathbb{P}_\eta(\eta_t \in \cdot) - \mu\|_{\mathrm{TV}} := \frac{1}{2}\sum_{\xi\in\Omega_N} |\mathbb{P}_\eta(\eta_t = \xi) - \mu(\xi)|$$

- Maximal distance to equilibrium at time $t \geq 0$

$$d(t) := \max_{\eta\in\Omega_N} \|\mathbb{P}_\eta(\eta_t \in \cdot) - \mu\|_{\mathrm{TV}}$$

- ε-**mixing time** for $0 < \varepsilon < 1$

$$t_{\mathrm{mix}}^N(\varepsilon) := \inf\{t \geq 0 : d(t) < \varepsilon\}$$

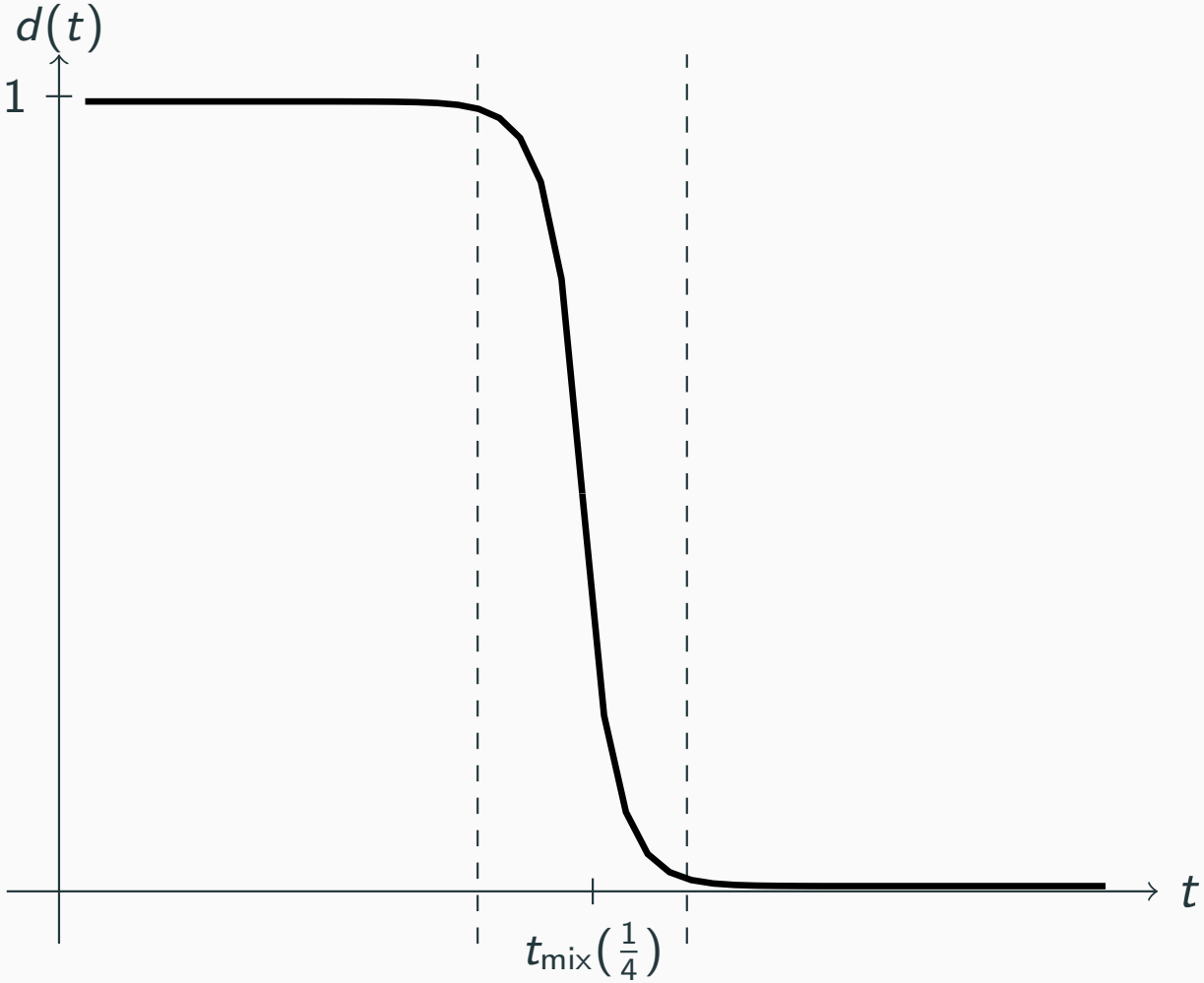
Cutoff phenomenon

A sequence of Markov chains $(\eta_t^N)_{t\geq 0}$, satisfies **cutoff** if the distance to equilibrium drops from near 1 to near 0 over a time interval which is asymptotically smaller than the mixing time, i.e.

$$\lim_{N\to\infty} \frac{t_{\text{mix}}^N(\varepsilon)}{t_{\text{mix}}^N(1-\varepsilon)} = 1$$

See "Markov Chains and Mixing Times", David Levin, Yuval Peres, Elizabeth Wilmer.

$(\eta_t^N)_{t\geq 0}$ satisfies **pre-cutoff** if the first order of the ε-mixing times can be bounded within two constants which do not depend on ε,


$d(t)$
1
t
$t_{\mathrm{mix}}(\frac{1}{4})$

Example: one particle

Take a lazy simple symmetric random walk on the segment $[1, 2, \ldots, N]$. Then $t_{\text{mix}}^N(\varepsilon)$ is of order N^2 and there is no cutoff.
Take a lazy simple random walk with drift on the segment $[1, 2, \ldots, N]$. Then

$$\lim_{N\to\infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N} = \frac{1}{2p-1}\,.$$

In particular, the cutoff phenomenon occurs.

Previous results

Let $\alpha = \beta = \gamma = \delta = 0$, $p = \frac{1}{2}$ and start with k particles.

Theorem (David Wilson, 2001; Hubert Lacoin, 2016)

Let $2k \leq N$ for all $N \in \mathbb{N}$ and $k = k(N) \to \infty$ with $N \to \infty$. Then

$$\lim_{N \to \infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N^2 \log k} = \frac{1}{\pi^2}$$

for all $0 < \varepsilon < 1$. In particular, the cutoff phenomenon occurs.

Previous results

Let $\alpha = \beta = \gamma = \delta = 0$, $p > \frac{1}{2}$ and start with k particles.

Theorem (Itai Benjamini, Noam Berger, Christopher Hoffman, Elchanan Mossel, 2002; Cyril Labbé, Hubert Lacoin 2019)

For the asymmetric simple exclusion process with $k \in \{1, \dots, N-1\}$ particles and $p \neq \frac{1}{2}$, we have that

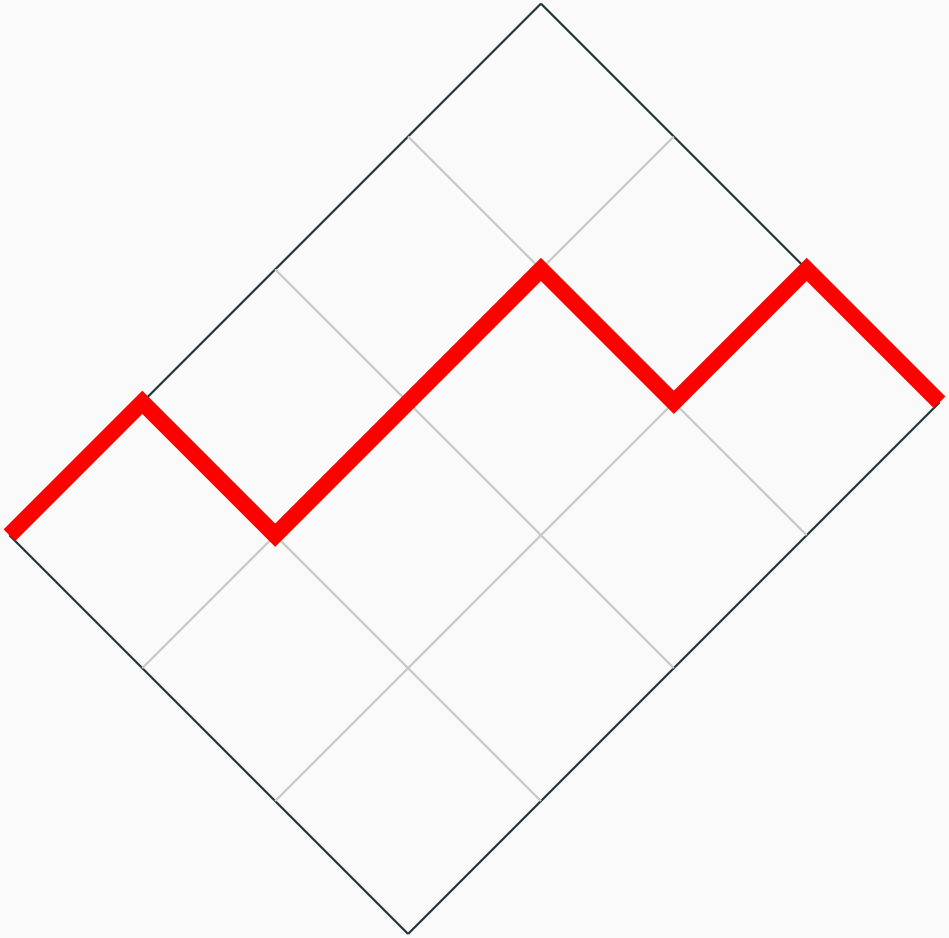
$$\lim_{N \to \infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N} = \frac{const}{2p-1}$$

for all $0 < \varepsilon < 1$, where const depends on the ratio of k and N. In particular, the cutoff phenomenon occurs.

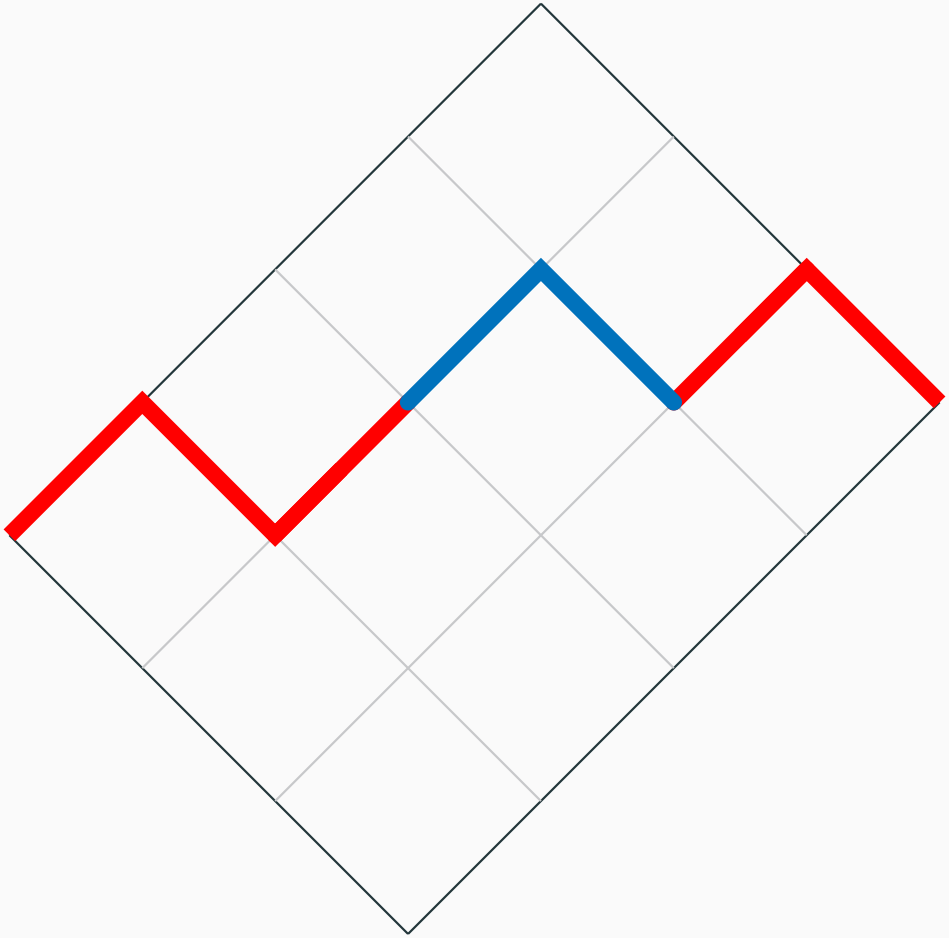
Differences in our model

- Number of particles is not conserved
- Process is in general not reversible
- In general no closed formula for μ (a notable exception comes later)

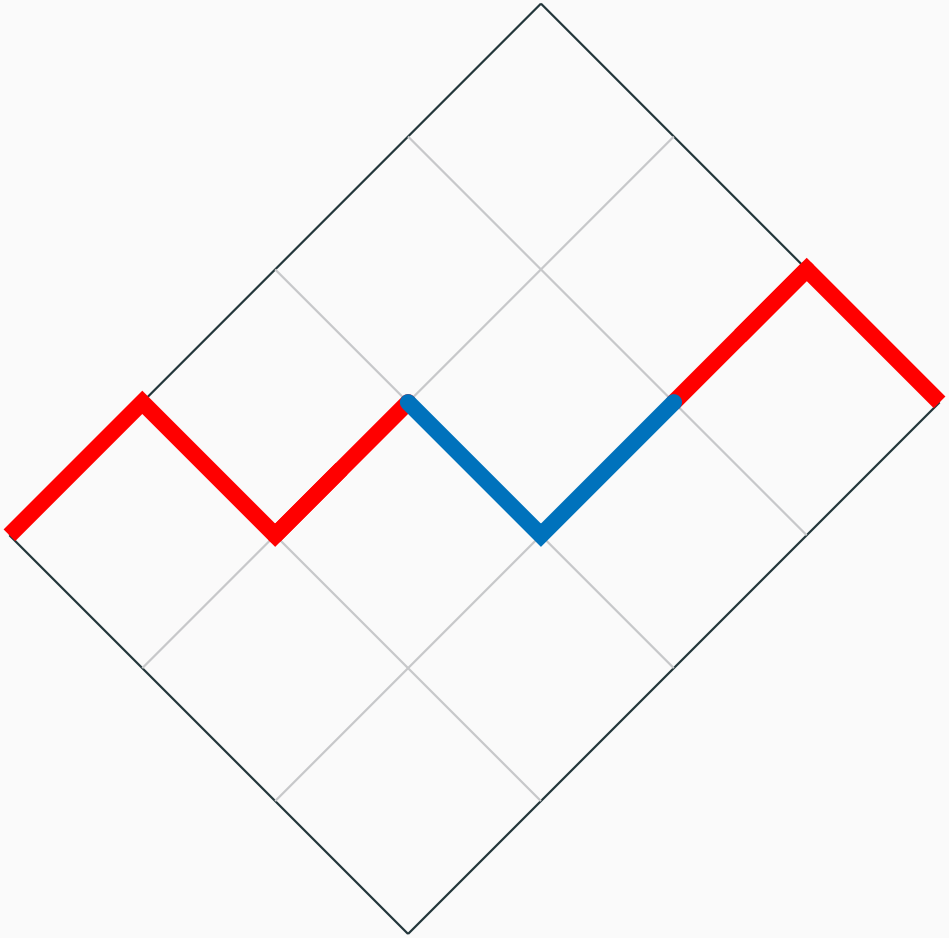
Height function representation



Height function representation



Height function representation



Mixing times and coupling

The height functions give a partial order $\succeq_{\mathsf{h}}$.

Lemma

Let P *be a coupling which is monontone w.r.t.* $\succeq_{\mathsf{h}}$*, let* τ *denote the first time, at which the processes started from* 1 *and* 0*, respectively, coincide in the coupling* P*. If for some* $s \geq 0$

$$\mathsf{P}(\tau \geq s) \leq \varepsilon$$

holds, then the ε*-mixing time satisfies* $t^{N}_{\mathsf{mix}}(\varepsilon) \leq s$*.*

Results

Theorem (Symmetric two - sided)

For $p = \frac{1}{2}$, the ε-mixing time of the simple exclusion process with open boundaries satisfies

$$\frac{1}{\pi^2} \leq \liminf_{N\to\infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N^2 \log(N)} \leq \limsup_{N\to\infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N^2 \log(N)} \leq C$$

for all $\varepsilon \in (0,1)$ and some constant $C = C(\alpha, \beta, \gamma, \delta)$.

Results

Theorem (Symmetric one - sided)

For $p = \frac{1}{2}$, suppose that $\max(\alpha, \gamma) = 0$ and $\min(\beta, \delta) > 0$ holds. Then for all $\varepsilon \in (0, 1)$, the ε-mixing time of the simple exclusion process with open boundaries satisfies

$$\lim_{N \to \infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N^2 \log(N)} = \frac{4}{\pi^2}\,.$$

In particular, the cutoff phenomenon occurs.

Results

Theorem (Asymmetric one - sided)

Suppose that $p > \frac{1}{2}$, $\min(\alpha, \beta) = 0$ *and* $\max(\alpha, \beta) > 0$ *holds. Moreover,* $\gamma, \delta \geq 0$ *are arbitrary. Then for all* $\varepsilon \in (0, 1)$, *the* ε*-mixing time of the simple exclusion process with open boundaries satisfies*

$$\frac{1}{2p-1} \leq \liminf_{N\to\infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N} \leq \limsup_{N\to\infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N} \leq C$$

for some constant $C = C(p, \alpha, \beta, \gamma, \delta)$.

Results

Theorem (Reverse bias regime)

Suppose that $\max(\alpha, \beta) = 0$ *and* $p \in \left(\frac{1}{2}, 1\right)$ *holds. Then for all* $\varepsilon \in (0, \frac{1}{2})$*, we have that*

$$\lim_{N \to \infty} \frac{\log\left(t_{\text{mix}}^{N}(\varepsilon)\right)}{N} = \log\left(\frac{p}{1-p}\right)$$

holds whenever $\min(\gamma, \delta) = 0$ *and* $\max(\gamma, \delta) > 0$*. If* $\min(\gamma, \delta) > 0$ *holds, we have that*

$$\lim_{N \to \infty} \frac{\log\left(t_{\text{mix}}^{N}(\varepsilon)\right)}{N} = \frac{1}{2} \log\left(\frac{p}{1-p}\right) .$$

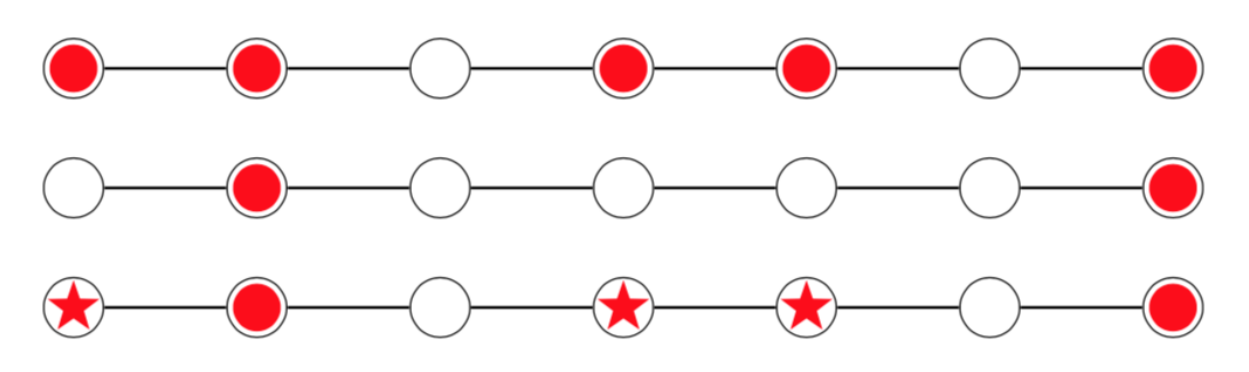
Simulations

Due to Dominik Schmid, thanks also to Patrik Ferrari!
https://sites.google.com/view/dominik-schmid/simulations/sep-open-boundaries

Ingredients of the proofs

- Lower bounds for hitting times:
  - "approximative" version of Wilson's lemma (continuous time variant of a theorem by Danny Nam and Evita Nestoridi)
- Upper bounds for hitting times:
  - second class particles
  - current
  - comparison with multi-species exclusion processes
  - censoring inequality (Yuval Peres and Peter Winkler)

Second class particles



- empty sites have less priority than second class particles, second class particles have less priority than first class particles
- update edges: exchange values at x and $x+1$ with prob. p if x has higher priority than $x+1$ and with prob. $1-p$ otherwise
- If we couple the two processes, the disagreements move as second class particles!

Current

Define the **current** J_t^N as follows:
Assume $p \in \left(\frac{1}{2}, 1\right]$, $\min(\alpha, \beta) > 0$.
J_t^{N+} number of particles which have entered at the left side until time t,
J_t^{N-} number of particles which have exited at the left side until time t,

$$J_t^N := J_t^{N+} - J_t^{N-} \quad \text{for all } t \geq 0$$

Now, take $a = a(\alpha, \gamma, p)$ and $b = b(\beta, \delta, p)$, given by

$$a = \frac{1}{2\alpha}\left(2p - 1 - \alpha + \gamma + \sqrt{(2p - 1 - \alpha + \gamma)^2 + 4\alpha\gamma}\right), b = \dots$$

Intuition: our process interpolates between two Bernoulli-product measures on the integers with densities $\frac{1}{1+a}$ and $\frac{b}{1+b}$, respectively.

Example

If $p = 1, \gamma = 0, \delta = 0$, $0 < \alpha, \beta < 1$, then $\frac{1}{1+a} = \alpha$ and $\frac{b}{1+b} = 1 - \beta$.

Theorem (Richard Brak, Sylvie Corteel, John Essam, Robert Parviainen, Andrew Rechnitzer, 2006)

Suppose that $\min(\alpha, \beta) > 0$ *and* $a = \frac{1}{b}$ *holds for* a *and* b *as before, which means* $\frac{1}{1+a} = \frac{b}{1+b}$.
Then for every configuration $\eta \in \Omega_N$*, we have that*

$$\mu(\eta) = \left(\frac{1}{1+a}\right)^{|\eta|} \left(\frac{a}{1+a}\right)^{N-|\eta|}$$

where $|\eta| := \sum_{i=1}^{N} \eta(i)$ *denotes the number of particles in configuration* η.

Lemma (Masaru Uchiyama, Tomohiro Sasamoto, Miki Wadati)

The current $(J_t^N)_{t\geq 0}$ *of the simple exclusion process with open boundaries satisfies*

$$\lim_{t\to\infty} \frac{J_t^N}{t} = J_N$$

almost surely for a sequence $(J_N)_{N\in\mathbb{N}}$ *with* $\lim_{N\to\infty} J_N = J$*, where*

$$J = J(a,b,p) := \begin{cases} (2p-1)\frac{a}{(1+a)^2} & \text{if } a > \max(b,1) \quad \text{“low density”} \\ (2p-1)\frac{b}{(1+b)^2} & \text{if } b > \max(a,1) \quad \text{“high density”} \\ (2p-1)\frac{1}{4} & \text{if } \max(a,b) \leq 1 \quad \text{“max current”.} \end{cases}$$

Then the current $(J_t^N)_{t\geq 0}$ *of the simple exclusion process with open boundaries satisfies*

$$\lim_{t\to\infty} \frac{J_t^N}{t} = J_N$$

almost surely for some sequence $(J_N)_{N\in\mathbb{N}}$ *with* $\lim_{N\to\infty} J_N = J$*.*

Example

If $p = 1, \gamma = 0, \delta = 0$, $0 < \alpha, \beta < 1$ we get

$$J = J(\alpha, \beta) := \begin{cases} \alpha(1-\alpha) & \text{if } \alpha < \frac{1}{2}, \alpha < \beta \text{ “low density”} \\ \beta(1-\beta) & \text{if } \beta < \frac{1}{2}, \beta < \alpha \text{ “high density”} \\ \frac{1}{4} & \text{if } \alpha \geq \frac{1}{2}, \beta \geq \frac{1}{2} \text{ “max current”}. \end{cases}$$

Theorem (Asymmetric two-sided)

For parameters $\alpha, \beta, \gamma, \delta \geq 0$ and $p > \frac{1}{2}$, suppose we are in the low density phase or in the high density phase. Then there exists a constant $C = C(a, b, p) > 0$ such that the ε-mixing time of the simple exclusion process with open boundaries satisfies

$$\frac{1}{2p-1} \leq \liminf_{N\to\infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N} \leq \limsup_{N\to\infty} \frac{t_{\text{mix}}^N(\varepsilon)}{N} \leq C$$

for all $\varepsilon \in (0, 1)$.

Conjecture

When $p \neq \frac{1}{2}$ and $\max(a, b) < 1$, i.e. in the maximal current phase, we have that the ε-mixing time of the simple exclusion process with open boundaries is of order $N^{\frac{3}{2}}$ for all $\varepsilon \in (0, 1)$. Moreover, the cutoff phenomenon does **not** *occur.*

Reason for the conjecture:
We believe that the typical time for all second class particles to leave the segment is of order $N^{\frac{3}{2}}$, using a comparison to the typical fluctuations of a second class particle on $\mathbb{Z}$ in a Bernoulli-$\frac{1}{2}$-product measure, see Márton Balász and Timo Seppäläinen.

Open problems

Plenty!

– Higher dimensions?

– General graphs?

Recent progress for symmetric exclusion processes by Joe P. Chen and Rodrigo Marinho. Progress for the symmetric case with open boundaries announced by Joe P. Chen, Milton Jara, Rodrigo Marinho.

Thanks for your attention!