A stochastic variational approach to viscous Burgers equations

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Contents

Martingale

- Fix a probability space $(\Omega, \mathcal{F}, P)$ and an increasing σ-algebra $(\mathcal{F}_t)_{t\geq 0}$ $(\mathcal{F}_s \subset \mathcal{F}_t, \forall s < t)$
- A stochastic process $X : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is adapted if $X(t)$ is $\mathcal{F}_t$-measurable for every t.
- A (real valued) adapted process $M(t)$ is a martingale if
  (i) $E|M(t)| < \infty$ for all t
  (ii) $E(M(t)|\mathcal{F}_s) = M(s)$ a.s. for all $0 \leq s < t$
  where E denotes expectation and $E(\cdot|\mathcal{F}_s)$ conditional expectation with respect to $\mathcal{F}_s$.
- For martingales M, N, we define covariation $[M, N]_t = \lim_n \sum (M(t_{i+1}) - M(t_i))(N(t_{i+1}) - N(t_i))$.

- A real-valued process $X(t)$ is a semi-martingale if it is of the form

$$X(t) = X(0) + M(t) + A(t)$$

where M is a martingale and A an adapted process of bounded variation with $A(0) = 0$.

- Itô stochastic integral

$$\int_0^t X(s)dY(s) = \lim_n \sum X(t_i)[Y(t_{i+1}) - Y(t_i)]$$

- covariation of semi-martingale=covariation of their martingale part

Theorem (Itô’s formula)

If X is a continuous semi-martingale and $f \in C^2(\mathbb{R})$, then $f(X)$ is a continuous semi-martingale satisfying

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) + \frac{1}{2}\int_0^t f''(X(s))d[X,X]_s$$

Remark

Brownian motion $W(t)$ is a continuous martingale such that $[W, W]_t = t$.

Definition

If ξ is a semi-martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, we define D_t as follows

$$D_t F(\xi_t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[E_t F(\xi(t+\epsilon)) - F(\xi(t))], \forall F \in C^\infty(\mathbb{R}^d)$$

where E_t denotes conditional expectation with respect to $\mathcal{F}_t$.

Example

For the diffusion process

$$g_t^u = g_0^u + \sqrt{2\nu}\int_0^t dW_s + \int_0^t u(s, g_s^u)ds, \tag{1}$$

by Itô's formula, we obtain
$f(g_t^u, t) = f(g_0^u, 0) + \int_0^t (\partial_t + u \cdot \nabla + \nu\Delta) f(g_s^u, s)ds + \sqrt{2\nu}\int_0^t \nabla f(g_s^u, s)dW_s$.
Since $EM(t) = EM(s)$ for the martingale M, we have
$Df(g_t^u, t) = (\partial_t + u \cdot \nabla + \nu\Delta) f(g_t^u, t)$.

Theorem (N.V.Krylov,M.Rockner(2005),X.Zhang(2011))

Assume

$$\int_0^T \Big(\int_{\mathbb{R}} |b(t,x)|^p dx\Big)^{\frac{q}{p}} dxdt < +\infty, \forall T > 0$$

with $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{2}{q} < 1$. *Then there exists a unique strong solution* $X_t(x)$ *to SDE*

$$dX_t = b_t(X_t)dt + dW_t.$$

Moreover, for almost all ω *and all* $t \geq 0$*,*

$$x \mapsto X_t(\omega, x) \text{ is a homeomorphism on } \mathbb{R}.$$

Corollary (1)

Let u belong to the space $L^q([0,T];L^q(\mathbb{R}))$. When $q>3$, there exists a stochastic process $g_u(t,x)$, strong solution of the stochastic differential equation

$$\begin{cases} dg^u(t,x) = u(t,g^u(t,x))dt + \sqrt{2\nu}dW_t \\ g^u(0,x) = x, x \in \mathbb{R} \end{cases} \tag{2}$$

that defines a homeomorphism on $\mathbb{R}$.

Proof.

When $u \in L^q([0,T];L^q(\mathbb{R}))$ (i.e. $\int_0^T \int_{\mathbb{R}} |u(t,x)|^q dxdt < +\infty$) and $q>3$, the drift satisfies the condition

$$\int_0^T \left(\int_{\mathbb{R}} |u(t,x)|^p dx\right)^{\frac{p'}{p}} dxdt < +\infty$$

with $p, p' \in (1,\infty)$, $\frac{1}{p} + \frac{2}{p'} < 1$ (for $p = p' = q$). □

Several important sets

- $\mathcal{S}$ denotes the set of continuous semi-martingales taking values in the homeomorphism group of $\mathbb{R}$.

- $\mathcal{S}_0$ denotes the subset of $\mathcal{S}$ consisting of diffusions g^u such that

$$dg_t^u = \sqrt{2\nu}dW_t + u(t, g_t^u)dt, \quad g_0^u = e$$

  with $u \in L^q([0, T] \times \mathbb{R})$.

- $\tilde{\mathcal{S}}_0$ denotes the subset of $\mathcal{S}_0$ consisting of diffusions whose drift u satisfies the condition $\int_0^T \int_{\mathbb{R}} u(t,x)\psi(t,x)dtdx \geq c$, for a fixed constant $c > 0$ and a function $\psi \in L^p([0, T] \times \mathbb{R})$ with $p = \frac{q}{q-1}$.

Critical semi-martingales

To a vector field $v \in C^1([0,T] \times C_c^\infty(\mathbb{R}))$ with $v(0,\cdot) = v(T,\cdot) = 0$, we associate e_t^v the solution of the ordinary differential equation

$$\frac{d}{dt} e_t^v(x) = \dot{v}(t, e_t^v(x)), \qquad e_0^v(x) = x.$$

Definition

Let J be a real-valued functional defined on $\mathcal{S}$. Consider left and right derivatives of J at a semi-martingale $\xi \in \mathcal{S}$ along directions $e_\cdot^v$, $v \in C^1([0,T] \times C_c^\infty(\mathbb{R}))$ with $v(0,\cdot) = 0$, namely

$$(D_l)_{e_\cdot^v} J[\xi] = \frac{d}{d\epsilon}|_{\epsilon=0} J[e_\cdot^{\epsilon v} \circ \xi(\cdot)]$$

$$(D_r)_{e_\cdot^v} J[\xi] = \frac{d}{d\epsilon}|_{\epsilon=0} J[\xi(\cdot) \circ e_\cdot^{\epsilon v}]$$

A semi-martingale ξ is said to be critical for J if

$$(D_l)_{e_\cdot^v} J[\xi] = (D_r)_{e_\cdot^v} J[\xi] = 0$$

for all v as above.

Action functional

- For $g \in \mathcal{S}$, we define the action functional as follows:

$$A_q[g] \triangleq \frac{1}{q} E \int_0^T ||(D_t g(t)) \circ (g(t))^{-1}(\cdot)||_{L^q}^q dt.$$

- If $g^u \in \mathcal{S}_0$, its drift is of the form $D_t g_t^u = u(t, g_t^u)$. For such a process the action functional becomes

$$\begin{aligned} A_q[g^u] &= \frac{1}{q} E \int_0^T ||(D_t g_t^u) \circ (g_t^u)^{-1}(\cdot)||_{L^q}^q dt \\ &= \frac{1}{q} \int_0^T ||u(t, \cdot)||_{L^q}^q dt. \end{aligned}$$

Assume that $3 < q < \infty$ is even.

Theorem

Let g^u be solution of a stochastic differential equation of the form (1). Then g^u is critical for the action functional A_q if and only if the vector field $u(t)$ satisfies in the weak sense the following equation

$$\partial_t(u^{q-1}) = -\frac{q+1}{q}(u^q)' + \nu(u^{q-1})''. \tag{3}$$

Remark

Let $v = u^{q-1}$, we have $u^q = v^{\frac{q}{q-1}}$. Then equation (3) becomes

$$\partial_t v = -\frac{q+1}{q}(v^{\frac{q}{q-1}})' + \nu v'', \tag{4}$$

which is the viscous Burgers equation.

As the metric is right-invariant we only need to consider left derivatives. Let $\varepsilon > 0$. Since $e_0^v(x) = x$, we have

$$e_t^{\varepsilon v} = e + \varepsilon \int_0^t \dot{v}(s, e_s^{\varepsilon v}) ds$$

and

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} e_t^{\varepsilon v} = \int_0^t \dot{v}(s, x) ds = v(t, x).$$

We denote $e_\varepsilon(t, x) = e_t^{\varepsilon v}(x)$. Since q is even, we deduce that

$$
\begin{aligned}
& \frac{d}{d\varepsilon}|_{\varepsilon=0} A_q[e_\varepsilon \circ g^u] \\
= \quad & \frac{d}{d\varepsilon}|_{\varepsilon=0} \frac{1}{q} E \int_0^T \|D[e_\varepsilon(t, g_t)] \circ (e_\varepsilon(g_t))^{-1}(\cdot)\|_{L^q}^q dt \\
= \quad & E \int_0^T \int \frac{d}{d\varepsilon}|_{\varepsilon=0} \frac{1}{q} |D[e_\varepsilon(t, g_t)] \circ (e_\varepsilon(g_t))^{-1}(x)|^q dxdt \\
= \quad & E \int \int |u(t,x)|^{q-2} < \frac{d}{d\varepsilon}|_{\varepsilon=0} \{D[e_\varepsilon(t, g_t)] \circ (e_\varepsilon(g_t))^{-1}\}, u(t,x) > dtdx \\
= \quad & E \int \int |u(t,x)|^{q-2} < \partial_t v + [uv' - u'v] + \nu v'', u(t,x) > dtdx \\
= \quad & - \int \int \partial_t(|u|^{q-2}u) v dtdx + \nu \int \int (|u|^{q-2}u)'' v dtdx \\
& - \int \int \frac{q+1}{q} (|u|^q)' v dtdx,
\end{aligned}
$$

Since v is arbitrary satisfying $v(0,\cdot) = v(T,\cdot) = 0$,

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} A_q[e_\varepsilon \circ g^u] = 0$$

is equivalent to

$$\partial_t(u^{q-1}) = -\frac{q+1}{q}(u^q)' + \nu(u^{q-1})''$$

weakly.

If the admissible variations g_t^ε satisfy the following equation

$$dg_t^\varepsilon = \sqrt{2\nu}dW_t + [\partial_t e_t^{\epsilon v} + u \cdot (e_t^{\epsilon v})' + \nu(e_t^{\epsilon v})''](g_t^\varepsilon)dt, \tag{5}$$

we consider the notion of criticality with respect to this new class of admissible variations. Then the following result holds

Theorem

Let g_t be solution of a stochastic differential equation of the form (1). Then g_t is critical for the action functional A_q if and only if the vector field $u(t)$ satisfies in the weak sense the following equation

$$\partial_t(u^{q-1}) = -(u^q)' + \nu(u^{q-1})''. \tag{6}$$

As the admissible variations considered in the second variational principle preserve the class $\mathcal{S}_0$, we obtain the following result,

Theorem

There exists a semi-martingale $g(t)$ in the class $\tilde{\mathcal{S}}_0$ which realizes the minimum of the action functional A_q. Then the corresponding drift function $u(t,\cdot)$ is such that $v = u^{q-1}$ satisfies the viscous Burgers equation

$$\partial_t v = -(v^{\frac{q}{q-1}})' + \nu v'' \tag{7}$$

in the weak sense.

Proof

The action functional is bounded below. Let α be its infimum in the (non empty set) $\tilde{\mathcal{S}}_0$. We consider $g^m(t)$ a minimizing sequence and write $g^m(t) = g^{u_m}(t)$, with $u_m \in L^q([0,T];L^q(\mathbb{R}))$. We have the convergence $A_q[g^m(\cdot)] \to \alpha$ as $m \to \infty$. The sequence

$$A_q[g^m(\cdot)] = \frac{1}{q}\|u_m\|^q_{L^q([0,T];L^q(\mathbb{R}))}$$

is bounded, therefore there exists a subsequence u_{m_j} of u_m that converges with respect to the weak topology, more precisely there exists u such that

$$u_{m_j} \to u, \quad \text{weakly } in\ L^q([0,T];L^q).$$

The limit function u satisfies the assumptions of Corollary 1. We can therefore define a diffusion process g^u in $\mathcal{S}_0$ as solution of the equation (1). Since the norm is weakly lower semi-continuous, we have

$$A_q[g^u(\cdot)] \leq \lim_{j\to\infty} A_q[g^{m_j}(\cdot)].$$

We deduce that $A_q[g^u(\cdot)] = \alpha$ and g^u is a minimum.
On the other hand, since u_{m_j} converges weakly to u, we will have $\int_0^T \int u(t,x)\psi(t,x)dtdx \geq c$ which allows to consider nontrivial limit functions.
The time-dependent vector field $u(t,\cdot)$ satisfies equation (6).

- We make a change of variables for (4) and (7).
  Let $\tilde{v}(t,\theta) = -v(T-t,\theta)$, we obtain the following equation:

$$\partial_t \tilde{v} = -c\tilde{v}^{\frac{1}{q-1}}\tilde{v}' - \nu\tilde{v}''.$$

- Let $Y_t = \tilde{v}(t, \sqrt{2\nu}W_t)$, $Z_t = \tilde{v}'(t, \sqrt{2\nu}W_t)$. By Itô's formula we have,

$$\begin{aligned} dY_t &= \sqrt{2\nu}Z_t dW_t - cY_t^{\frac{1}{q-1}} Z_t dt \\ Y_T &= u_0^{q-1}(\sqrt{2\nu}W_T) \end{aligned}$$

Maximal bounded solution

- If $X|_{[0,t]\times\Omega} : [0,t]\times\Omega \to \mathbb{R}$ satisfies $X \in \mathcal{B}([0,t])\times\mathcal{F}_t$, we say X is a progressively measurable process.

- $\mathcal{H}_T^\infty(\mathbb{R})$ denotes the set of one-dimensional progressively measurable processes which are almost surely bounded for almost every t.

- $\mathcal{H}_T^2(\mathbb{R})$ denotes the set of progressively measurable processes $(Z_t)_{0\le t\le T}$ such that $E\int_0^T |Z_s|^2 ds < \infty$.

- We shall say that $(Y,Z) \in \mathcal{H}_T^\infty(\mathbb{R}) \times \mathcal{H}_T^2(\mathbb{R})$, or simply $Y \in \mathcal{H}_T^\infty(\mathbb{R})$, is a maximal bounded solution of

$$Y_t = \xi + \int_t^T f(t,\omega,Y_s,Z_s)ds - \int_t^T Z_s dW_s \tag{8}$$

if for all other solution $(\hat{Y},\hat{Z}) \in \mathcal{H}_T^\infty(\mathbb{R}) \times \mathcal{H}_T^2(\mathbb{R})$ we have $\hat{Y} \le Y$.

Theorem (J.P.Lepeltier,J.San Martin,1997)

Assume $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$, $f \in \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2)$ *satisfies:*

$$\forall t, \omega, f(t, \omega, \cdot, \cdot) \text{ is continuous}$$

$$\forall t, \omega, y, z, |f(t, \omega, y, z)| \leq l(y) + c|z|^2 \text{ for some finite constant } c$$

If $l : \mathbb{R} \to \mathbb{R}_+$ *is a positive function such that*

$$\int_{-\infty}^{0} \frac{1}{l(y)} dy = \int_{0}^{+\infty} \frac{1}{l(y)} dy = \infty,$$

then the backward stochastic differential equation

$$Y_t = \xi + \int_t^T f(t, \omega, Y_s, Z_s) ds - \int_t^T Z_s dW_s$$

has a maximal bounded solution (Y, Z).

Corollary (2)

There exists a maximal bounded solution (Y, Z) *to the following backward stochastic differential equation*

$$Y_t = \xi + c\int_t^T Y_s^{\frac{1}{q-1}} Z_s ds - \sqrt{2\nu}\int_t^T Z_s dW_s \tag{9}$$

with final value $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$*.*

Proof.

Let $f(y,z) = cy^{\frac{1}{q-1}}z$. Then $|f(y,z)| \le C(y^{\frac{2}{q-1}} + z^2)$. We denote $l(y) = y^{\frac{2}{q-1}}$. Then $l(y)$ satisfies the conditions

$$\int_{-\infty}^0 \frac{1}{l(y)}dy = \int_0^{+\infty} \frac{1}{l(y)}dy = \infty.$$

□

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Thank you for your attention!