Representation Theories of the Symmetric Group and the Rook Monoid

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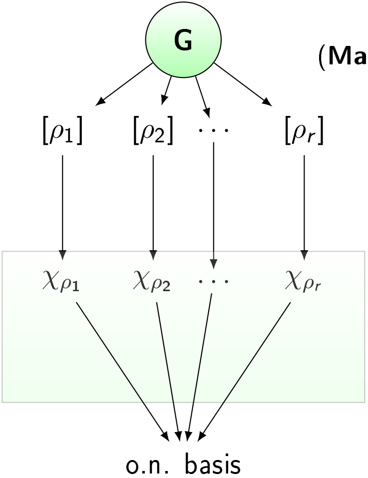
Finite Groups Representation Theory



Finite Group

(**Masche:** Representations are completely reducible.)

Irreducible Representations

Irreducible Characters

Class functions

$$\langle f, h\rangle_G = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{h(g)}$$

Theorem

The number of irreducible representations of G (up to isomorphism) is equal to the number of conjugacy classes of G.

Theorem[Brauer]

Let V be a complex vector space of finite dimension.

1. If ρ is a representation of G over V, then V is a $\mathbb{C}G$-module respecting the product

$$\sigma v := \rho(\sigma)(v);$$

2. If V is a $\mathbb{C}G$-module with product σv, then the map $\rho(\sigma) : v \to \sigma g$ belongs to $GL(V)$ and

$$\begin{array}{rccc} \rho : & G & \longrightarrow & GL(G) \\ & \sigma & \mapsto & \rho(\sigma) \end{array}$$

is a representation of G over V, for all $\sigma \in G$ and for all $v \in V$.

The special case of the Symmetric Group

n.º of isoclasses of Irr. Rep. = n.º of conjugacy classes

$\Downarrow$

Two permutations are conjugate if and only if they have the same cycle type

$\Downarrow$

n.º of conjugacy classes of S_n = n.º of partitions of n

Isoclasses of Irreducible Representations of $\mathcal{S}_n$

(1) $n = 1$

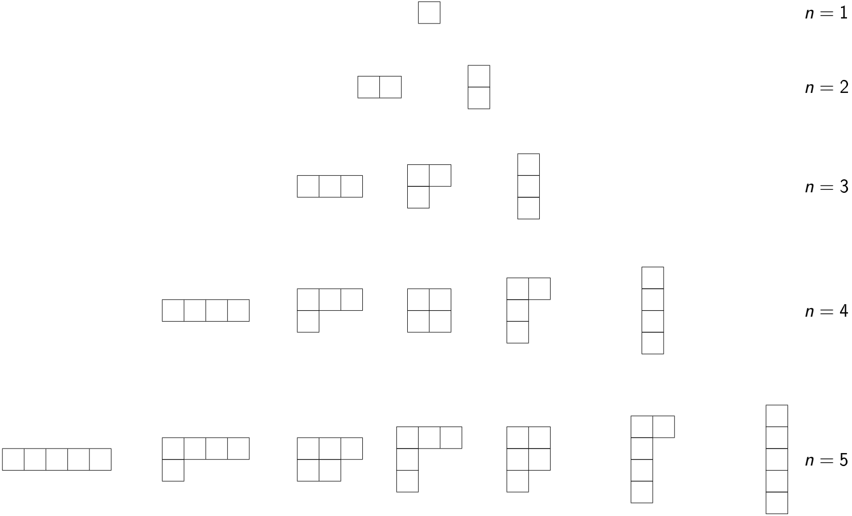
(2) (1,1) $n = 2$

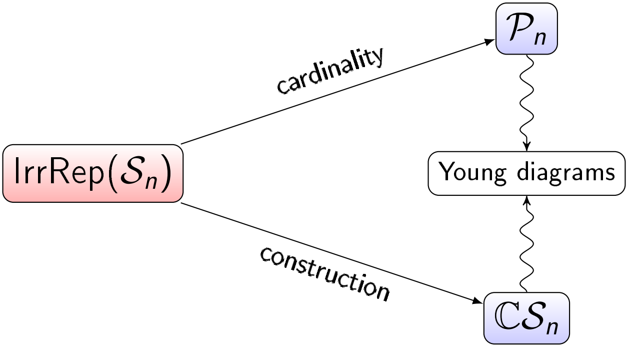
(3) (2,1) (1,1,1) $n = 3$

(4) (3,1) (2,2) (2,1,1) (1,1,1,1) $n = 4$

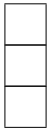
(5) (4,1) (3,2) (3,1,1) (2,2,1) (2,1,1,1) (1,1,1,1,1) $n = 5$

Isoclasses of Irreducible Representations of $\mathcal{S}_n$



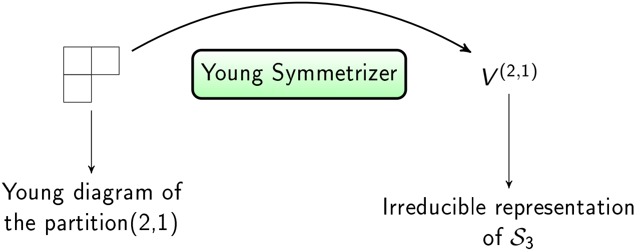

IrrRep($\mathcal{S}_n$)
cardinality
$\mathcal{P}_n$
Young diagrams
construction
$\mathbb{C}\mathcal{S}_n$

Example: Character table of $\mathcal{S}_3$

Conjugacy Classes	1 2 3  1 2 3 (1) (*)(*)(*)	1 2 3  1 2 3 (12) 1 2 3  1 2 3 (13) 1 2 3  1 2 3 (23) (* *)(*)	1 2 3  1 2 3 (123) 1 2 3  1 2 3 (132) (* * *)
 (3) trivial			
 (2,1) standard			
 (1,1,1) sign			

Classic Approach

$$\left\{\begin{matrix}\lambda \\ \text{partition of } n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}S^{\lambda} \\ \text{Specht module}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{classe of irreducible} \\ \mathcal{S}_n\text{−representations}\end{matrix}\right\}$$



Constructing the Specht Modules of $\mathcal{S}_3$

$\lambda_1 = (3)$	$\lambda_2 = (2,1)$	$\lambda_3 = (1,1,1)$
↓	↓	↓
$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$
↓	↓	↑
$R_{\lambda_1} = \mathcal{S}_3$	$R_{\lambda_2} = \{(1),(1\ 2)\}$	$R_{\lambda_3} = \{(1)\}$
$C_{\lambda_1} = \{(1)\}$	$C_{\lambda_3} = \{(1),(1\ 3)\}$	$C_{\lambda_3} = \mathcal{S}_3$
↓	↓	↓

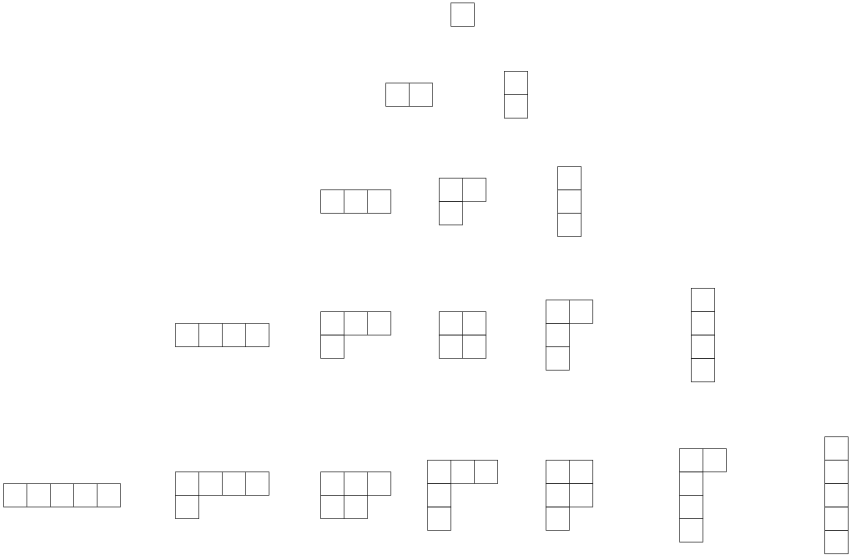
Young symmetrizers

$$\mathfrak{s}_{\lambda_i} = \left(\sum_{\sigma \in R_{\lambda_i}} \sigma\right)\left(\sum_{\sigma \in C_{\lambda_i}} \mathrm{sign}(\sigma)\sigma\right) \in \mathbb{C}\mathcal{S}_3$$

↓	↓	↓
$\mathbb{C}\mathcal{S}_3 \cdot \mathfrak{s}_{\lambda_1}$	$\mathbb{C}\mathcal{S}_3 \cdot \mathfrak{s}_{\lambda_2}$	$\mathbb{C}\mathcal{S}_3 \cdot \mathfrak{s}_{\lambda_3}$
$\mathbb{C} \sum_{\sigma \in \mathcal{S}_3} \sigma$	$\mathbb{C}\mathfrak{s}_{\lambda_2} + \mathbb{C}x$	$\mathbb{C} \sum_{\sigma \in \mathcal{S}_3} \mathrm{sgn}(\sigma)\sigma$

Theorem

When λ ranges over all distinct partitions of n, $\{\mathbb{C}\mathcal{S}_n \cdot \mathfrak{s}_\lambda\}$ is a full set of non-isomorphic simple $\mathbb{C}\mathcal{S}_n$-modules.



$\mathfrak{S}_{(1)}$

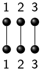
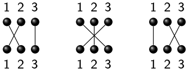
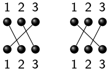
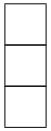
$\mathfrak{S}_{(2)}$ $\mathfrak{S}_{(1,1)}$

$\mathfrak{S}_{(3)}$ $\mathfrak{S}_{(2,1)}$ $\mathfrak{S}_{(1,1,1)}$

$\mathfrak{S}_{(4)}$ $\mathfrak{S}_{(3,1)}$ $\mathfrak{S}_{(2,2)}$ $\mathfrak{S}_{(2,1,1)}$ $\mathfrak{S}_{(1,1,1,1)}$

$\mathfrak{S}_{(5)}$ $\mathfrak{S}_{(4,1)}$ $\mathfrak{S}_{(3,2)}$ $\mathfrak{S}_{(3,1,1)}$ $\mathfrak{S}_{(2,2,1)}$ $\mathfrak{S}_{(2,1,1,1)}$ $\mathfrak{S}_{(1,1,1,1,1)}$

Example: Character table of $\mathcal{S}_3$

Conjugacy Classes	 (1) (*)(*)(*)	 (12) (13) (23) (* *)(*)	 (123) (132) (* * *)
 (3) trivial	1	1	1
 (2,1) standard	2	0	-1
 (1,1,1) sign	1	-1	1

Theorem[Wedderburn-Artin]

$$\mathbb{C}\mathcal{S}_n \simeq \mathbb{C}^{n_1 \times n_1} \times \ldots \times \mathbb{C}^{n_r \times n_r} \simeq \bigoplus_{\lambda \in \mathcal{S}_n^\wedge} \mathrm{GL}(V^\lambda)$$

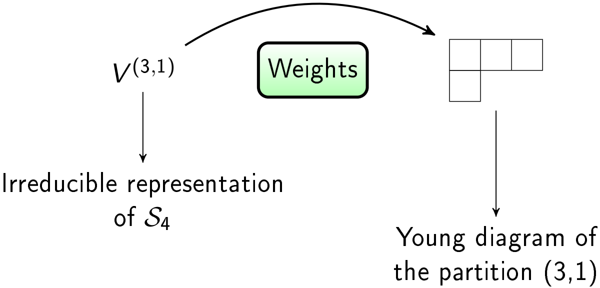
$$\mathbb{C}\mathcal{S}_3 \simeq \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \simeq \mathrm{GL}(\mathbb{C}\mathcal{S}_3 \cdot \mathfrak{s}_{\lambda_1}) \oplus \mathrm{GL}(\mathbb{C}\mathcal{S}_3 \cdot \mathfrak{s}_{\lambda_2}) \oplus \mathrm{GL}(\mathbb{C}\mathcal{S}_3 \cdot \mathfrak{s}_{\lambda_3}).$$

Theorem

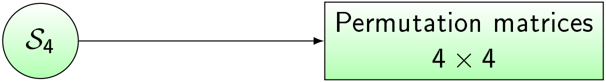
Let V be an irreducible $\mathbb{C}\mathcal{S}_n$-module.
The restriction $V|_{\mathcal{S}_{n-1}}$ is multiplicity-free.

Different Approach

$$\left\{\begin{array}{c}\text{classe of irreducible}\\ \mathcal{S}_n-\text{representations}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Spectrum}\\ \text{of } \mathbb{C}\mathcal{S}_n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Resume}\\ \text{of } \mathcal{S}_n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\lambda\\ \text{partition of } n\end{array}\right\}$$



Example: The standard representation of $\mathcal{S}_4$



$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$

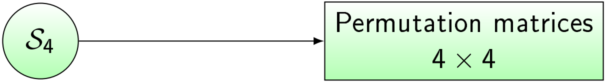
$$\rho : S_4 \to GL(V)$$

where for all $\sigma \in \mathcal{S}_4$

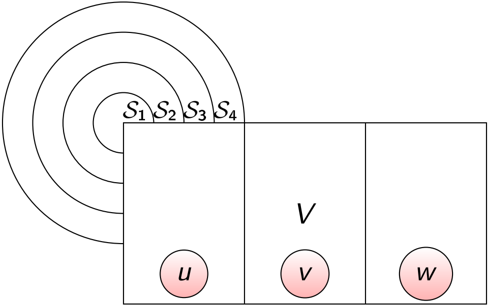
$$\rho(\sigma)(x_1, x_2, x_3, x_4) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}, x_{\sigma^{-1}(4)}).$$

ρ is an irreducible representation of $\mathcal{S}_4$

Example: The standard representation of $\mathcal{S}_4$



$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$



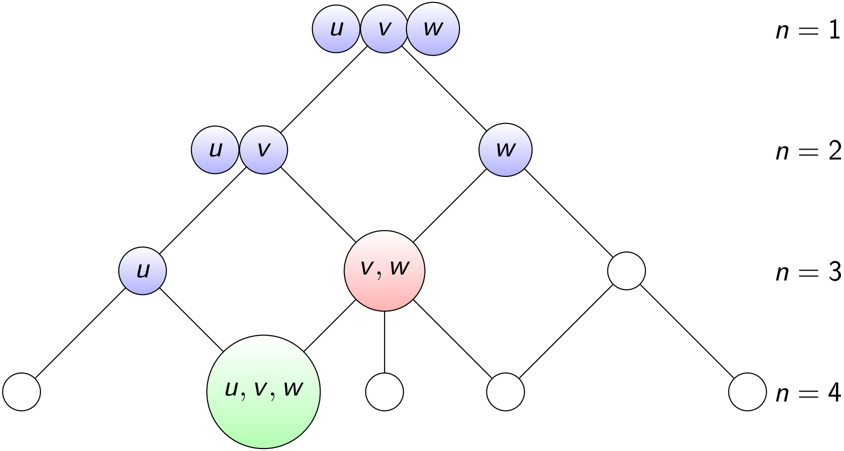
Gelfand-Zetlin Basis

Let V be a simple $\mathbb{C}S_n$-module whith canonical decomposition

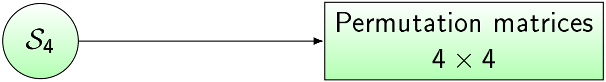
$$V^{\rho}|_{S_1} = \bigoplus_{T} V^{T}.$$

For each T_ρ we can choose a vector v_{T_ρ} form each V^{T_ρ} obtaining a basis $\{v_{T_\rho}\}$ of V,which we call **Gelfand-Zetlin basis**.

Branching Graph



Example: The standard representation of $\mathcal{S}_4$



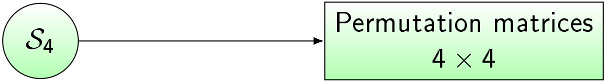
$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$

$$V = <(1,1,1,-3)> \oplus <(1,1,-2,0)> \oplus <(1,-1,0,0)>.$$

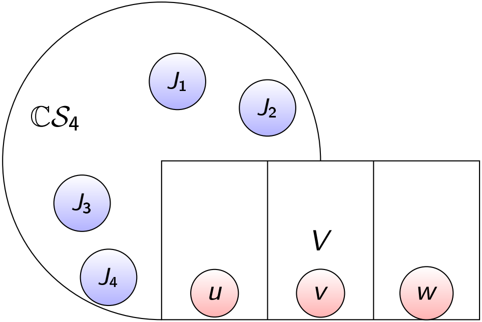
Let $u = (1,1,1,-3)$, $v = (1,1,-2,0)$, $w = (1,-1,0,0)$ and $\text{GZ} = \{u, v, w\}$ which is a basis of V.

(1)	$(1\ 2)$	$(2\ 3)$	$(3\ 4)$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{4}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix}.$

Example: The standard representation of $\mathcal{S}_4$



$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$

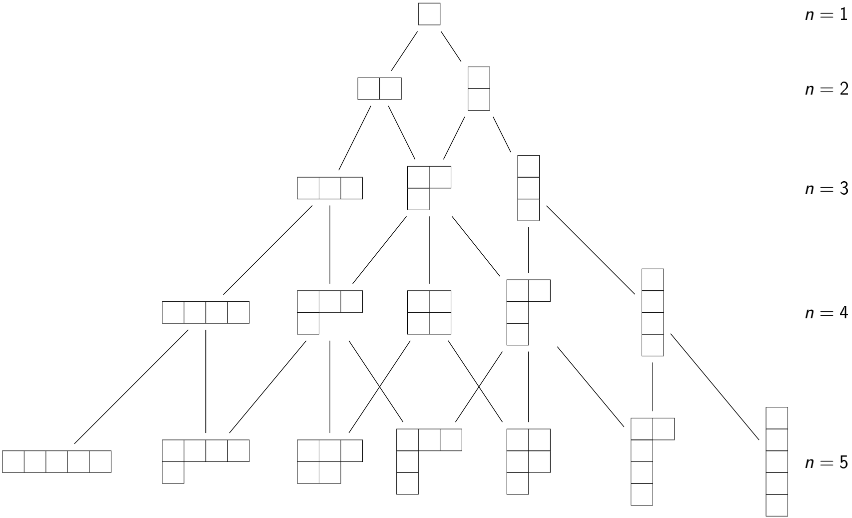


$J_1 = 0$
$J_2 = (1\ 2)$
$J_3 = (1\ 3) + (2\ 3)$
$J_4 = (1\ 4) + (2\ 4) + (3\ 4)$
Jucys-Murphy elements

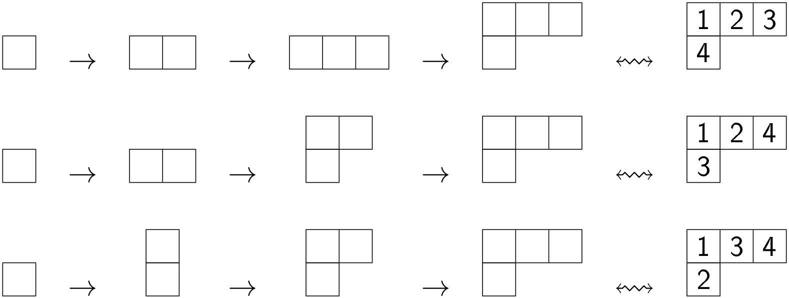
Example: The standard representation of $\mathcal{S}_4$

0	$(1\ 2)$	$(1\ 3)+(2\ 3)$	$(1\ 4)+(2\ 4)+(3\ 4)$
$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$	$\begin{bmatrix}2&0&0\\0&-1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}-1&0&0\\0&2&0\\0&0&2\end{bmatrix}$.

$$\left.\begin{aligned}\gamma(u)&=(0,1,2,-1)\\ \gamma(v)&=(0,1,-1,2)\\ \gamma(w)&=(0,-1,1,2)\end{aligned}\right\}\text{weights}$$


$n = 1$
$n = 2$
$n = 3$
$n = 4$
$n = 5$

Example



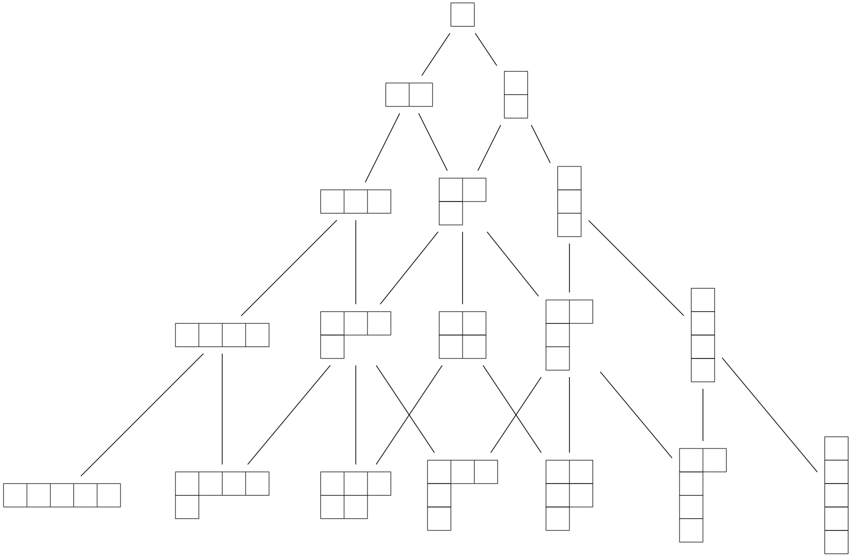
Example

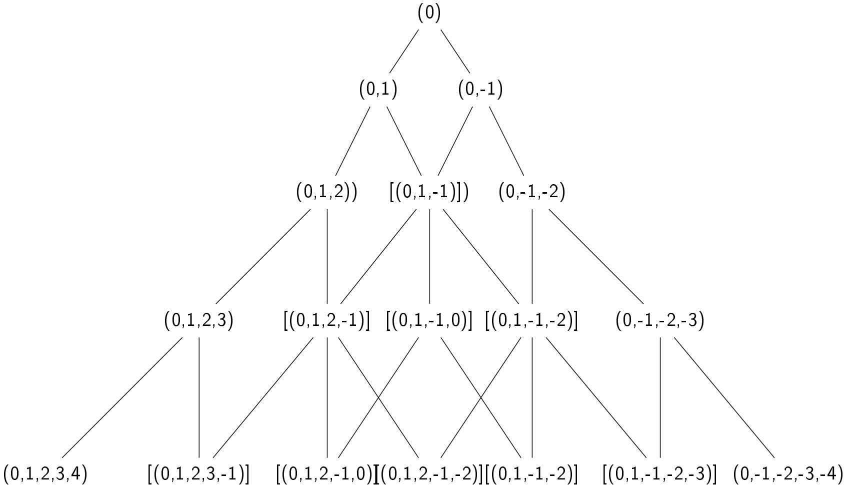
Consider

$$Q_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \quad Q_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad Q_3 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

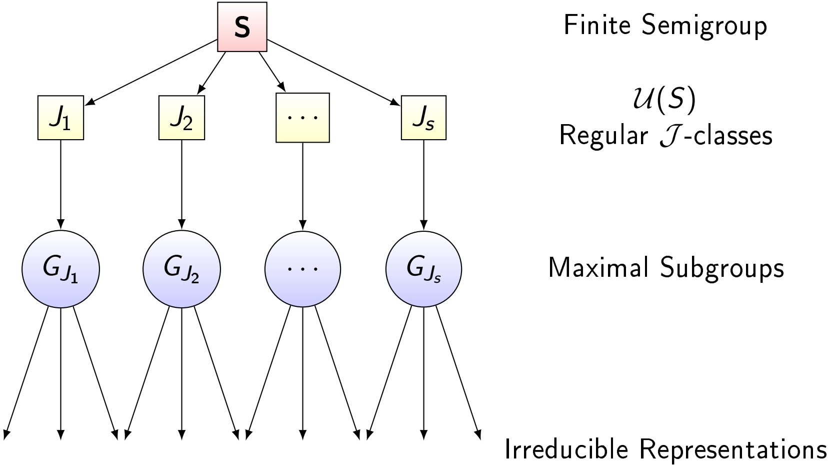
We have:

$$\left.\begin{array}{l} \delta(Q_1) = (0,1,2,-1) \\ \delta(Q_2) = (0,1,-1,2) \\ \delta(Q_3) = (0,-1,1,2) \end{array}\right\} \text{contents} = \text{weights} \left\{\begin{array}{l} \gamma(u) = (0,1,2,-1) \\ \gamma(v) = (0,1,-1,2) \\ \gamma(w) = (0,-1,1,2) \end{array}\right.$$




(0)
(0,1)
(0,-1)
(0,1,2))
[(0,1,-1)])
(0,-1,-2)
(0,1,2,3)
[(0,1,2,-1)]
[(0,1,-1,0)]
[(0,1,-1,-2)]
(0,-1,-2,-3)
(0,1,2,3,4)
[(0,1,2,3,-1)]
[(0,1,2,-1,0)]
[(0,1,2,-1,-2)]
[(0,1,-1,-2)]
[(0,1,-1,-2,-3)]
(0,-1,-2,-3,-4)

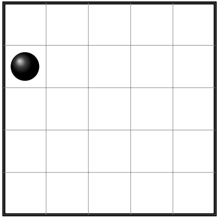
Finite Semigroups Representation Theory

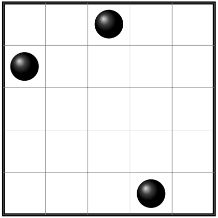


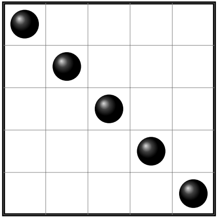
Theorem[Clifford,Munn,Ponizovskii]

The number of irreducible representations of S (up to isomorphism) is equal to the number of irreducible representations of its maximal subgroups G_J, with $J \in \mathcal{U}(S)$.

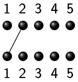
The Rook Monoid



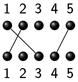




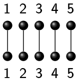
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



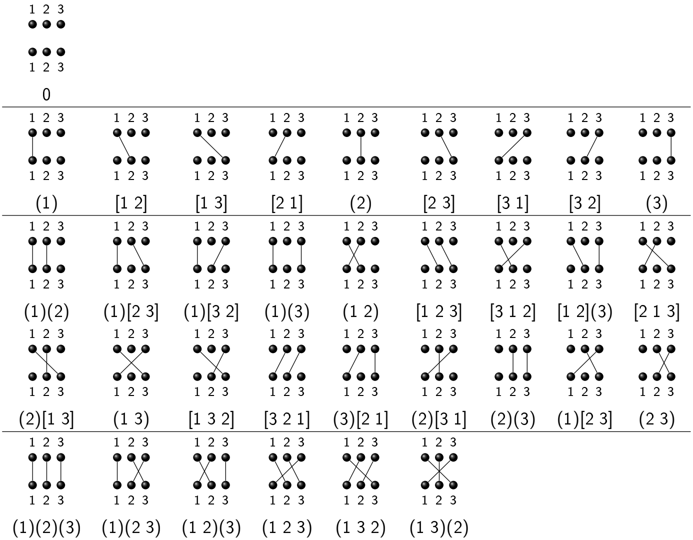
[2 1]



[2 1 3][5 4]



(1)(2)(3)(4)(5)


0
(1)
[1 2]
[1 3]
[2 1]
(2)
[2 3]
[3 1]
[3 2]
(3)
(1)(2)
(1)[2 3]
(1)[3 2]
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[1 2 3]
[3 1 2]
[1 2](3)
[2 1 3]
(2)[1 3]
(1 3)
[1 3 2]
[3 2 1]
(3)[2 1]
(2)[3 1]
(2)(3)
(1)[2 3]
(2 3)
(1)(2)(3)
(1)(2 3)
(1 2)(3)
(1 2 3)
(1 3 2)
(1 3)(2)

The special case of the Rook Monoid

n.º of isoclasses of Irr. Rep. = sum of the n.º of isoclasses of Irr. Rep of its maximal subgroup G_J

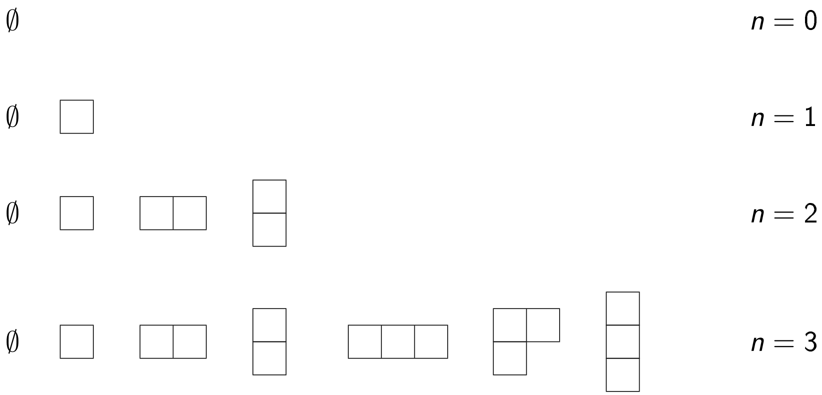
$$\Downarrow$$

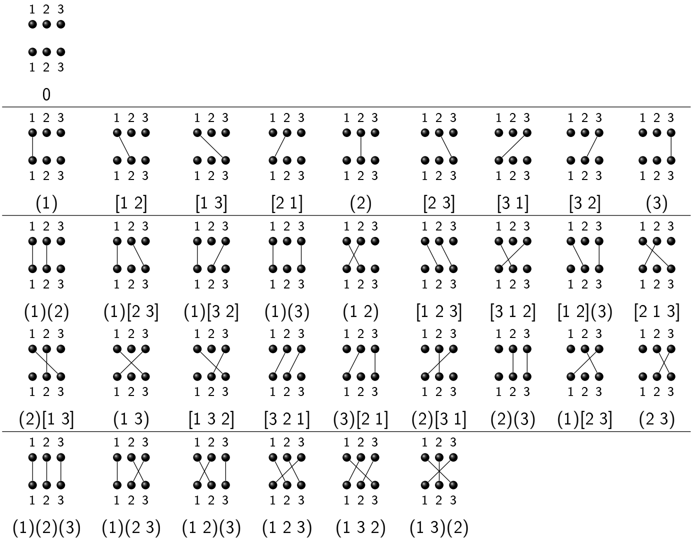
The list of the maximal subgroups G_J of $\mathcal{I}_n$ will be isomorphic to $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n$.

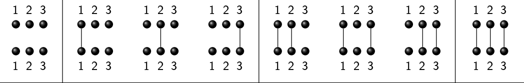
$$\Downarrow$$

$$|IrrRep(\mathcal{I}_n)| = \sum_{k=0}^{n} |IrrRep(\mathcal{S}_k)|$$

Isoclasses of Irr. Rep. of $\mathcal{I}_n$




0
(1) [1 2] [1 3] [2 1] (2) [2 3] [3 1] [3 2] (3)
(1)(2) (1)[2 3] (1)[3 2] (1)(3) (1 2) [1 2 3] [3 1 2] [1 2](3) [2 1 3]
(2)[1 3] (1 3) [1 3 2] [3 2 1] (3)[2 1] (2)[3 1] (2)(3) (1)[2 3] (2 3)
(1)(2)(3) (1)(2 3) (1 2)(3) (1 2 3) (1 3 2) (1 3)(2)



rank 0	rank 1			rank 2			rank 3
$\varepsilon_\emptyset$	$\varepsilon_{\{1\}}$	$\varepsilon_{\{2\}}$	$\varepsilon_{\{3\}}$	$\varepsilon_{\{1,2\}}$	$\varepsilon_{\{1,3\}}$	$\varepsilon_{\{2,3\}}$	$\varepsilon_{\{1,2,3\}}$

$$\eta_0 = \varepsilon_\emptyset$$

$$\eta_1 = -3\,\varepsilon_\emptyset + \varepsilon_{\{1\}} + \varepsilon_{\{2\}} + \varepsilon_{\{3\}}$$

$$\eta_2 = 3\,\varepsilon_\emptyset - 2\,\varepsilon_{\{1\}} - 2\,\varepsilon_{\{2\}} - 2\,\varepsilon_{\{3\}} + \varepsilon_{\{1,2\}} + \varepsilon_{\{1,3\}} + \varepsilon_{\{2,3\}}$$

$$\eta_3 = -\varepsilon_\emptyset + \varepsilon_{\{1\}} + \varepsilon_{\{2\}} + \varepsilon_{\{3\}} - \varepsilon_{\{1,2\}} - \varepsilon_{\{1,3\}} - \varepsilon_{\{2,3\}} + \varepsilon_{\{1,2,3\}}$$

$$\mathbb{C}\mathcal{I}_n \simeq \mathbb{C}\mathcal{I}_n\eta_0 \oplus \ldots \oplus \mathbb{C}\mathcal{I}_n\eta_n \simeq M_{\binom{n}{0}}(\mathbb{C}\mathcal{S}_0) \oplus \ldots \oplus M_{\binom{n}{n}}(\mathbb{C}\mathcal{S}_n)$$

In this case:

$$\mathbb{C}\mathcal{I}_3 \simeq M_1(\mathbb{C}\mathcal{S}_0) \oplus M_3(\mathbb{C}\mathcal{S}_1) \oplus M_3(\mathbb{C}\mathcal{S}_2) \oplus M_1(\mathcal{S}_3)$$

Theorem[Munn]

Let

$$\mathcal{Q}_n = \bigcup_{r=0}^{n} \mathcal{P}_r.$$

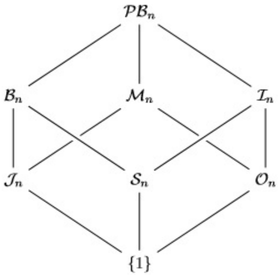
The set $\{\rho^{\lambda\star} : \lambda \in \mathcal{Q}_n\}$ is a full set of inequivalent irreducible representations of $\mathcal{I}_n$, where

$$\rho^{\lambda\star}(\sigma) = \sum_{|K|=r,\mathrm{rank}(\varepsilon_K\sigma=r)} \rho^{\lambda}(p(\varepsilon_K\sigma))E_{\mathrm{dom}(\varepsilon_k\sigma),\mathrm{codom}(\varepsilon_K\sigma)}.$$

The character table of $\mathcal{I}_3$

Cicle patterns	–	(*)	(*)(*)	(* *)	(*)(*)(*)	(* *)(*)	(* * *)
∅ -	1	1	1	1	1	1	1
(1)	0	1	2	0	3	1	0
(2)	0	0	1	1	3	1	0
(1,1)	0	0	1	-1	3	-1	0
(3)	0	0	0	0	1	1	1
(2,1)	0	0	0	0	2	0	-1
(1,1,1)	0	0	0	0	1	-1	1

Where to go?



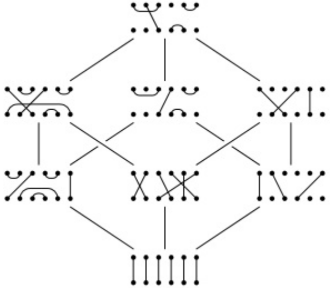


Figure: Taken from *Motzkin monoids and partial Brauer monoids*(I. Dolinka, J. East, R. Gray) - 2015.

Open Problems

- Construct a supercharacter theory for the rook monoid.
- Which are the irreducible representations of $\mathcal{S}_n$ over a field with arbitrary characteristic?
- How many commutation classes does the longest element of $\mathcal{S}_n$ have?
- Brauer's Problem List.

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