THE MATHEMATICS OF MAKING A MESS

(AN INTRODUCTION TO RANDOM WALK ON GROUPS)

(NOTES FOR A TALK IN LISBON, OCT 13, 2020)

1) THE BASIC SET UP

LET G BE A FINITE GROUP

LET $S = S^{-1}$ BE A GENERATING SET ($id \in S$)

A RANDOM WALK ON G, STARTS AT id AND

$$X_{n+1} = \epsilon_{n+1} X_n$$

WITH ϵ_i iid FROM

$$P(g) = \begin{cases} 1/|S| & \text{IF } g \in S \\ 0 & \underline{} \end{cases}$$

$$P * P(g) = \sum_{h \in G} P(h) P(gh^{-1}), \quad P^{k} = P * P^{k-1}$$

$$U(g) = \frac{1}{|G|}$$

2) THEOREM (POINCARÉ ≐ 1890)

$$P^{k}(g) \xrightarrow[\infty]{k} U(g)$$

MORE GENERALLY, IF SUPP P IS NOT IN A COSET OF A SUBGROUP

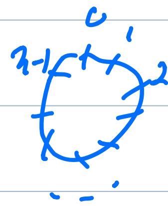
3) HOW FAST?

$$\|P^h - u\| = \max_{A \subseteq G} |P^h(A) - u(A)| = \frac{1}{2} \sum_{g} |P^h(g) - u(g)|$$

GIVEN $\epsilon > 0$, HOW LARGE h SO $\|P^h - U\| < \epsilon$?

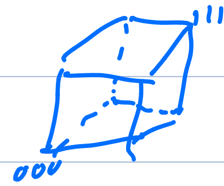
4) <u>EXAMPLES</u>

(a) $G = \mathbb{Z}_n$, $S = \{0, 1, -1\}$



ORDER n^2 STEPS ARE N+S.

(b) $G = \mathbb{Z}_2^n$, $S = \{0, e_1, \dots, e_n\}$



ORDER $d \log d$ STEPS N+S

(EHRENFESTS URN)

(c) $G = S_n$ (all permutations)

$S = \{\text{all transpositions}\}$

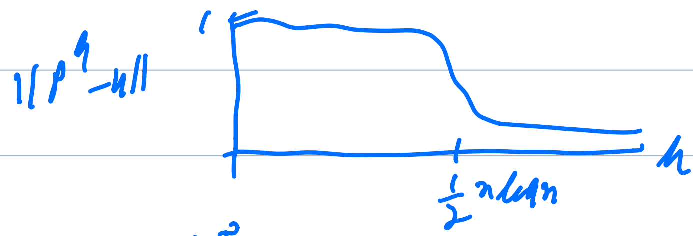
5) THEOREM (WITH SHAHSHAHANI) ON S_n, LET

$$P(\sigma) = \begin{cases} 1/n & \sigma = id \\ 2/n^2 & \sigma = (i,j) \\ 0 & - \end{cases}$$

LET $h = \frac{1}{2} n \log n + c n$

$\|P^h - u\| \le 2 e^{-2c}$ (MATCHING LOWER BOUND)

6) THERE IS A CUTOFF AT $\frac{1}{2} n \log n$



DEFINITION $\{G_n, P_n\}_{n=1}^{\infty}$ GROUPS

P_n HAS A CUTOFF AT k_n IF, $\forall\, \epsilon > 0$

$$\|P^{k_n(1+\epsilon)} - u\| \to 0$$

$$n \nearrow \infty$$

$$\|P^{k_n(1-\epsilon)} - u\| \to 1$$

CONJECTURE $\forall$ SEQUENCE OF GENERATING SETS OF S_n THERE IS CUTOFF.

(INDEED $\forall$ SEQUENCE OF FINITE SIMPLE GROUPS)

7) FOURIER ANALYSIS

A REPRESENTATION OF G IS $\rho : G \to GL(V)$

$$\rho(st) = \rho(s)\,\rho(t)$$

eg

$G = \mathbb{Z}_n \qquad \rho_j(k) = e^{2\pi i j k/n}$

$G = \mathbb{Z}_2^n \qquad \rho_x(y) = (-1)^{x \cdot y}$

$G = S_n, \qquad \rho_{Tr}(\sigma) = 1, \quad \rho_{SGN}(\sigma) = \mathrm{SGN}(\sigma), \quad \rho_{n-1}(\sigma) \in$ PERMUTATION MATRICES

$d_\rho = \dim V$ — DIMENSION OF ρ

ρ is IRREDUCIBLE if $\nexists$ $W \subseteq V$, NONTRIVIAL
$\rho(s)W \subseteq W$ ALL $s \in G$.

LET P BE A PROBABILITY ON G.

$$\hat{P}(\rho) = \sum_{g} P(g)\,\rho(g)$$

FACTS . $\widehat{P * P}(\rho) = \hat{P}(\rho)^2$

. $\hat{u}(\rho) = \begin{cases} 0 & \rho \text{ IRREDUCIBLE, NOT TRIVIAL} \\ 1 & \rho = \text{TRIV.} \end{cases}$

. $P^{k}(g) = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \, \mathrm{Tr}(\rho(g^{-1})\,\hat{P}(\rho))$

SO SHOW $P^{k} \Rightarrow u$ VIA $\hat{P}(\rho)^{k} \to 0$.

LEMMA (UPPER BOUND LEMMA) $4\|P^{k}-u\|^2 \le \sum_{\rho \ne 1} d_\rho \|\hat{P}(\rho)\|^{k}$ (TRACE NORM)

PROOF $4\|P^{k}-u\|^2 = \left(\sum |P(g)-u(g)|\right)^2 \le |G| \sum |P^{k}(g)-u(g)|^2$

$$= \sum_{\rho \ne 1} d_\rho \|\hat{P}(\rho)\|^{2k} \text{ (PLANCHEREL).}$$

8) BACK TO RANDOM TRANSPOSITIONS ON S_n

KEY FACT P IS CONSTANT ON CONJUGACY CLASSES

$$P(\eta^{-1}\sigma\eta) = P(\sigma)$$

SCHUR'S LEMMA (ANY SUCH G ON ANY SUCH GROUP)

$$\hat{P}(\rho) = cI, \quad c = \mathrm{Tr}\,\hat{P}(\rho)/d_\rho$$

e.g. $P(\sigma) = \begin{cases} 1/n & \sigma = \text{id} \\ 2/n^2 & \sigma = (i,j) \\ 0 & \text{—} \end{cases}$

$$\hat{P}(\rho) = c\, I \qquad c = \left(\frac{1}{n} + \frac{n-1}{n}\frac{\chi_\rho(12)}{d_\rho}\right)$$

$$\chi_\rho(\sigma) = \mathrm{Tr}(\rho(\sigma)) \quad \text{CHARACTER}$$

SO

$$4\|P^k - u\|^2 \le \sum_{\substack{\rho \in \hat{S}_n \\ \rho \ne 1}} d_\rho^2 \left(\frac{1}{n} + \frac{n-1}{n}\frac{\chi_\rho(12)}{d_\rho}\right)^{2k}$$

4) FINALLY, WE'VE GONE AS FAR AS WE CAN "WITHOUT KNOWING ANYTHING"

- WHAT IS $\hat{G}$
- WHAT IS d_ρ
- WHAT IS $\chi_\rho(12)$

LOOK AT CHARACTER TABLES, USUALLY $\chi_\rho(12)/d_\rho$ SMALL ($\doteq 1/2$)

IF SO

$$\sum_\rho d_\rho^2\left(\frac{1}{n} + \frac{n-1}{n}\frac{\chi_\rho}{d_\rho}\right)^{2k} \doteq \sum_\rho d_\rho^2\left(\frac{1}{2}\right)^{2k} = \frac{n!}{2^{2k}},$$

SO $k \gg n\log n$ MAKES THIS SMALL.

BUT $n-1$ DIMENSIONAL REPRESENTATION

$$d_{n-1,1} = n-1, \quad \chi_\rho = n-3, \quad \left(\frac{1}{n} + \frac{n-1}{n}\frac{\chi_\rho}{d_\rho}\right) = \left(1 - \frac{2}{n}\right)^{2k}$$

HAVE TO MAKE

$$(n-1)^2\left(1 - \frac{2}{n}\right)^{2k} \le e^{2\log n - \frac{4k}{n}} \quad \text{SMALL}$$

$k = \frac{1}{2}n\log n + cn$ MAKES IT $\le e^{-4c}$.

10) COMPARISON THEORY

SAY YOU HAVE A GROUP G, GENERATING SET S

ALSO HAVE A GENERATING SET $\tilde{S}$ AND

'KNOW EVERYTHING' ABOUT $\tilde{P}$

THEN, IF UNDERSTAND HOW TO WRITE ELTS OF S WITH $\tilde{S}$

YOU CAN GET GOOD ESTIMATES FOR P.

EXAMPLE $G = S_n$, $S = \{(12)\ (12\ldots n)\}$ VERY DIFFERENT FROM $\{(i,j)\}$.

GET $\frac{1}{6} e^{-k/n^3 \log n} \leq \| P^k - U \| \leq 6 e^{-k/n^3 \log n}$

INDEED THIS RESULT HOLDS FOR ALMOST EVERY PAIR $\{\sigma, \tau\} \subseteq S_n$!

(HELFGOT, SERESS, ZUK) (WE CONJECTURE CUTOFF)

11) PROGRAM

- FIND A (FAMILY OF) GROUPS YOU LIKE
- PICK A GENERATING SET OF INTEREST S
- FIND A SMALL GENERATING CONJUGACY CLASS (OR SET OF THEM) $\tilde{S}$
- DO THE WORK REQUIRED TO UNDERSTAND $P_{\tilde{S}}$
- DO THE WORK TO UNDERSTAND THE 'GEOMETRY OF THE CAYLEY GRAPH OF (G,S) IN TERMS OF $\tilde{S}$.'

- SEND ME A COPY!

12) PROGRESS (i) PART 1 IS IN PRETTY GOOD SHAPE (CONJUGACY)

(WORK OF GLUCK, LIEBECK, SHALEV, GURALNICK, TIEP, ...)

FOR FINITE GROUPS OF LIE TYPE.

$GL_n(q)$, $U_n(q)$, ... THERE ARE EVEN SOME

CUTOFF RESULTS (RANK TO N+S)

BUT PART 2 REMAINS TO BE DONE

SPECIFIC PROBLEM: ALL SUCH GROUPS ARE

GENERATED BY TWO GENERATORS. TAKE

STEINBERGS GENERATORS + GO FORWARD

(b) FOR LOW-CLASS NILPOTENT GROUPS (: BDD # GENERATORS, RANK)

eg HEISENBERG GROUP $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ $x, y, z \in C_n$.

$(\text{DIAMETER})^2$ N + S (SALOFF-COSTE)

FOURIER ANALYSIS
- POSSIBLE (CARLOS ANDRE - SUPERCHARACTERS)
- DIFFICULT

(c) P-GROUPS OF MAXIMAL CLASS, HIGH COCLASS

JONATHAN HERMAN, SAM THOMAS

(d) SOLVABLE GROUPS, eg $x \mapsto ax+b$ (mod p)

- FASCINATING (HILDEBRAND, US, BRAINARD, VARJU, EBERHARDT...)
- LARGELY OPEN.

(e) THERE IS PROGRESS: PROVING CUTOFF ON S_n FOR

- TOP TO RANDOM (WITH ALDOUS)
- RANDOM TO RANDOM (BERNSTEIN, NESTORIDI)

- ADJACENT TRANSPOSITIONS (LACOIN)
- BERNOULLI-LAPLACE,
- GELFAND PAIRS

(3) <u>OTHER TECHNIQUES</u> COUPLING, STATIONARY TIMES, LOG-SOBOLEV, GRAPH THEORY (BERESTYCKI, SCHRAMM), METHOD OF MIRACLES

(4) WE WORK ON ALL KINDS OF GROUPS; COMPACT, DISCRETE, LIE, P-ADIC, ON INFINITE GROUPS REPLACE 'RATES OF CONVERGENCE' WITH 'RATE OF DECAY' OF $P^{*k}(id)$, DIFFERENT METRICS

(5) IN ADDITION TO 'RATES OF CONVERGENCE' WE DO

- FIRST HITTING TIME (TO POINT, SET)
- COVER TIMES (SIZE OF RANGE)
- BIRTHDAY PROBLEMS
- BEHAVIOUR BEFORE CONVERGENCE

(6) REAL PROBLEM: ALL OF ABOVE IS 'WARM UP' FOR AN ACTUALLY IMPORTANT PROBLEM;

TAKE <u>ANY</u> <u>ACTUAL</u> MARKOV CHAIN USED BY A PHYSICIST, CHEMIST, BIOLOGIST, STATISTICIAN AND TRY TO PROVE <u>ANYTHING</u> <u>USEFUL</u> ABOUT ITS MIXING TIME.

REFERENCES

LEVIN, D. AND PERES, Y. 'MARKOV CHAINS AND MIXING TIMES'
(AMS + FREE ON DAVID LEVIN'S HOME PAGE)

DIACONIS, P. 'THE MARKOV CHAIN MONTE CARLO REVOLUTION'
(BULL. AMS + ON MY HOME PAGE)

RANDOM WALKS ON GROUPS - CHARACTERS AND GEOMETRY.

(EASYEST TO FIND ON MY HOME PAGE)

SALOFF-COSTE, L. PROBABILITY ON GROUPS: RANDOM
WALKS AND INVARIANT DIFFUSIONS (NOTICES AMS)

+ I'M STILL HARD AT WORK ON THIS
PLEASE LOOK AT MY HOME PAGE
FOR LATEST ///