Geometrical diffeomorphism invariant observables for General Relativity and their applications

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in collaboration with:
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Plan of the talk

Introduction

- crash course on canonical General Relativity
- problem of observables

Construction

- observer's observables
- Poisson algebra of the observables

Application

- phase space reduction — gauge fixing
- comments on quantum reduction to spherical symmetry

General Relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \longleftrightarrow \begin{aligned} S &= S_{HE}[g_{\mu\nu}] + S_{\text{matter}} \\ S_{HE}[g_{\mu\nu}] &= \int_{\mathcal{M}} {}^4 R \sqrt{-g} \end{aligned}$$

Introduction — 3 + 1 split

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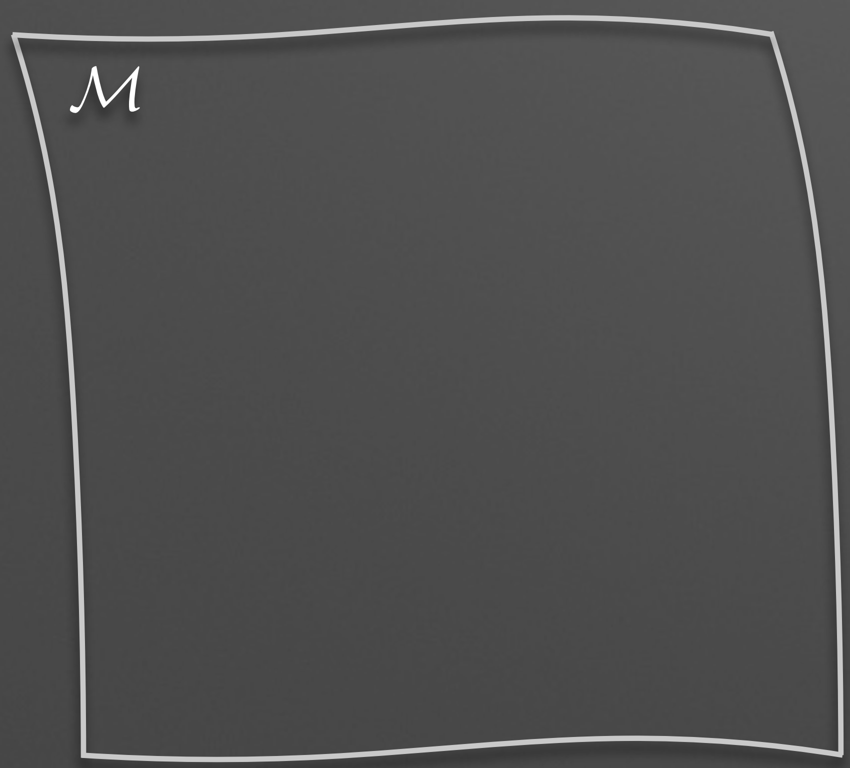
3 + 1 split of spacetime — the ADM formalism



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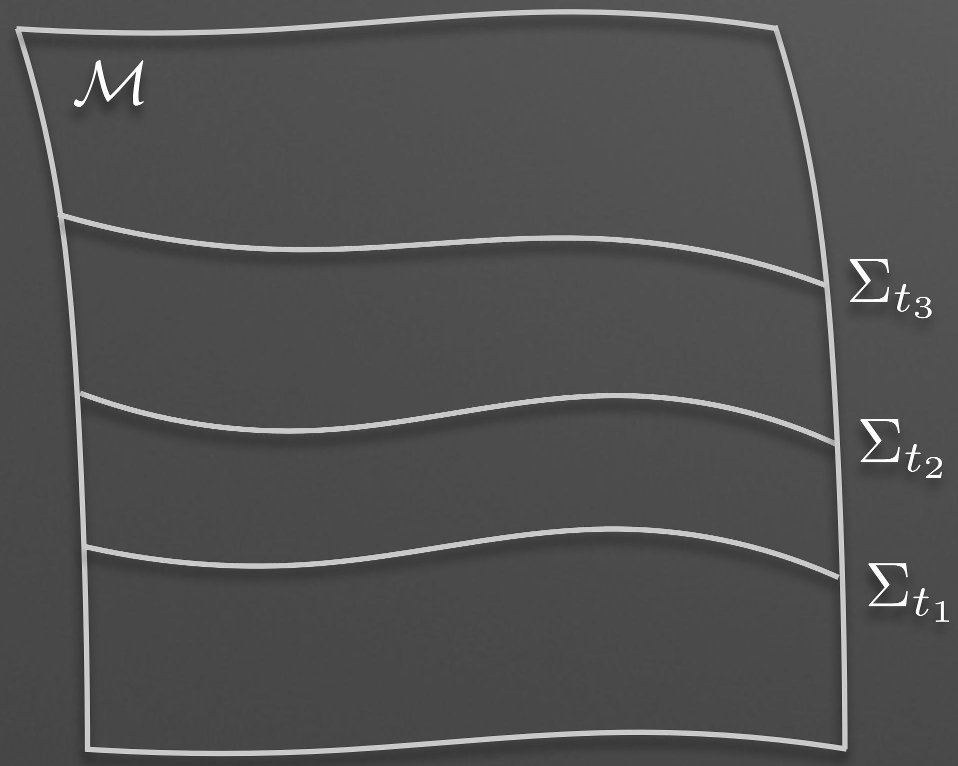

$$\Sigma_t := \{\text{level sets of } t\}$$

Introduction — $3+1$ split

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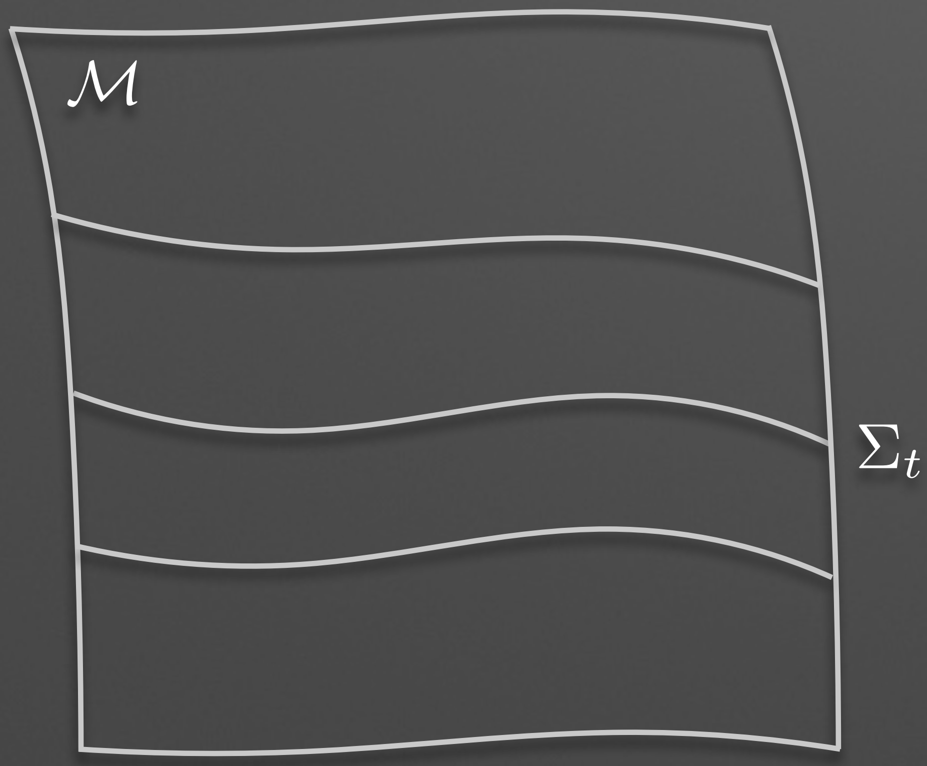
$3+1$ split of spacetime — the ADM formalism



$$(x^\mu) \longrightarrow (t, x^i)$$

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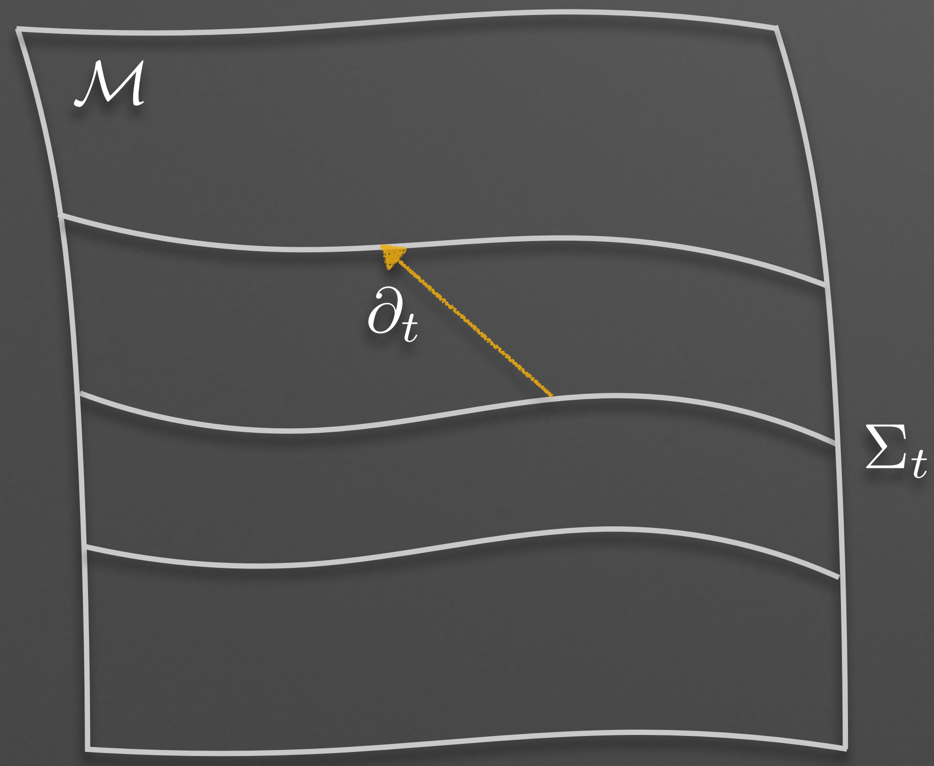

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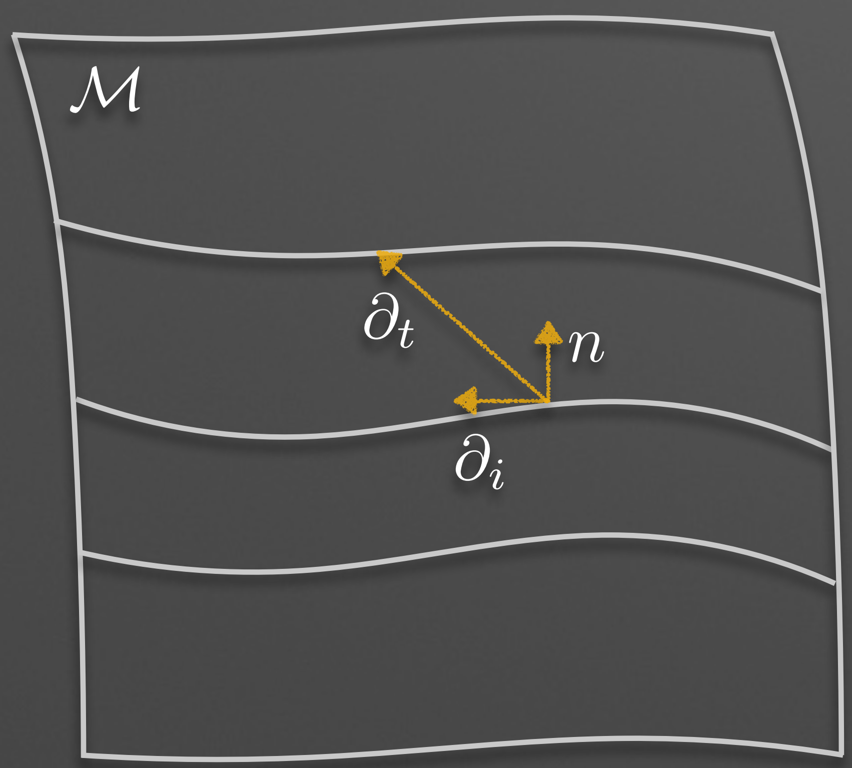

$$\partial_t = Nn + N^i \partial_i$$

N is lapse function
N^i is shift vector

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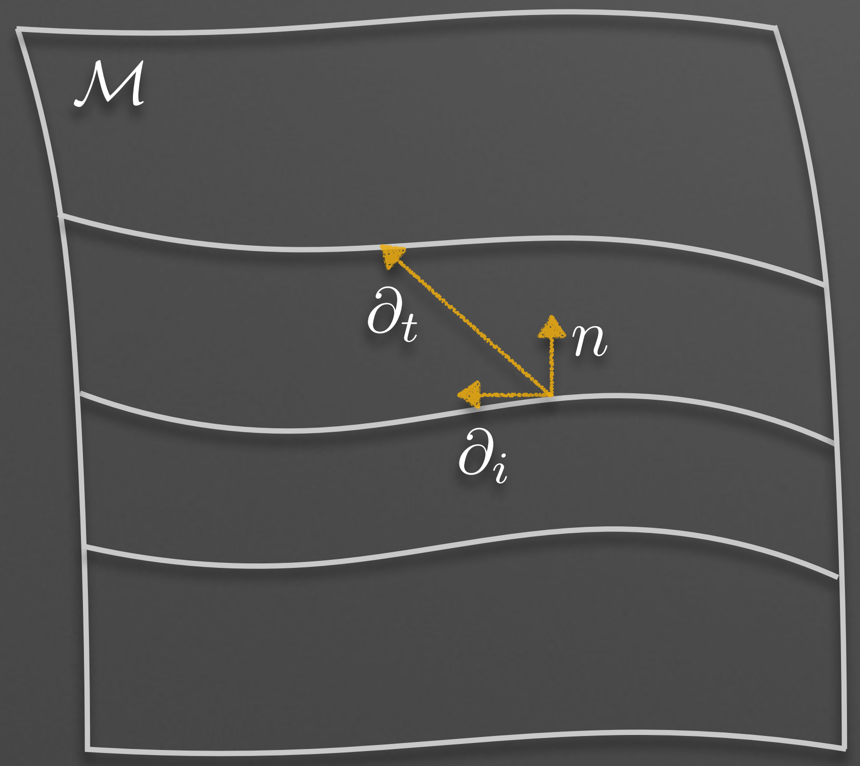

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N is lapse function $\quad$ q_{ij} is spatial metric
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$$g_{\mu\nu} \longrightarrow \begin{bmatrix} -N^2 + N^i N_i & N_i \\ N_i & q_{ij} \end{bmatrix}$$

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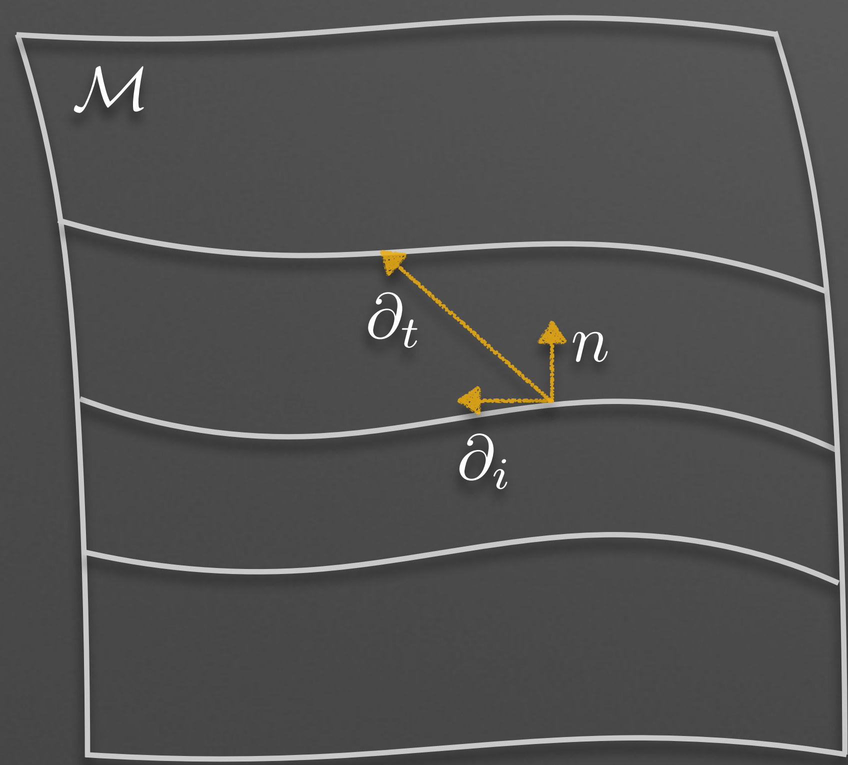

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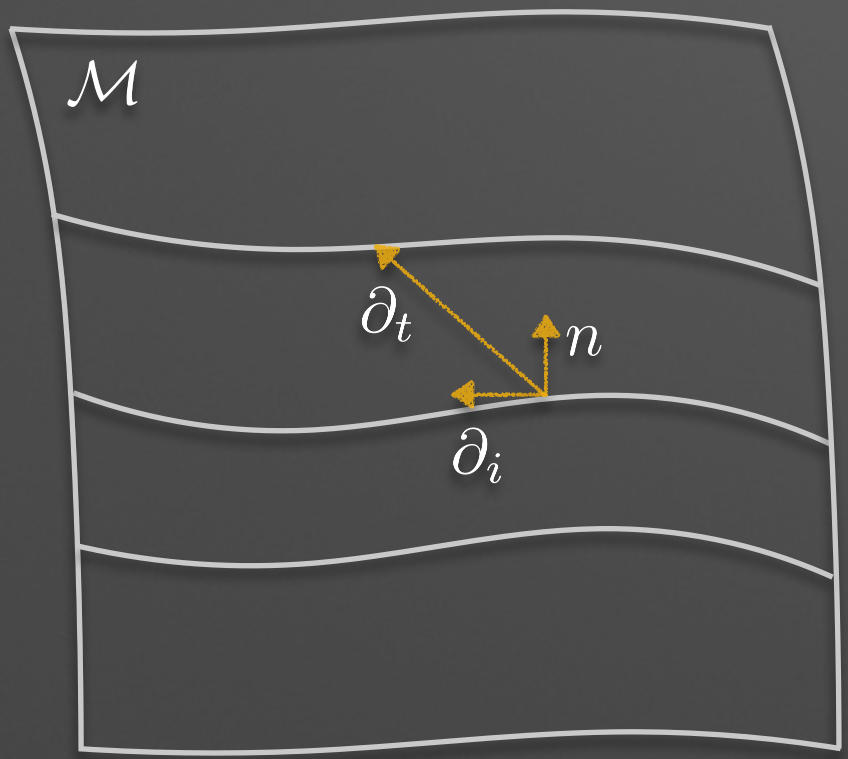

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$$S_{HE} = \int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x N \sqrt{\det q} \left({}^3R + K^{ij} K_{ij} - (K^i{}_i)^2 \right)$$

Introduction — canonical GR

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Legendre transform

$$p^{ij} := \frac{\delta L_{HE}}{\delta \dot{q}_{ij}} = \sqrt{\det q}(q^{ik}q^{jl} - q^{ij}q^{kl})K_{kl}$$

$$\frac{\delta L_{HE}}{\delta \dot{N}} = 0 = \frac{\delta L_{HE}}{\delta \dot{N}^i}$$

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Hamiltonian is

$$\mathcal{H} = H[N] + C[\vec{N}] \qquad \text{with} \qquad \begin{aligned} H[N] &= \int_\Sigma N \left(\frac{1}{\sqrt{q}} (2p^{ab}p_{ab} - (p^a{}_a)^2) - \frac{\sqrt{q}}{2} {}^3R \right) \\ C[\vec{N}] &= \int_\Sigma N^i \left(-2D_j p^j{}_i \right) \end{aligned}$$

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Poisson bracket

$$\{q_{ij}(\sigma),\ p^{kl}(\sigma')\} = \delta^k_{(i}\delta^l_{j)}\delta(\sigma, \sigma')$$

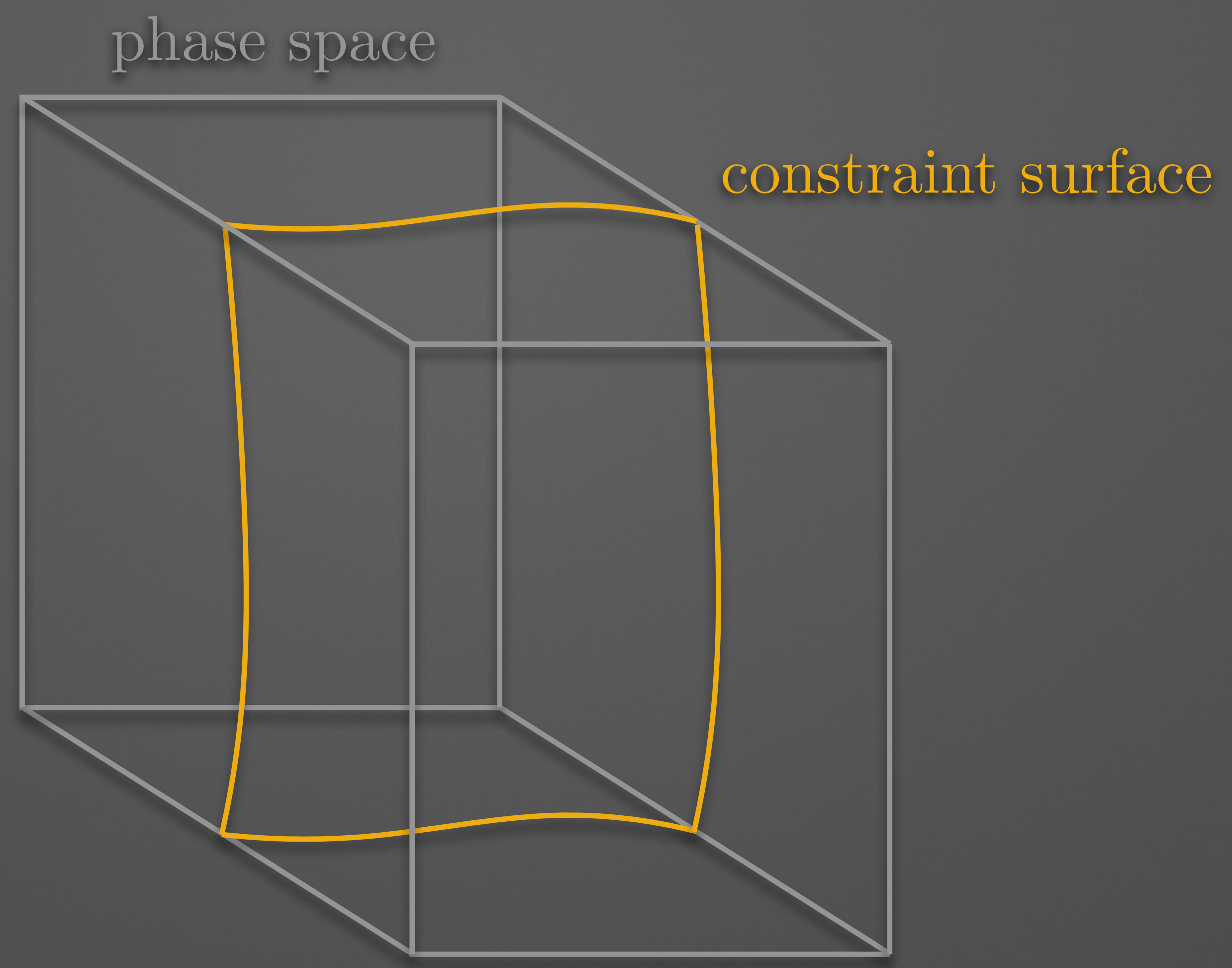
Dirac algebra

$$\{C[\vec{N}],\ C[\vec{N}']\} = C[(N^i\partial_i N'^j - N'^i\partial_i N^j)\partial_j]$$

$$\{C[\vec{N}],\ H[N]\} = H[N^i\partial_i N]$$

$$\{H[N],\ H[N']\} = C[q^{ij}(N\partial_i N' - N'\partial_i N)\partial_j]$$

Introduction — observables



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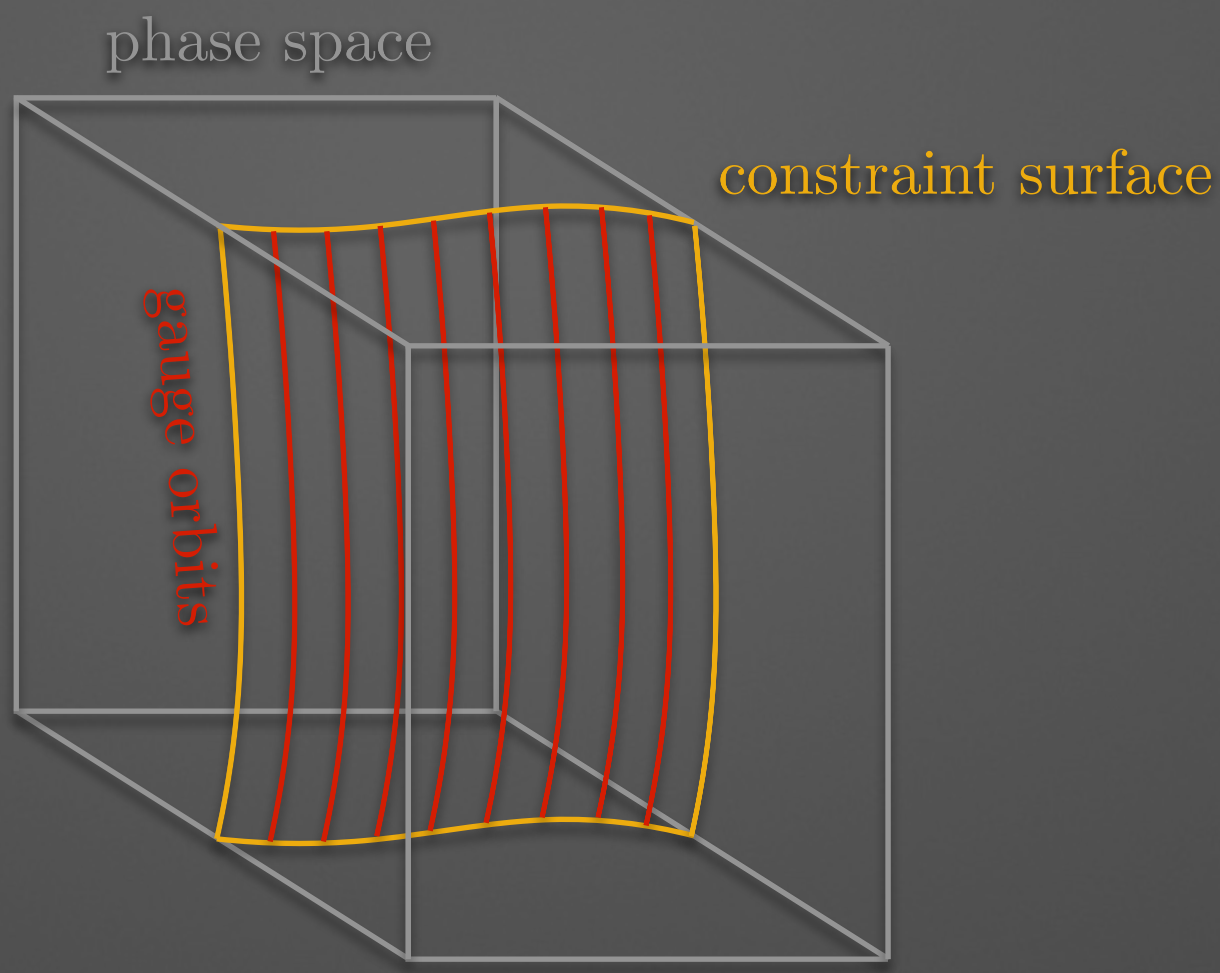

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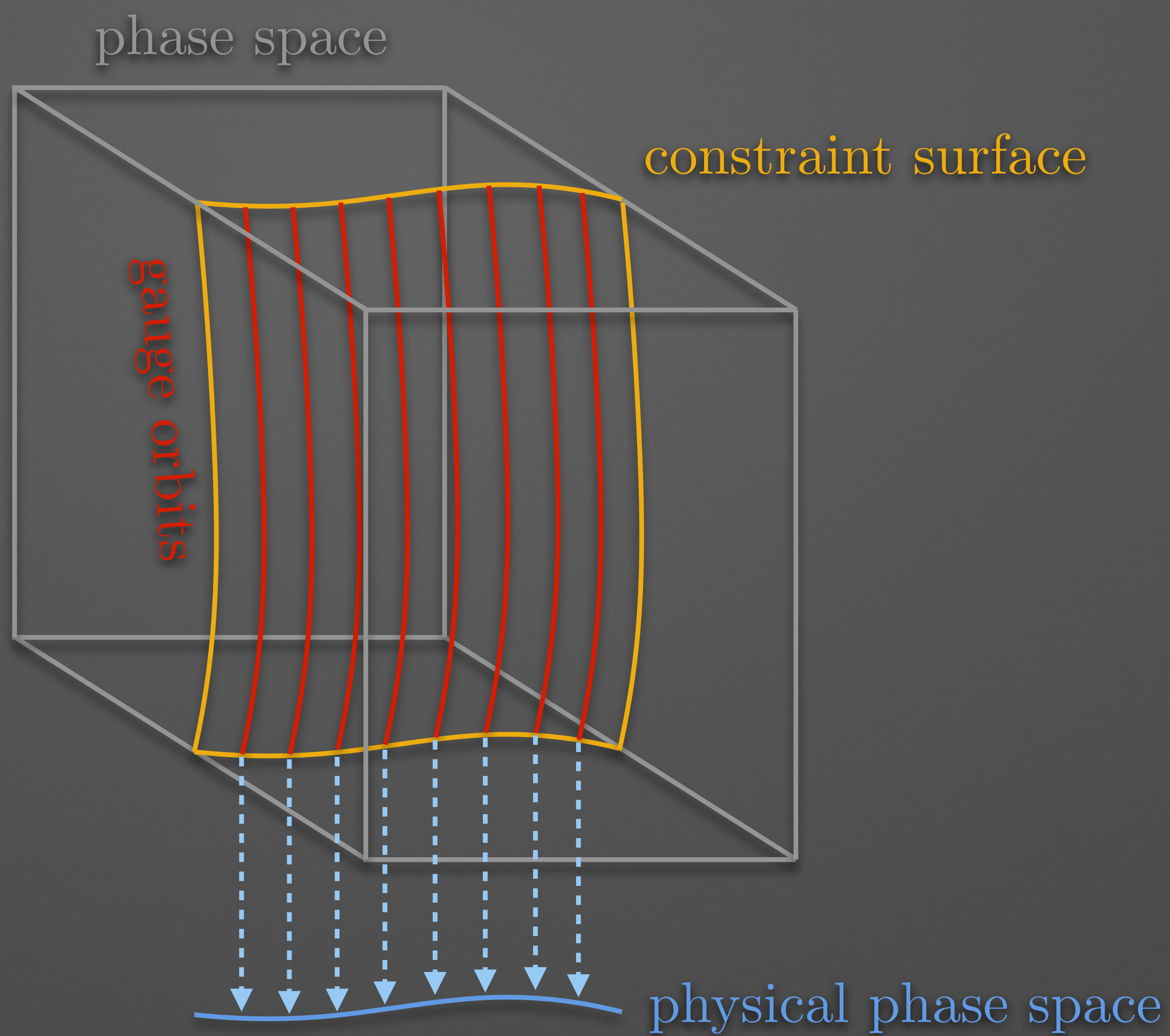

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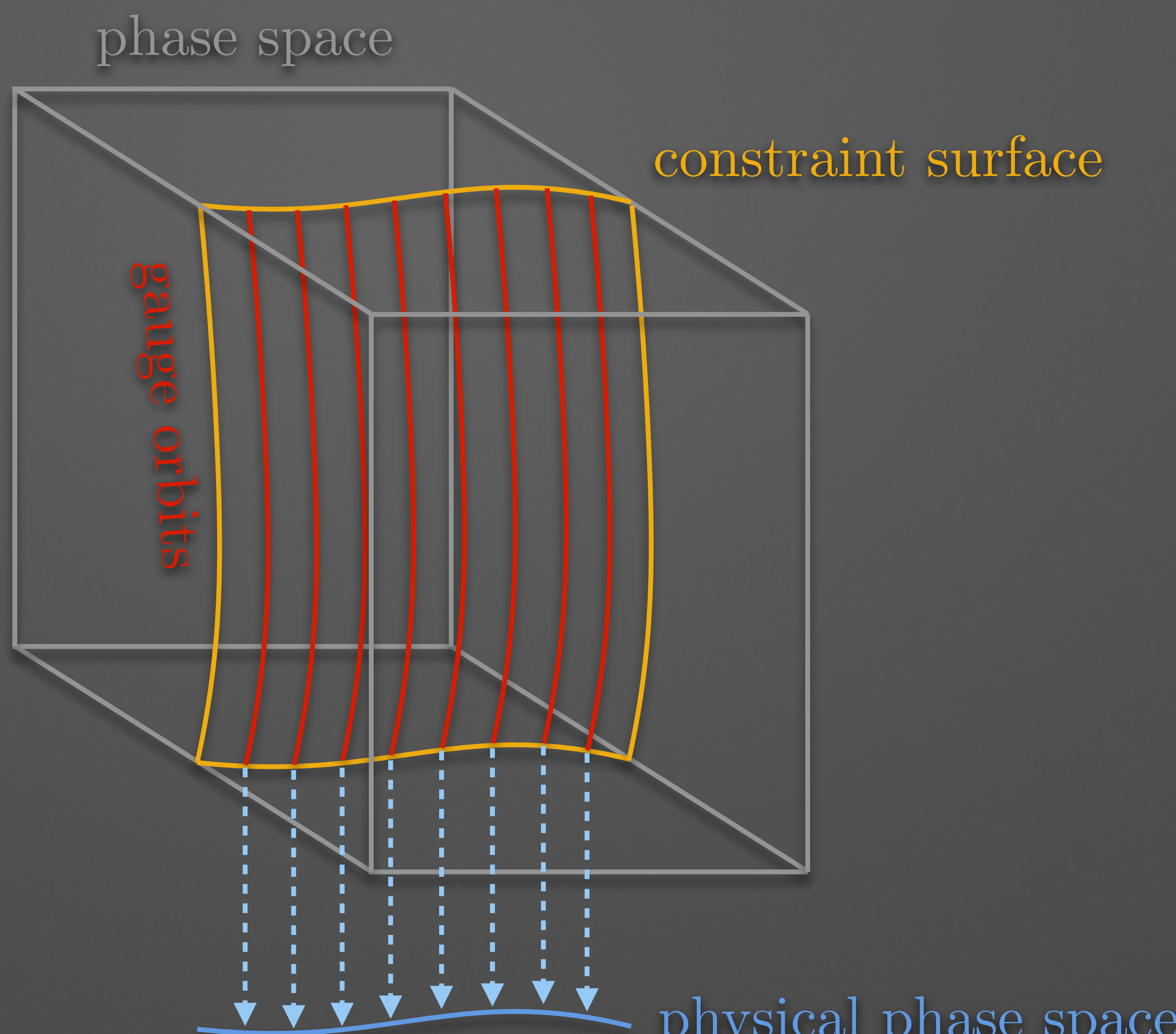
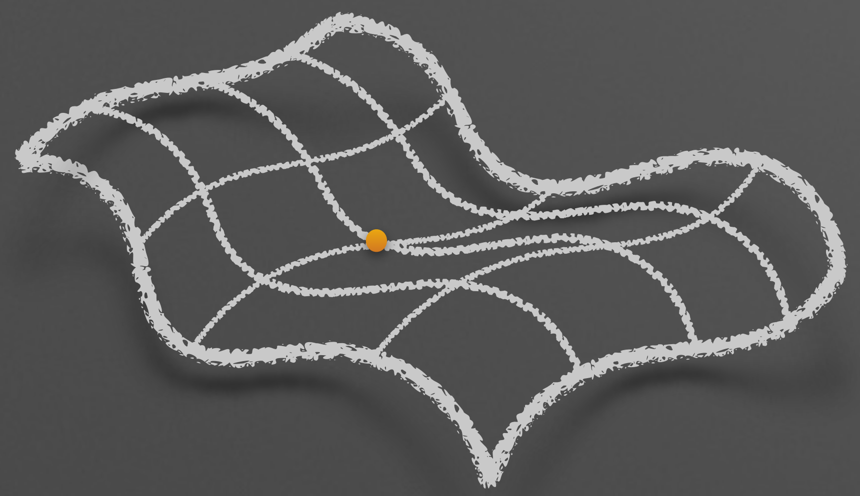


observables — function(al)s on the phase space

Dirac observables — gauge independent observables

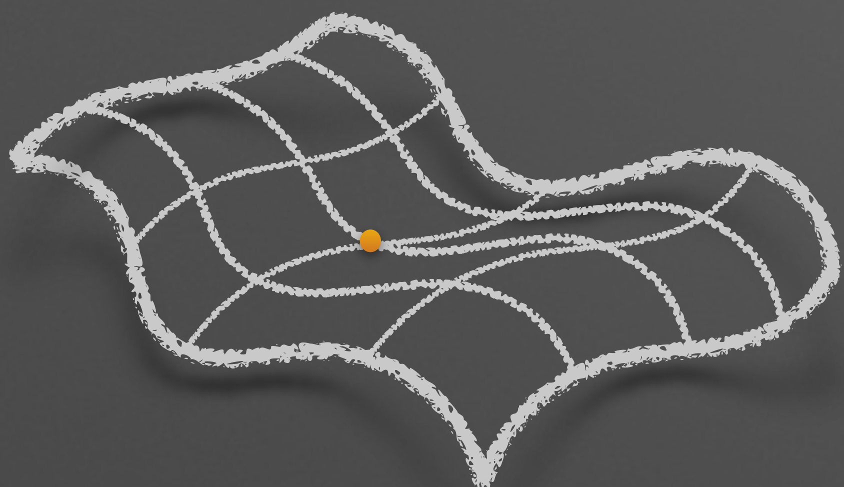
for GR: Dirac observables are observables invariant under diffeomorphisms curvature scalars



Bergmann & Komar

Introduction — observables — relational approaches

curvature scalars



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dust fields

Brown & Kuchař

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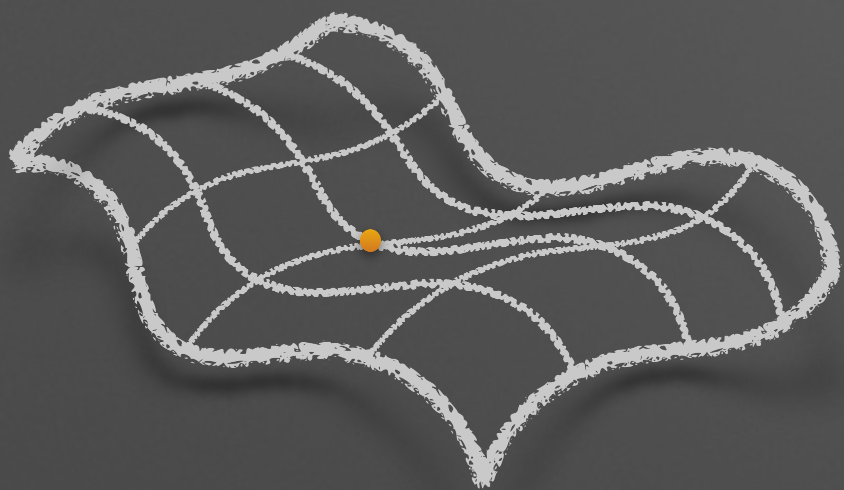

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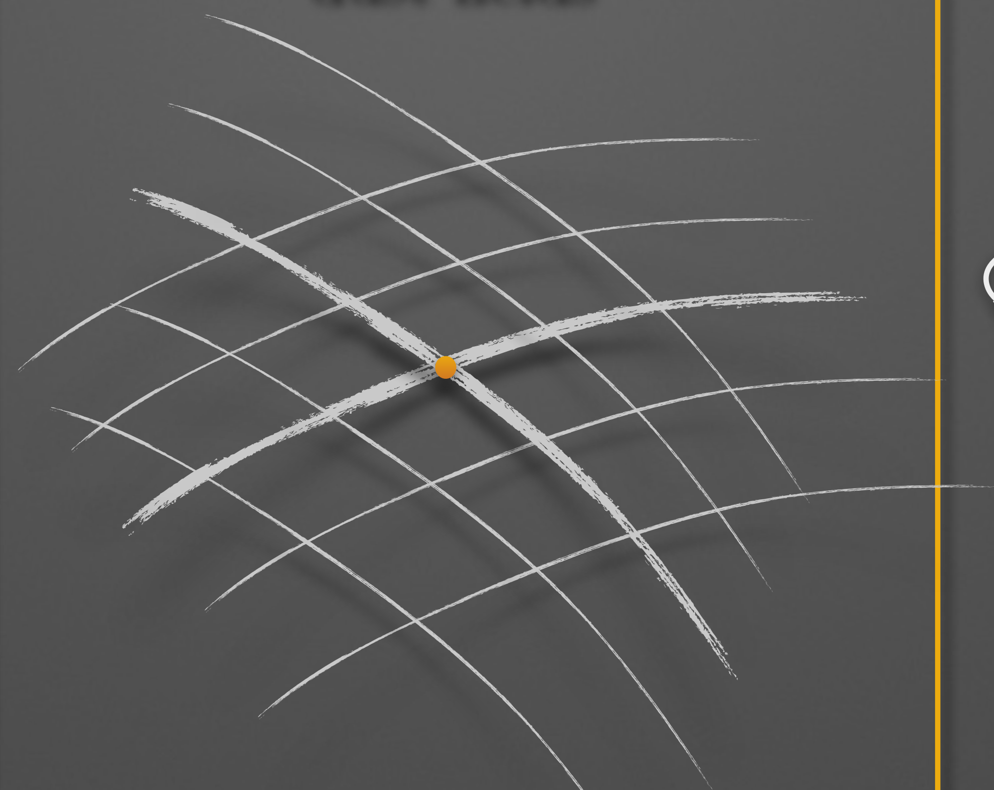

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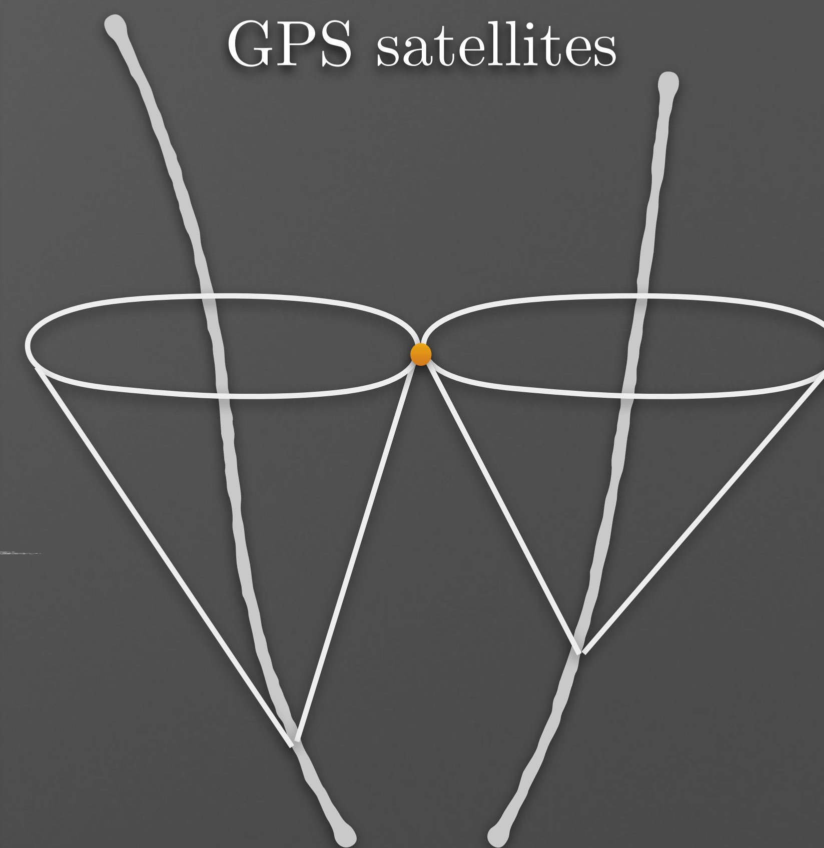
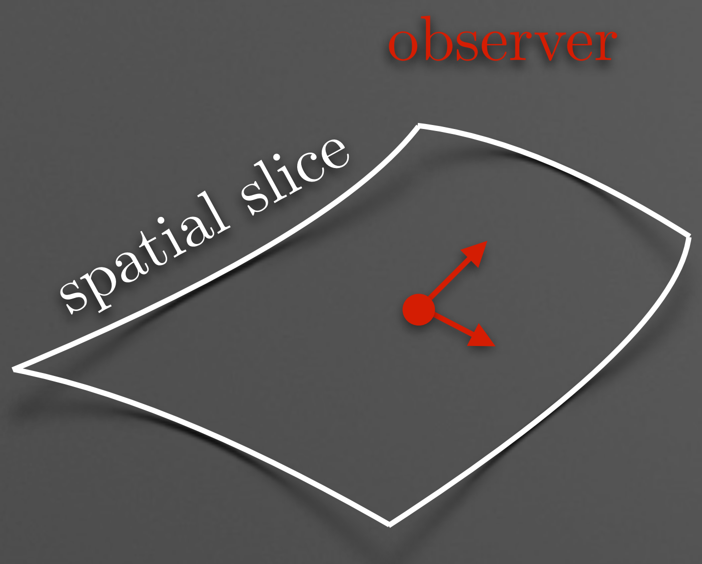
GPS satellites



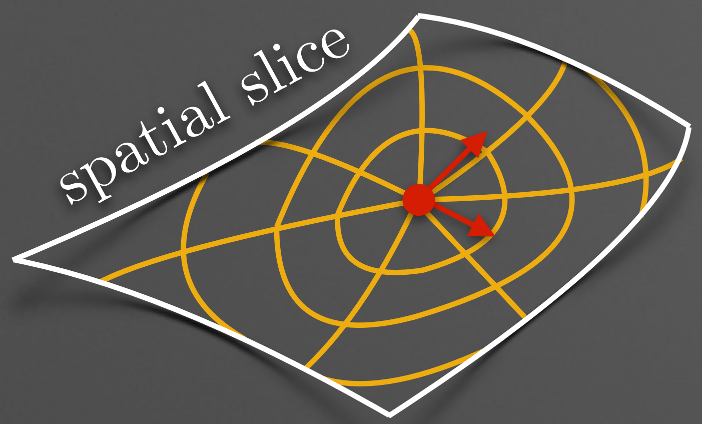
Rovelli

Construction — observer's observables



Construction — observer's observables

observer



adapted coordinates

(r, θ)

Construction — observer's observables

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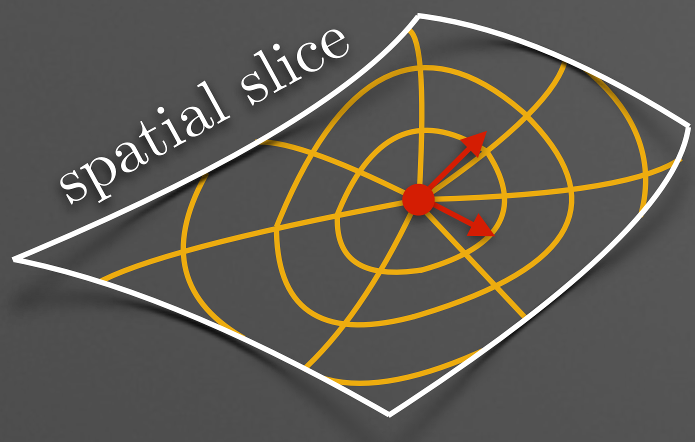

adapted coordinates

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observables

$$Q_{ab}(r,\theta) : (q,p,\phi,\pi) \mapsto q_{ab}(r,\theta)$$

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Construction — observer's observables

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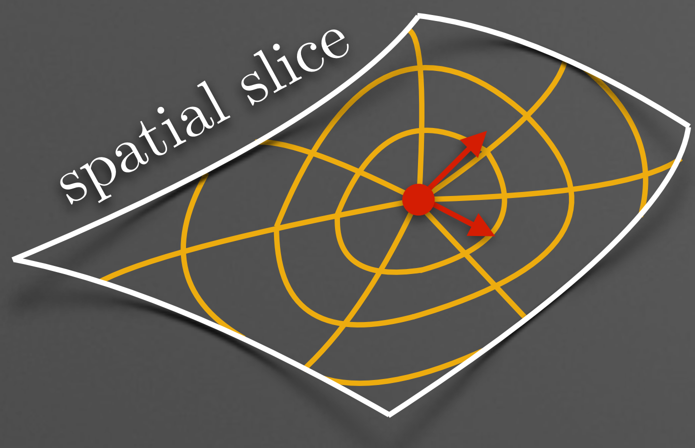

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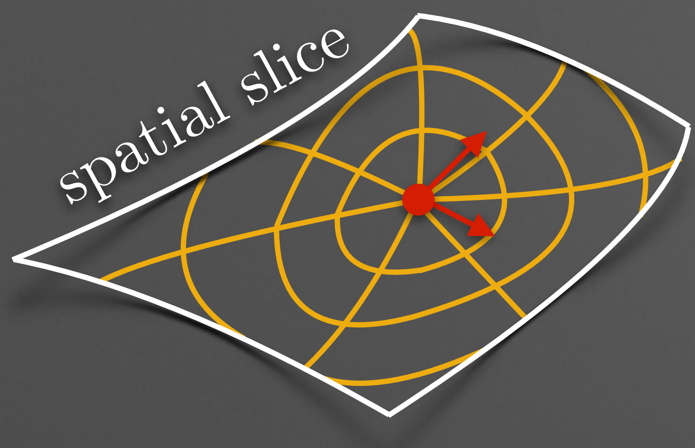
Construction — observer's observables



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by construction

$$Q_{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{Q_{AB}}} \\ 0 & & \end{bmatrix}$$

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Construction — observer's observables

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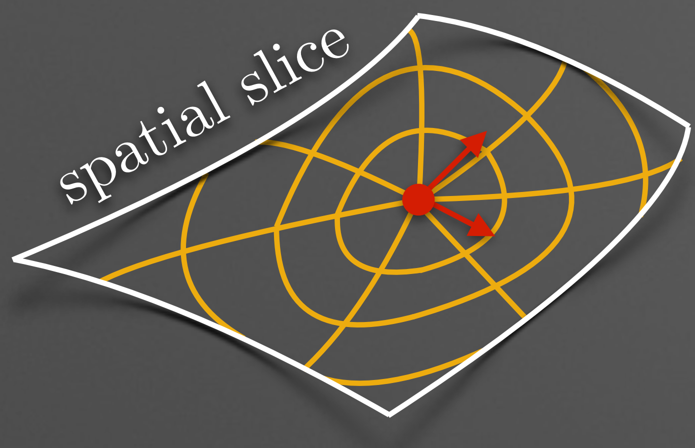

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Poisson algebra

$$\{Q_{AB},\ P^{CD}\} = \delta\delta\delta \qquad \{\Phi,\ P^{AB}\} = 0, \text{ etc.}$$
$$\{\Phi,\ \Pi\} = \delta$$
$$\{\cdot,\ P^{ra}\} = \text{nontrivial}$$

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Construction — observer's observables 2

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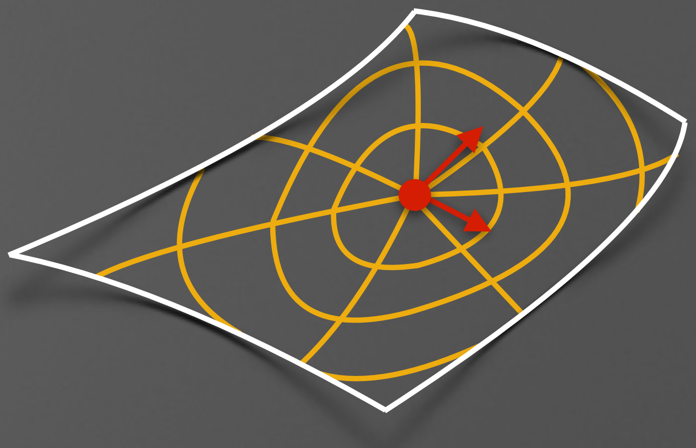

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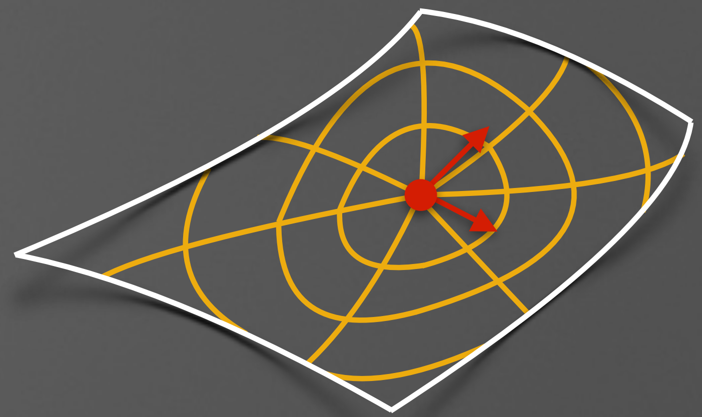

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- find a shift generating a diffeomorphism s.t. $(\mathcal{L}_{\vec{N}}q)_{ra} = \delta q_{ra}$
- preserving the observer: $N^I(\sigma_0) = 0, \quad \partial_J N^I(\sigma_0) = \triangle$

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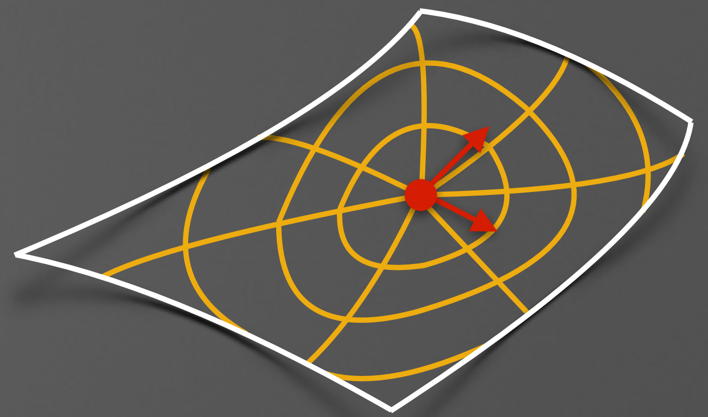

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remarkably this can be done!

$$\vec{N} = \left[\frac{1}{2}\int_0^r dr' \delta q_{rr}(r',\theta)\right]\partial_r + \left[\int_0^r dr' q^{AB}(r',\theta)\left(\delta q_{rB}(r',\theta) - \frac{1}{2}\int_0^{r'} dr''\partial_B \delta q_{rr}(r'',\theta)\right)\right]\partial_A$$

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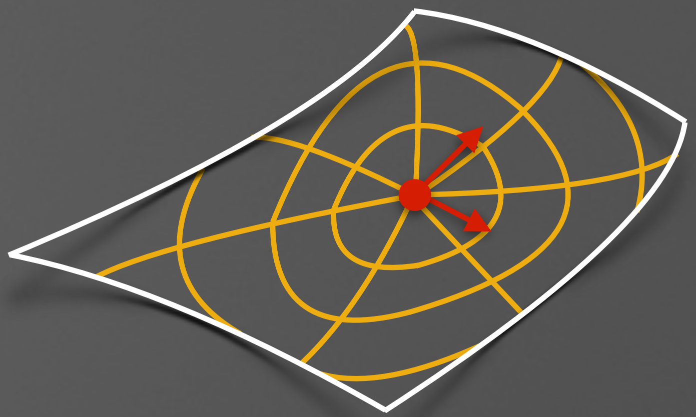

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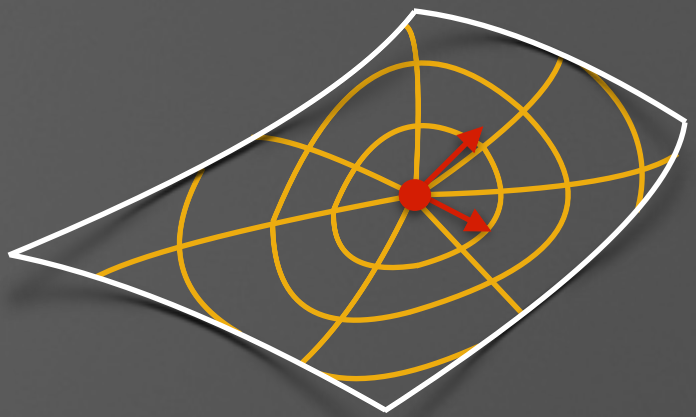
Application — gauge fixing GR

$$q_{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{q_{AB}} \end{bmatrix} \quad \text{i.e.} \quad q_{ra} = \delta_{ra}$$

$\boxed{\text{the radial gauge}}$



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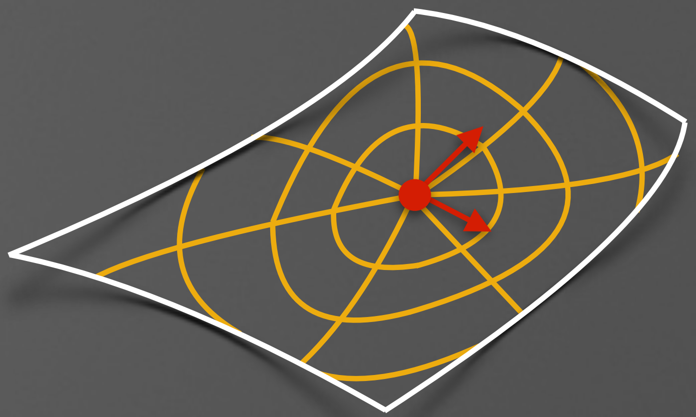

To impose the gauge you need a Hamiltonian which preserves it, namely s.t.

$$\{q_{ra},\ H[N] + C[\vec{N}]\} \approx 0\ , \quad \text{which leads to} \quad (\mathcal{L}_{\vec{N}}q)_{ra} = -\{q_{ra},\ H[N]\}\ .$$

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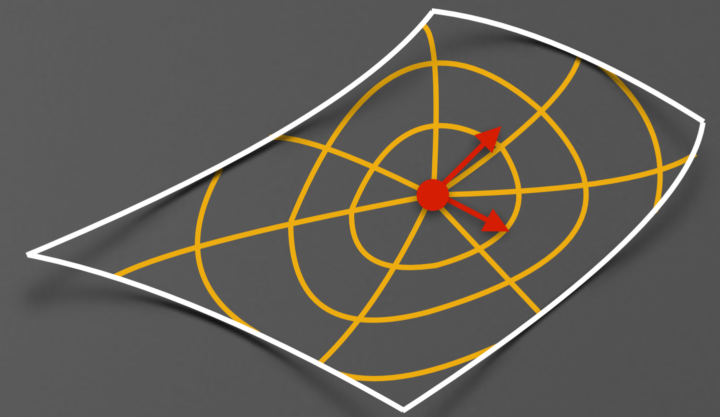

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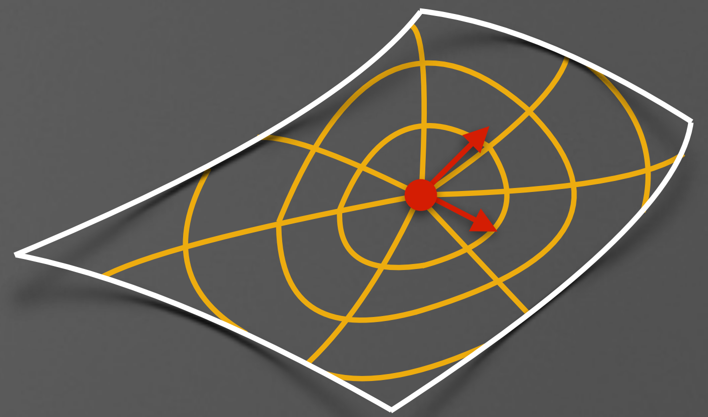
The vector constraint $C_a \approx 0$ can be solved for
$$\begin{aligned} p^{rr} &= \text{function}(q_{AB}, p^{AB}, \phi, \pi) \\ p^{rA} &= \text{function}(q_{AB}, p^{AB}, \phi, \pi) \end{aligned}\ .$$

Application — gauge fixing GR

$$q_{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & q_{AB} \end{bmatrix} \quad \text{i.e.} \quad q_{ra} = \delta_{ra}$$

the radial gauge



To impose the gauge you need a Hamiltonian which preserves it, namely s.t.

$$\{q_{ra},\ H[N] + C[\vec{N}]\} \approx 0\ , \quad \text{which leads to} \quad (\mathcal{L}_{\vec{N}} q)_{ra} = -\{q_{ra},\ H[N]\}\ .$$

The vector constraint $C_a \approx 0$ can be solved for
$$\begin{aligned} p^{rr} &= \text{function}(q_{AB}, p^{AB}, \phi, \pi) \\ p^{rA} &= \text{function}(q_{AB}, p^{AB}, \phi, \pi) \end{aligned}\ .$$

The resulting theory is parametrised by $(q_{AB}, p^{AB}, \phi, \pi)$ with the Hamiltonian constraint

$$\widetilde{H}[N] = \int N \left(\frac{2}{\sqrt{q}} G - \frac{\sqrt{q}}{2} {}^{(3)}R + h^{\text{matt}} \right) \quad \text{where}$$

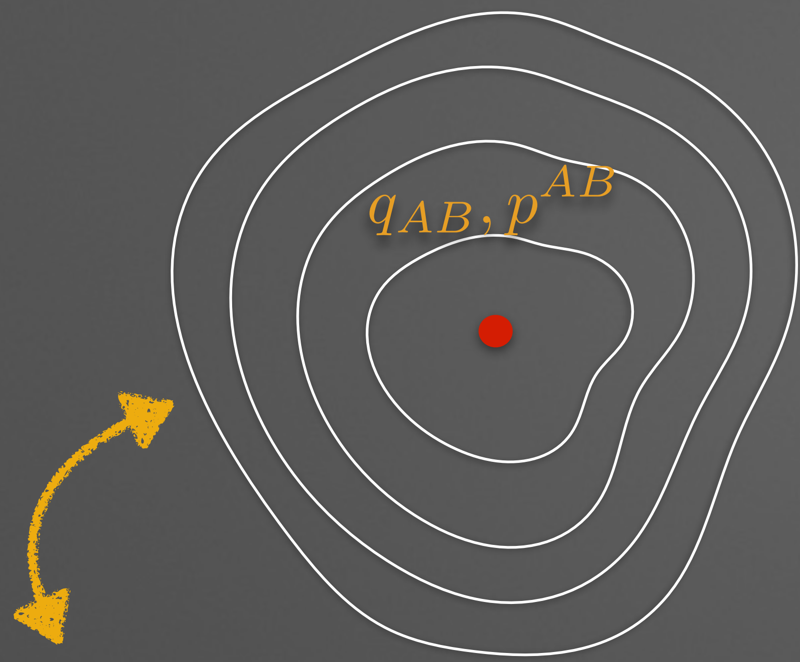
$$\begin{cases} G = \frac{1}{2}({p^r}_r)^2 + 2q^{AB} {p^r}_A {p^r}_B - q_{AB} p^{AB} {p^r}_r + (q_{AC} q_{BD} - \frac{1}{2} q_{AB} q_{CD}) p^{AB} p^{CD} \\ {}^{(3)}R = {}^{(2)}R - q^{AB} q_{AB,rr} - \frac{3}{4} {q^{AB}}_{,r} q_{AB,r} - \frac{1}{4}(q^{AB} q_{AB,r})^2 \end{cases}$$

$${p^r}_r = \frac{1}{2} \int_0^r p^{AB} q_{AB,r} + \int_0^r \mathcal{D}_A \left(q^{AB} \int_0^{r'} \mathcal{D}_C {p^C}_B \right) \qquad {p^r}_A = -\int_0^r \mathcal{D}_B {p^B}_A$$

Application — defining spherical symmetry on the quantum level

$$\begin{tikzcd} \text{General Relativity} \arrow[r] \arrow[d] & \text{Loop Quantum Gravity} \\ \text{midisuperspace model} \arrow[r] & \text{quantum spherically symmetric models} \end{tikzcd}$$

Application — defining spherical symmetry on the quantum level

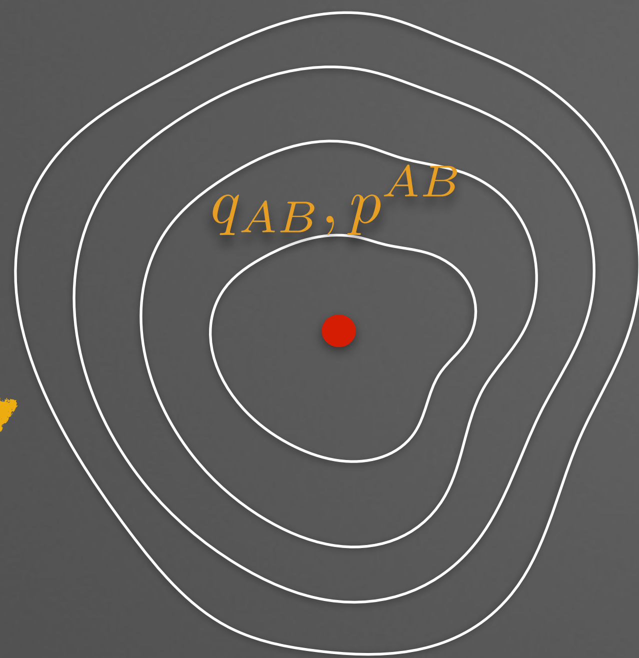


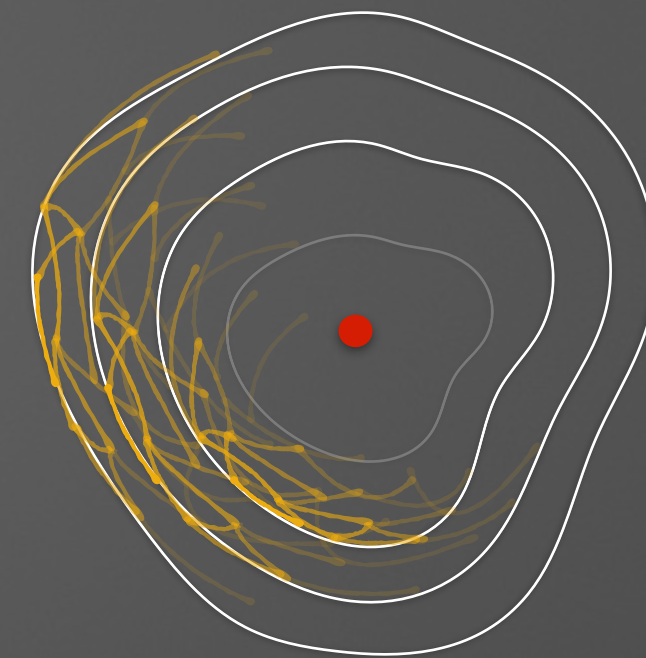
General Relativity → Loop Quantum Gravity

↓

midisuperspace model → quantum spherically symmetric models

Application — defining spherical symmetry on the quantum level





General Relativity → Loop Quantum Gravity

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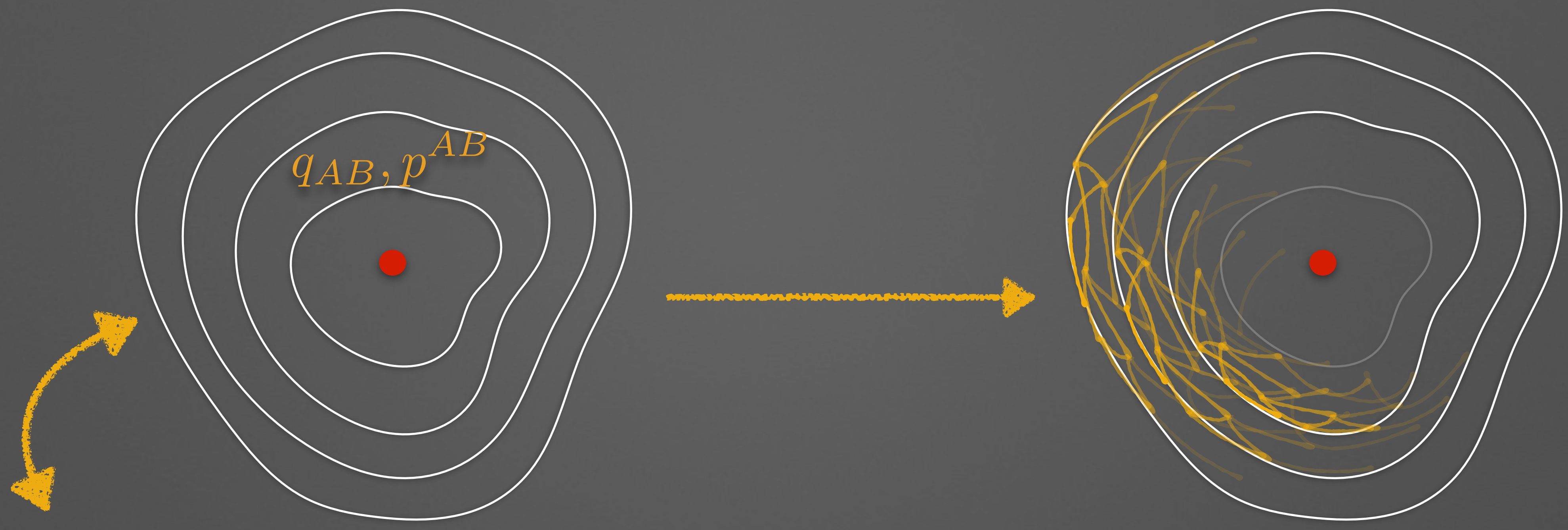
midisuperspace model → quantum spherically symmetric models

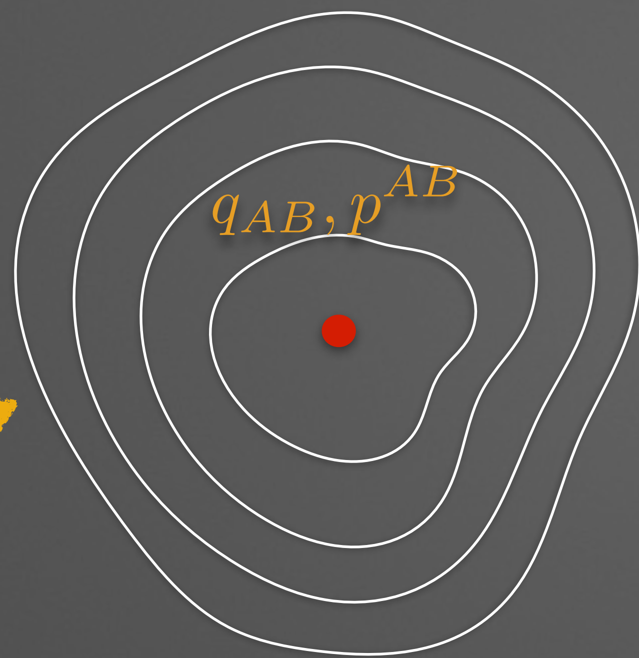
Application — defining spherical symmetry on the quantum level

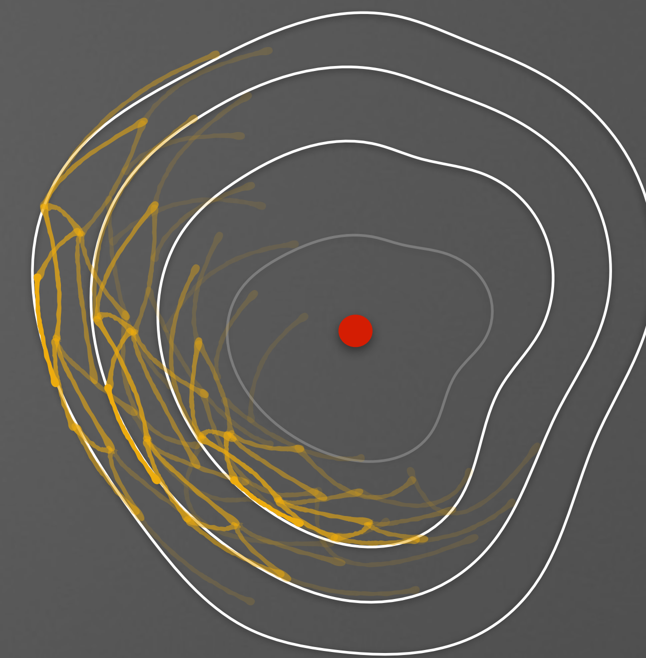


General Relativity → Loop Quantum Gravity

midisuperspace model → quantum spherically symmetric models

Application — defining spherical symmetry on the quantum level





General Relativity → Loop Quantum Gravity

midisuperspace model → quantum spherically symmetric models

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Thank you for your attention

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