Schur positivity and ribbon shapes with interval support in the dominance lattice

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Outline

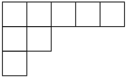
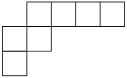
1. Introduction

2. Classification of a disjoint union of stair ribbons or products of stair ribbon Schur functions with interval support

(Skew) Schur functions

- Schur functions s_λ, λ a partition, are considered to be the most important and interesting basis for the ring of symmetric functions $\mathbb{C}m_\lambda$ (the vector space spanned by all the m_λ).

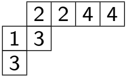
  $m_{21} = m_{21}(x) = m_\lambda(x_1, x_2, \dots) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + \cdots$
- Given partitions $\mu \subseteq \lambda$, $A := \lambda/\mu$.

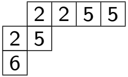
$\mu = 1 =$  $\lambda = 521 =$  $A = 521/1 =$ 

- The skew-Schur function s_A is the generating function for SSYT T of shape A

$$s_A(x) = \sum_T x^T,$$

where the sum is over all SSYT T of shape A, $x^T = x_1^{\#1's\ in T} x_2^{\#2's\ in T} \dots$ is the monomial weight of T.

$T =$ 

$x^T = x_1x_2^2x_3^2x_4^2$ $U =$ 

$x^U = x_2^3x_5^3x_6$

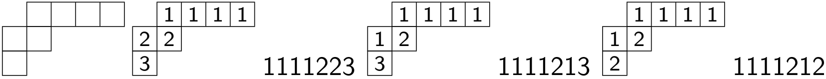
The Littlewood-Richardson rule

- s_A is a symmetric function and can be expressed as a linear combination of Schur functions.

$$s_A = \sum_{\nu} c_A^{\nu} s_{\nu},$$

where $c_A^{\nu} := c_{\mu,\lambda}^{\nu} \geq 0$ is the number of SSYT of shape A and content ν, satisfying the Littlewood-Richardson rule.

- **The LR rule**. A SSYT T is said to be an LR-filling if, as we read the entries of T from right to left along rows and top to bottom, the number of appearances of i always stays ahead of the number of appearances of $i+1$, for $i = 1, 2, \ldots$.

$A =$ 

$$c_A^{421} = 1 \quad c_A^{511} = 1 \quad c_A^{52} = 1$$

$$s_A = s_{421} + s_{511} + s_{52}$$

Dominance order on partitions

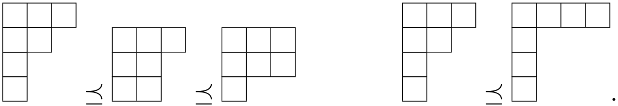
- The *dominance order* $\preceq$ on partitions of N, $\lambda = (\lambda_1, \ldots, \lambda_l)$, $\mu = (\mu_1, \ldots, \mu_s)$ is defined by setting $\lambda \preceq \mu$ if

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i,$$

  for $i = 1, \ldots, l$, where we set $\mu_i = 0$ if $i > l$.

  The set of partitions of size N equipped with the dominance order is a lattice with maximum element (N) and minimum element (1^N).

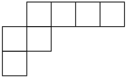
- $\lambda \preceq \mu$ if and only if the Young diagram of μ is obtained by *"lifting"* at least one box in the Young diagram of λ.

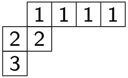


$$\lambda \preceq \mu \Leftrightarrow \mu' \preceq \lambda'$$

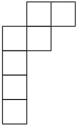
Schur interval

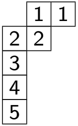
- $c_A^{\nu} = c_{A'}^{\nu'}$
-

$A =$ 



$r(A) = 421 \quad c_A^{\nu} > 0 \Rightarrow r(A) = 421 \preceq \nu$

$A' =$ 



$c_A^{\nu} = c_{A'}^{\nu'} > 0 \Rightarrow r(A') = c(A) = 22111 \preceq \nu'$

$$c_A^{\nu} > 0 \Rightarrow r(A) = 421 \preceq \nu \preceq c(A)' = 52$$

$$[r(A) = 421, c(A)' = 52] = \{421, 43, 511, 52\}.$$

Schur interval

- $c_A^{\nu} = c_{A'}^{\nu'}$
-

$$A = \begin{array}{|c|c|c|c|c|}\hline \multicolumn{1}{c|}{} & & & & \\ \hline & & \multicolumn{3}{c}{} \\ \cline{1-2} & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline \multicolumn{1}{c|}{} & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{3}{c}{} \\ \cline{1-2} 3 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \qquad r(A) = 421 \quad c_A^{\nu} > 0 \Rightarrow r(A) = 421 \preceq \nu$$

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$$c_A^{\nu} > 0 \Rightarrow r(A) = 421 \preceq \nu \preceq c(A)' = 52$$

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- $$A = \begin{array}{|c|c|c|c|c|}\cline{4-5} \multicolumn{3}{c|}{} & & \\ \hline & & & & \multicolumn{1}{c}{} \\ \cline{1-4} & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|c|}\cline{4-5} \multicolumn{3}{c|}{} & 1 & 1 \\ \hline 1 & 1 & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-4} 3 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|c|}\cline{4-5} \multicolumn{3}{c|}{} & 1 & 1 \\ \hline 1 & 1 & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-4} 2 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|c|}\cline{4-5} \multicolumn{3}{c|}{} & 1 & 1 \\ \hline 1 & 1 & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-4} 3 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|c|}\cline{4-5} \multicolumn{3}{c|}{} & 1 & 1 \\ \hline 1 & 1 & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-4} 2 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}$$

$$[421, 52] = \{421, 43, 511, 52\}.$$

Skew Schur function support

- $$s_A = \sum_{r(A) \preceq \nu \preceq c(A)'} c^{\nu}_{\mu,\lambda} s_{\nu} = s_{r(A)} + \cdots + c^{\nu}_{A} s_{\nu} + \cdots + s_{c(A)'} = s_{A^{\pi}}$$
$$s_{A'} = \sum_{c(A) \preceq \nu' \preceq r(A)'} c^{\nu}_{\mu,\lambda} s_{\nu'} = s_{c(A)} + \cdots + c^{\nu}_{A} s_{\nu'} + \cdots + s_{r(A)'}$$
- The support of a skew shape A, $\mathrm{supp} A$, considered as a subposet of the *dominance lattice*, has a top element and a bottom element uniquely defined by the shape A,
$$r(A), c(A)' \in \mathrm{supp} A = \{\nu : c^{\nu}_{A} > 0\} \subseteq [r(A), c(A)']$$
$$c(A), r(A)' \in \mathrm{supp} A' = \{\nu' : c^{\nu}_{A} > 0\} \subseteq [c(A), r(A)']$$

$$c^{c(A)'}_{A} = c^{r(A)}_{A} = 1$$

Skew Schur function support

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  $$c(A), r(A)' \in \mathrm{supp}A' = \{\nu' : c^{\nu}_{A} > 0\} \subseteq [c(A), r(A)']$$
  $$c^{c(A)'}_{A} = c^{r(A)}_{A} = 1$$
- The support of s_A is the support of A.

Problems

Given the skew shape A and $\nu \in [r(A), c(A)']$

1. How does the shape of A govern the positivity of c_A^ν?
   How does the shape of A govern the support of A?

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2. Under what conditions do we have $c_A^\nu > 0$ whenever $\nu \in [r(A), c(A)']$?

Problems

Given the skew shape A and $\nu \in [r(A), c(A)']$

1. How does the shape of A govern the positivity of c_A^ν?
   How does the shape of A govern the support of A?
2. Under what conditions do we have $c_A^\nu > 0$ whenever $\nu \in [r(A), c(A)']$?

   Which skew shapes have interval support?

   A., *The admissible interval for the invariant factors of a product of matrices*, Linear and Multilinear Algebra (1999).

If A is a skew shape with two or more components and A has interval support, then the components of A are ribbon shapes.

Ribbon shapes and disjoint unions of ribbon shapes

- Which are the ribbon shapes with interval support?

Ribbon shapes and disjoint unions of ribbon shapes

- Which are the ribbon shapes with interval support?
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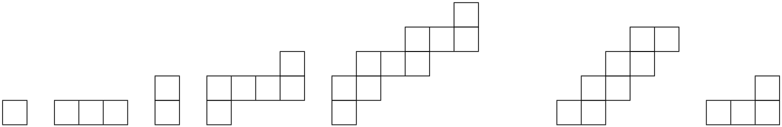
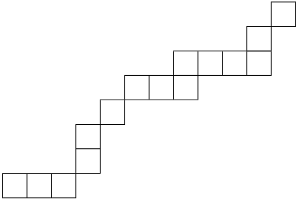
  - **Our answer.** *Ribbons whose column (row) lengths are at most two.*

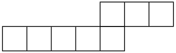
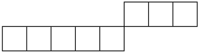


Disjoint union of similar ribbons



Example. *(Disjoint union) Ribbons such that all columns and row lenghts differ by at most one.*

Skew Schur functions and support

- $A =$  $B =$ 

  $s_A = s_{53} + s_{62} + s_{71}$ $s_B = s_{53} + s_{62} + s_{71} + s_8$

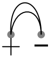
  $c(A)' = 71$

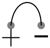
  $|$

  62

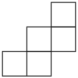
  $|$

  $r(A) = 53$

Skew Schur functions and support

- $A =$  $\qquad s_A = s_{221} + s_{311} + s_{32}$

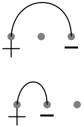
$c(A)' = 32$

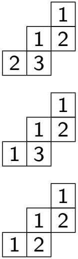
|

311

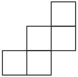
|

$r(A) = 221$





Skew Schur functions and support

- $A =$  $s_A = s_{221} + s_{311} + s_{32}$

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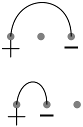
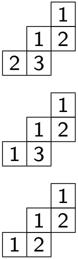
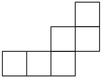
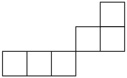




$0 \leq 1 - 1, \quad 1 \leq 2 + 1 - 2$

Skew Schur functions and support

- $A =$  $B =$ 

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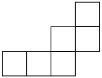
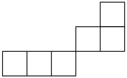
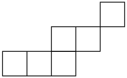
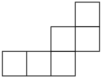
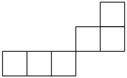
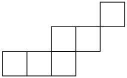
- $A =$  $B =$  $C =$ 

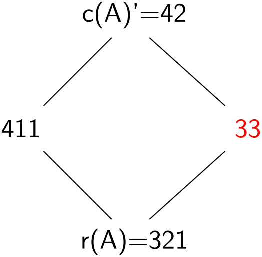
$$s_A = s_{321} + s_{411} + 0 s_{33} + s_{42}$$
$$s_B = s_{321} + s_{411} + 0 s_{33} + s_{42} + s_{51}$$

Skew Schur functions and support

- $A =$  $B =$  $C =$ 

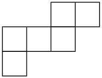
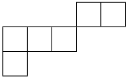
$$s_A = s_{321} + s_{411} + 0s_{33} + s_{42}$$

$$s_B = s_{321} + s_{411} + 0s_{33} + s_{42} + s_{51} \quad s_C = s_{321} + s_{411} + 1s_{33} + 2s_{42} + s_{51}$$

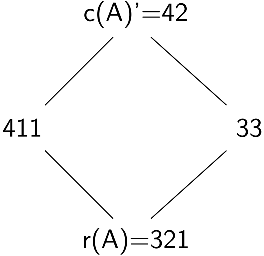


$$1 \not\leq 1 - 1, \qquad 1 \leq 2 + 1 - 2$$

Skew Schur functions and support

- $A =$  $B =$ 

$s_A = s_{321} + s_{411} + s_{33} + s_{42}$ $s_B = s_{321} + s_{411} + s_{33} + s_{42} + s_{51}$



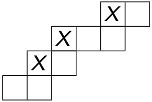
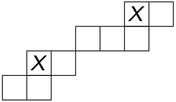
Stair ribbons and disjoint union of stair ribbon shapes

Definition

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ be a composition.
R_α denotes a skew-shape consisting of s row strips (α_i), $i = 1, \ldots, s$, right to left, so that any two of them overlap at most in one column, and the size of each column is at most two.
Let $0 \leq p < s$ be the number of columns of size two. When $p = s - 1$, R_α is a ribbon and one writes $R_\alpha =< \alpha >$. Otherwise, it is a disjoint union of $s - p$ ribbons.

$$\operatorname{supp} R_\alpha \subseteq [\alpha^+; (|\alpha| - p, p)],\ \alpha^+ = (\alpha_1^+, \ldots, \alpha_s^+) \text{ the decreasing rearrangement of } \alpha.$$

$R_{(2,3,2,2)}$

$< 2322 >=$  $p = 3$, $< 23 > \oplus < 22 >=$  $p = 2$

$[\alpha^+ = 32^3; 63]$ $[\alpha^+ = 3\,2^3; 72]$

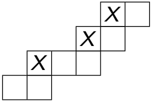
Stair ribbons and disjoint union of stair ribbon shapes

Definition

Given the composition $\alpha = (\alpha_1, \ldots, \alpha_s)$, and a skew shape R_α, let

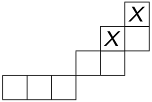
$$R^1_\alpha := R_\alpha, \text{ and } R^{i+1}_\alpha := R^i_\alpha \setminus < \alpha^+_i >, \ i = 1, \ldots, s-1,$$

giving priority to the rightmost row strip $< \alpha^+_i >$ of R_α, in case of equal size.
The *overlapping sequence* of R_α is the non increasing sequence of nonnegative integers $p_1 = p, p_2, \ldots, p_{s-1}, p_s = 0$, where p_i is the number of columns with size two of R^i_α, $1 \leq i \leq s$.
Note that $0 \leq p_{i+1} \leq p_i \leq s - i \leq \sum_{j=i+1}^{s} \alpha^+_j$, for $i = 1, \ldots, s-1$.

$< 2232 >=$ 

$p_1 = 3, p_2 = 1, p_3 = p_4 = 0$

$\mathrm{supp} R \subseteq [32^3; 63]$

$< 1,2,2 > \oplus < 3 >=$ 

$p_1 = 2 = p_2, p_3 = p_4 = 0$

$\mathrm{supp} R \subseteq [\alpha^+ = 32^21; 62]$

Support criterion for a disjoint union of stair ribbons

Theorem

Given the composition $\alpha = (\alpha_1, \dots, \alpha_s)$, consider R_α with overlapping sequence $(p_1, \dots, p_{s-1}, 0)$. Then

$$c^{\nu}_{R_\alpha} > 0 \text{ if and only if } \begin{cases} \nu \in [\alpha^+; (|\alpha| - p, p)] \Leftrightarrow \alpha^+ \preceq \nu \preceq (|\alpha| - p, p) \\ \nu_i \leq \sum_{j=i}^{s} \alpha_j^+ - p_i, \; i = 1, \dots, s. \end{cases}$$

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Equivalently, $c^\nu_{R_\alpha} > 0$ if and only if, $\nu \in [\alpha^+; (|\alpha| - p, p)]$, and

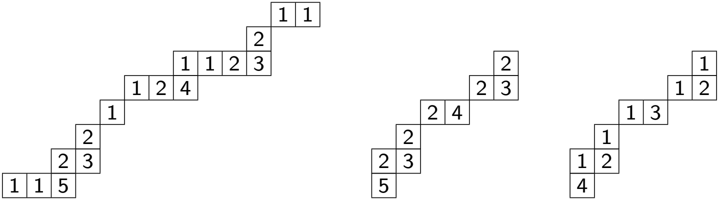
$$0 \leq \epsilon_i \leq \sum_{j=i+1}^{s} \alpha_j^+ - p_i, \; i = 1, \ldots, s-1,$$

where ϵ_i is the number of lifted boxes from the last $s - i$ rows of α^+ to the ith row α_i^+.

Sketch of proof for the *"only if part"*

$$c^{\nu}_{R_\alpha} > 0 \ \text{ only if } \ \begin{cases} \nu \in [\alpha^+; (|\alpha| - p, p)] \\ \nu_i \leq \sum_{j=i}^{s} \alpha_j^+ - p_i, \ i = 1, \ldots, s. \end{cases}$$

By induction on $s \geq 1$. If $s = 1$, then $p = 0$, and $\nu = (\alpha_1) = \alpha^+ = (|\alpha|)$.
Let $s \geq 2$ and assume the claim true for α with $1 \leq k < s$ parts.
Fix a $\nu = (\nu_1, \ldots, \nu_u, 0^{s-u})$ LR filling of R_α. If $u = 1$, then $p = 0$, $\nu = (|\alpha|)$ and there is nothing to prove.
Otherwise, $u \geq 2$ and delete all the boxes of R_α filled with 1.



Sketch of proof

The first row will disappear and some other rows will be shortened. We get another disjoint union of similar ribbons, $R_{\tilde{\alpha}}$, $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_t)$, $1 \leq u \leq t < s$, filled in the alphabet $\{2, \ldots, u\}$. Subtracting one unity to each entry of $R_{\tilde{\alpha}}$, we get a $\tilde{\nu} = (\nu_2, \ldots, \nu_u, 0^{s-u})$ LR filling of $R_{\tilde{\alpha}}$.
By induction

$$\nu_i \leq \sum_{j=i}^{t} \tilde{\alpha}_j^+ - \tilde{p}_i \leq \sum_{j=i}^{s} \alpha_j^+ - p_i,\ i = 2, \ldots, s.$$

Horn-Klyachko linear inequalities

- Let $N = \{1, 2, \ldots, n\}$, then for fixed d, with $1 \leq d \leq n$, let $I = \{i_1 > i_2 > \cdots > i_d\} \subseteq N$.

Horn-Klyachko linear inequalities

- Let $N = \{1, 2, \ldots, n\}$, then for fixed d, with $1 \leq d \leq n$, let $I = \{i_1 > i_2 > \cdots > i_d\} \subseteq N$.

- Let $I, J, K \subseteq N$ with $\#I = \#J = \#K = d$ and ordered decreasingly. One defines the partitions

$$\alpha(I) = I - (d, \ldots, 2, 1),$$

$$\beta(J) = J - (d, \ldots, 2, 1),$$

$$\gamma(K) = K - (d, \ldots, 2, 1).$$

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$$\gamma(K) = K - (d, \ldots, 2, 1).$$

- Let T_d^n be the set of all triples (I, J, K) with $I, J, K \subseteq N$ and $\#I = \#J = \#K = d$ such that $c_{\alpha(I),\beta(J)}^{\gamma(K)} > 0$.

Horn-Klyachko linear inequalities and Littlewood-Richardson coefficients

- $c^{\lambda}_{\mu,\nu} > 0$ if and only if the Horn-Klyachko inequalities are satisfied

$$\sum_{k=1}^{n} \lambda_k = \sum_{i=1}^{n} \mu_i + \sum_{j=1}^{n} \nu_j$$

$$\sum_{k\in K} \lambda_k \leq \sum_{i\in I} \mu_i + \sum_{j\in J} \nu_j$$

  for all triples $(I, J, K) \in T^n_d$ with $d = 1, \ldots, n-1$.
- *Lidskii- Wielandt inequalities and the dominance order*

$$\sum_{i\in I} \lambda_i \leq \sum_{i\in I} \mu_i + \sum_{i\leq d} \nu_i,$$

$$\sum_{i\in I} (\lambda_i - \mu_i) \leq \sum_{i\leq d} (\lambda_i - \mu_i)^+ \leq \sum_{i\leq d} \nu_i,$$

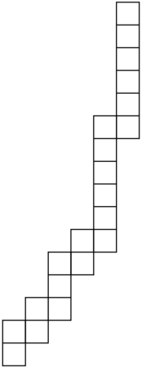
  for all $I \subseteq \{1, \ldots, n\}$ with $\#I = d$.

- $R =< 662322 > \;\; p = 5$

  $\alpha^+ = (6\,6\,3\,2\,2\,2) \preceq (777) \preceq (876) \preceq (21, 21-5)$

  $4 > \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - p_3 = 2+2+2-3 = 3, \;\; p_3 = 3 \Rightarrow (777) \notin \mathrm{supp} R$

  $\epsilon_3 = 3 = \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - p_3 = 2+2+2-3, \quad p_3 = 3$
  $\epsilon_2 = 1 < \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - p_2 = 3+2+2+2-3, \quad p_2 = 3$
  $\epsilon_1 = 2 < \alpha_2^+ + \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - p_1 = 6+3+2+2+2-5, p_1 = 5$
  $\Rightarrow (876) \in \mathrm{supp} R$



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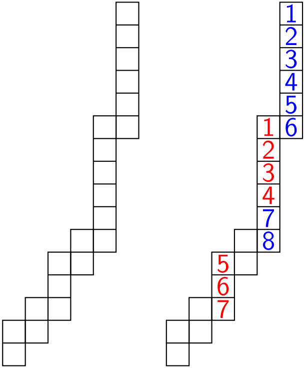

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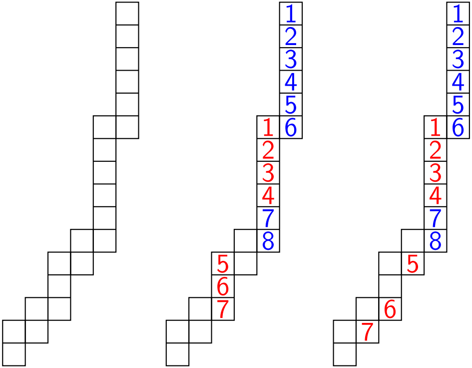

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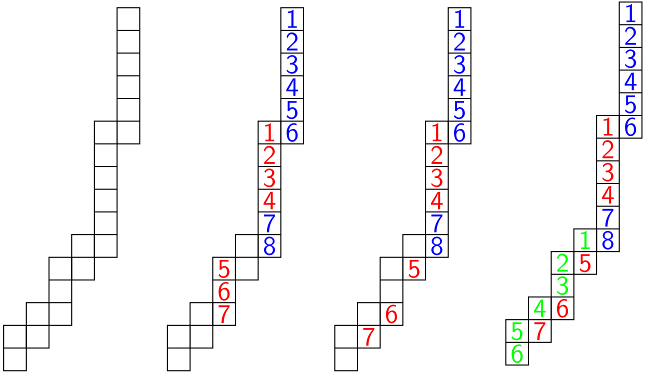

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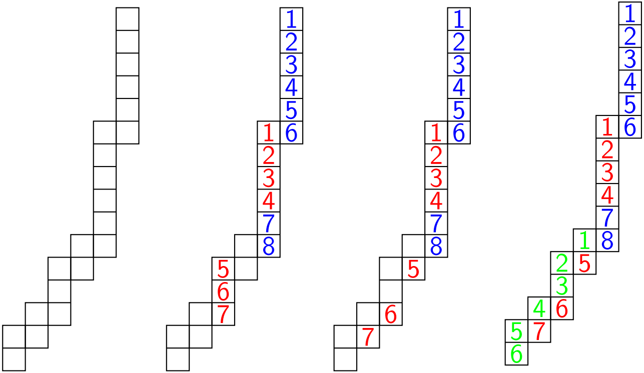

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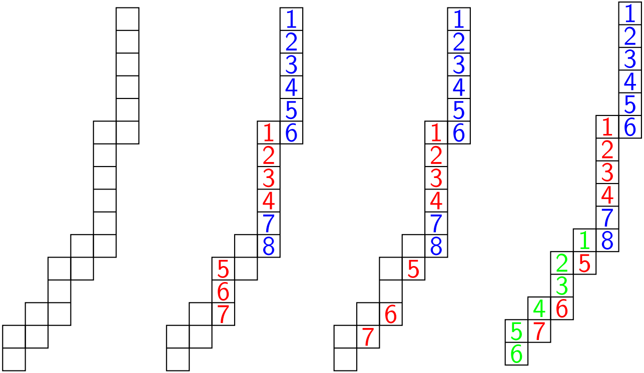


Classification of products of stair ribbon Schur functions with interval support

Theorem

Given the composition $\alpha = (\alpha_1, \ldots, \alpha_s)$, consider R_α with overlapping sequence $(p_1, \ldots, p_{s-1}, 0)$.
$\mathrm{supp} R_\alpha \subsetneq [\alpha^+; (|\alpha| - p, p)]$ if and only if for some $1 \leq i \leq s-2$ with $p_{i+1} \geq 1$, there exist integers $g_1, \ldots, g_i \geq 0$ with $\sum_{j=1}^{i} g_j \leq p_{i+1} - 1$, such that

$$\alpha_j^+ + g_j \geq \sum_{q=i+1}^{s} \alpha_q^+ - p_{i+1} + 1, \quad j = 1, \ldots, i.$$

In this case, $(\alpha_1^+ + g_1, \ldots, \alpha_i^+ + g_i, \sum_{q=i+1}^{s} \alpha_q^+ - p_{i+1} + 1, p_{i+1} - \sum_{j=1}^{i} g_j - 1)^+$ is not in the $\mathrm{supp} R_\alpha$.

Theorem

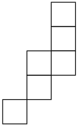
Given the composition $\alpha = (\alpha_1, \dots, \alpha_s)$, consider R_α with overlapping sequence $(p_1, \dots, p_{s-1})$.
$c^{\nu}_{R_\alpha} > 0$ whenever $\nu \in [\alpha^+; (|\alpha| - p, p)]$ if and only if for all $1 \leq i \leq s-2$ with $p_{i+1} \geq 1$, and for all integers $g_1, \dots, g_i \geq 0$ with $\sum_{j=1}^{i} g_j \leq p_{i+1} - 1$, one has always, for some $f \in \{1, \dots, i\}$,

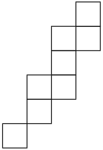
$$\alpha^+_f + g_f \leq \sum_{q=i+1}^{s} \alpha^+_q - p_{i+1}.$$

Corollary

- If $p_1 = 0$ or $p_2 = 0$, $\mathrm{supp} R_\alpha = [\alpha^+; (|\alpha| - p, p)]$.

$$< \alpha_1 > \oplus \cdots \oplus < \alpha_s >$$

$$< {\alpha_1}^+, \alpha_2^+ > \oplus \cdots \oplus < \alpha_s >$$ 

$$< {\alpha_1}^+, \alpha_2^+, \alpha_3^+ > \oplus \cdots \oplus < \alpha_s >$$  .

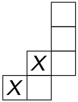
- If $p_2 = 1, p_3 = 0$, R_α has interval support except when

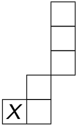
$$\alpha_1^+ \geq \sum_{q=2}^{s} \alpha_q^+$$

- If $p_2 = 1 = p_3$, $p_4 = 0$, R_α has interval support except when

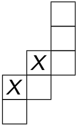
$$\alpha_1^+ \geq \sum^{s} \alpha_q^+ \quad \textit{or} \quad \alpha_1^+, \alpha_2^+ \geq \sum^{s} \alpha_q^+$$

$R_{(\alpha_1,\alpha_2,\alpha_3)}$ *has interval support except when* $\alpha_1 \geq \alpha_2 + \alpha_3$ *or* $\alpha_3 \geq \alpha_1 + \alpha_2$.



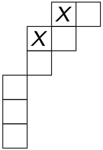


no

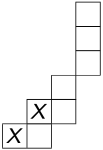


yes

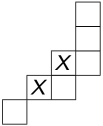
$s = 4$



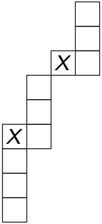
$p_1 = p_2 = 2$
$p_3 = 0$
no
$\alpha_1^+ + 1 \geq 2 + 2 + 1 - 2 + 1$



$p_1 = p_2 = 2$
$p_3 = 1$
no



$p_1 = 2, p_2 = 1$
$p_3 = 0$
yes
$\alpha_1^+ < \alpha_2^+ + \alpha_3^+ + \alpha_4^+$

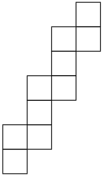
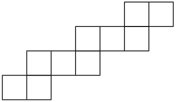


$p_1 = 2, p_2 = 1$
$p_3 = 0$
yes

Corollary

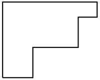
McNamara, van Willigenburg, 2011

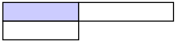
Ribbon shapes whose column and row lengths differ at most one have full support.

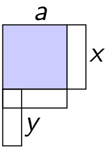
$R' =$  $R = < 2332 > =$ 

Corollary

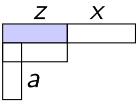
The Schur function product $s_\mu s_\nu$ has interval support if and only if one of the following is true: $\mu = (r_1, 1^{r_2})$ and $\nu = (s_1, 1^{s_2})$ are hooks such that $s_2 = r_2 = 1$, and either $r_1 = s_1 \geq 2$ or $s_1 = r_1 + 1$ (or vice versa).



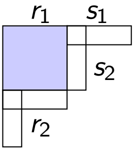




$a = 2$ and $1 \leq x \leq y + 1$,
or $a \geq 3$ and $x = 1$;



$a = 1$ and $1 \leq x \leq z$,
or $a \geq 2$ and $x = 1$;



$s_2 = r_2 = 1$, and either
$r_1 = s_1 \geq 2$ or
$s_1 = r_1 + 1$.

More examples

- $\alpha = (6,2,2,2,2,7,6), \quad \alpha^+ = (7,6,6,2,2,2,2),\ i = 3, \quad p_4 = 3$

$$\begin{array}{l} 7,6 \geq \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - p_4 + 1 = 2+2+2+2-2 \\ g_1 + g_2 + g_3 = 0 \end{array}.$$

$\nu = (7,6,6,6,2) \notin \mathrm{supp} R_\alpha$

$\nu = (7, 6+1, 6+1, 2+2+2+2-2) = (7,7,7,6) \notin \mathrm{supp} R_\alpha,$
$g_1 = 0, g_2 = g_3 = 1$
$\nu = (6+2, 7, 6, 2+2+2+2-2) = (8,7,6,6) \notin \mathrm{supp} R_\alpha,$
$g_1 = 0 = g_2, g_3 = 2$
$\nu = (7+2, 6, 6, 2+2+2+2-2) = (9,6,6,6) \notin \mathrm{supp} R_\alpha,$
$g_1 + g_2 + g_3 = 2.$

Note that $p_4 = 3 \Rightarrow 3 \leq p_2, p_3 \leq 4.$
If $p_3 = 4,\ 7+3, 6+3 \not\geq 6+2+2+2+2-3$