On Symplectic Inner and Outer Radii

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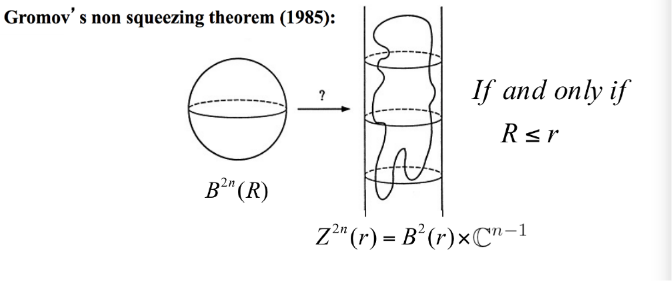
- Major driving force in Symplectic Topology.
- Goes back to Gromov's celebrated nonsqueezing theorem



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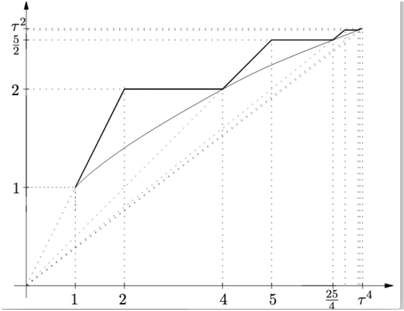

$a = 2$ "Symplectic Rigidity"

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Symplectic Inner and Outer Radii

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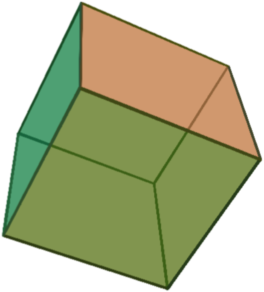
$B^4[r] \overset{s}{\hookrightarrow}$  $\overset{s}{\hookrightarrow} B^4[R]$

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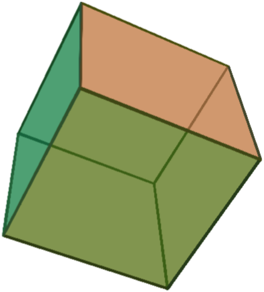
$B^4[r] \overset{s}{\hookrightarrow}$  $\overset{s}{\hookrightarrow} B^4[R]$

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Question: Which class of domains is "natural" to study?

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$$\|(x, y)\|_p = \begin{Bmatrix} \left(\|x\|_K^p + \|y\|_T^p\right)^{1/p}, & \text{for } 1 \leq p < \infty \\ \max\left\{\|x\|_K, \|y\|_Y\right\}, & \text{for } p = \infty \end{Bmatrix}$$

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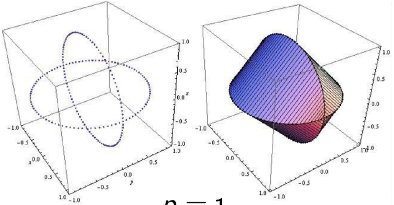

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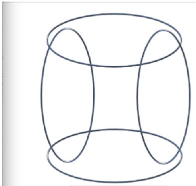

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Remark: The case $p = \infty$ was previously studied by V. Ramos, and is closely related with billiard dynamics!

Toric Domains

A **toric domain** X_Ω in $\mathbb{C}^2$ is the preimage of the region $\Omega \subset \mathbb{R}^2_{\geq 0}$ under the map $(z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2)$.

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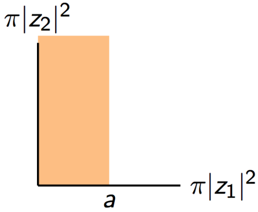
Example (Cylinder)



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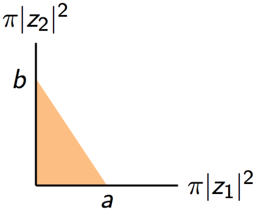

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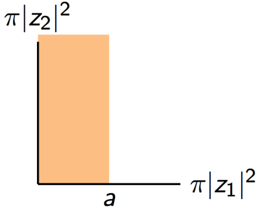

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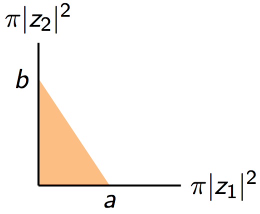

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is symplectomorphic to a **convex/concave** *toric domain* X_{Ω_p}.

The ℓ_p-sum of Disks as a Toric Domain

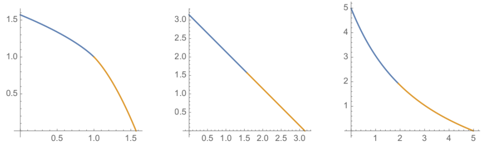
Theorem (O, Ramos)

$\mathbb{X}_p \overset{s}{\simeq} X_{\Omega_p}$ where $\Omega_p \subseteq \mathbb{R}^2_{\geq 0}$ is bounded by the axes and the curve

$$\begin{cases} \big(2\pi v + g_p(v), g_p(v)\big), & \text{for } v \in [0, (1/4)^{1/p}] \\ \big(g_p(-v), -2\pi v + g_p(-v)\big), & \text{for } v \in [-(1/4)^{1/p}, 0] \end{cases}$$

where $g_p : [0, (1/4)^{1/p}] \to \mathbb{R}$ is given by

$$g_p(v) = 2\int_{\left(\frac{1}{2}-\sqrt{\frac{1}{4}-v^p}\right)^{1/p}}^{\left(\frac{1}{2}+\sqrt{\frac{1}{4}-v^p}\right)^{1/p}} \sqrt{(1-r^p)^{2/p} - \frac{v^2}{r^2}}\,dr.$$



(a) $p = 1$ (b) $p = 2$ (c) $p = 6$

Figure 1: The set Ω_p for different values of p

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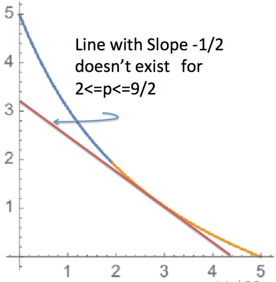

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Remark: $B \subset \mathbb{R}^2_{\geq 0}$ and X_B is a toric domain!

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$$V(x, y) = x \otimes y = y_1 x_2 - y_2 x_1$$

Note: One should be careful with certain *regularity issues* when applying the Arnold-Liouville theorem in this case!

The ℓ_p-sum of Disks as a Toric Domain

Let $\mathbb{X}_p = \left\{(x, y) \in \mathbb{R}^2_x \times \mathbb{R}^2_y \mid |x|^p + |y|^p < 1\right\}$, for $1 \leq p < \infty$

We have one natural Hamiltonian function

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Conclusion: By a *careful computation* of the action-angle coordinates, one gets the identification $\mathbb{X}_p \overset{s}{\simeq} X_{\Omega_p}$, where X_{Ω_p} is the **concave/convex** domain mentioned above.

Toric Domains in Disguise

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Question: Are there convex sets which are not symplectomorphic to toric domains?

Back to Symplectic Inner and Outer Radii

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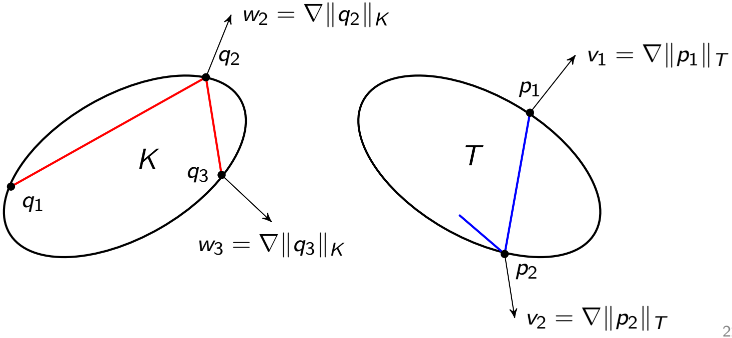
The dynamics on $\partial(K \times T)$ may be interpreted as **Finsler Bililard dynamics**, where K plays the role of a billiard table, and T defines a Minkowski geometry, which controls the billaird dynamics in K.

Characteristic foliation on $\partial(K \times T)$

Consider $H(x, y) = \max\{\|x\|_K, \|y\|_T\}$ (singular function)

The 1-level set is $\partial(K \times T)$.

$$\mathfrak{X}_H(x, y) = \begin{cases} (\nabla\|y\|_T, 0), & (x, y) \in int(K) \times \partial T, \\ (0, -\nabla\|x\|_K), & (x, y) \in \partial K \times int(T), \\ (?, ?) & (x, y) \in \partial(K) \times \partial(T) \end{cases}$$



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Remark I: Lagrangian products is a “natural" class to test symplectic embedding questions, since one (sometimes) has some “geometric understanding" of the corresponding dynamics.

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Remark II: In particular, if one can show that the symplectic inradius of $K \times K^*$ is 4, this would settle an 80-years old open conjecture in convex geometry known as “Mahler Conjecture"

(but this is a story for a different lecture...).

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THANK YOU VERY MUCH!