Eigenstate thermalization scaling

Many-body Hamiltonians are random matrices (?!?)

Eigenstate thermalization hypothesis (which observables?)

E.T.H. Scaling — local, less local & Behemoth operators

\[ \frac{J_2}{J_1} = 0.4, 0.5 \]

\[ \langle r \rangle \]

\[ \text{GOE} \]

\[ \text{Poisson} \]

\[ \text{XXZ} + NNN \]

\[ (L,N_p) = (17,8) \]

\[ J_1 = J_2 = 1, \Delta_1 = \Delta_2 = 0.8 \]

\[ n=8, M=2 \]

\[ n=7, M=6 \]

\[ n=1, M=12870 \]

(a) (b) (c)

(d) (e)
Beugeling, Moessner, Haque, P.R.E (2014)
Finite-size scaling of eigenstate thermalization

Beugeling, Andreanov, Haque, JSTAT (2015)
Entanglement & participation ratios, all eigenstates

Beugeling, Moessner, Haque, P.R.E (2015)
Off-diagonal matrix elements

Beugeling, Bäcker, Moessner, Haque, P.R.E (2018)
Eigenstate amplitudes (coefficients)

Haque and McClarty, P.R.B (2019)
ETH in Sachdev-Ye-Kitaev models

Khaymovich, Haque, McClarty, P.R.L. (2019)
Eigenstate Thermalization, Random Matrix Theory and Behemoths

Bäcker, Haque, Khaymovich, P.R.E (2019)
Multifractal dimensions

Entanglement in mid-spectrum states
REFERENCES

Beugeling, Moessner, Haque, P.R.E (2014)
Finite-size scaling of eigenstate thermalization

Beugeling, Andreanov, Haque, JSTAT (2015)
Entanglement & participation ratios, all eigenstates

Beugeling, Moessner, Haque, P.R.E (2015)
Off-diagonal matrix elements

Beugeling, Bäcker, Moessner, Haque, P.R.E (2018)
Eigenstate amplitudes (coefficients)

Haque and McClarty, P.R.B (2019)
ETH in Sachdev-Ye-Kitaev models

Khaymovich, Haque, McClarty, P.R.L. (2019)
Eigenstate Thermalization, Random Matrix Theory and Behemoths

Bäcker, Haque, Khaymovich, P.R.E (2019)
Multifractal dimensions

Entanglement in mid-spectrum states
Non-equilibrium dynamics of isolated quantum systems

Increasing # of experiments in the limit of “isolation”:

- Ultracold trapped atoms/ions
- NMR quantum computing
- Ultrafast pump-probe spectroscopy

Weiss group, Nature 2006

Wei, Ramanathan, Cappellaro, PRL 2018

Bloch group, Nature Phys 2013
The Many-Body Eigenspectrum

Standard many-body quantum physics

⇓

low-lying parts of the many-body spectrum.

Dynamics in isolation

⇓

No tendency toward ground state

Any part of spectrum can be important!

Motivates study of eigenstates in the middle of the many-body spectrum.

Mid-spectrum eigenstates are (often) somewhat ‘random’
### Relevant classes: GOE and GUE

**GOE**
- Symmetric matrix
- Real elements
- Elements random & gaussian-distributed

**GUE**
- Hermitian matrix
- Complex elements
- Real & imaginary parts are random and gaussian-distributed

### Properties

**Eigenstates:**
- coefficients are gaussian-distributed

**Eigenvalues:**
- level spacings \(s_i = E_{i+1} - E_i\)
- have Wigner-Dyson statistics

\[
\begin{align*}
P_{\text{GOE}}(s) &\propto se^{-\alpha_1 s^2} \\
P_{\text{GUE}}(s) &\propto s^2e^{-\alpha_2 s^2}
\end{align*}
\]

**Eigenvalues display level repulsion:**
- \(P(0) = 0\)
This talk: → lattice systems, finite Hilbert space

Eigenspectrum has a bottom and a top

Typically \( N_p \) particles in \( L \) sites

Particles → fermions, bosons, or up-spins

Thermodynamic limit: \( L \to \infty, \; N_p \to \infty, \; \text{constant} \; N_p/L \)
Typical many-body Hamiltonian, with few-body, local interactions:

- Hamiltonian matrix is sparse
- Elements are not random
**Many-Body Hamiltonians: random-matrix behavior?**

Typical many-body Hamiltonian, with few-body, local interactions:

Hamiltonian matrix is sparse

Elements are not random

Example:

\[
\begin{pmatrix}
0.4 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\
0.5 & -0.4 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0.4 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & -0.4 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0.5 & -0.4 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5 & 0 & -0.4 & 0.5 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5 & -0.4 & 0.5 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 & 0 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & -0.4 & 0.5 \\
0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & -0.4 & 0.5 \\
0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & -0.4 & 0.5 \\
0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & -0.4 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0.4
\end{pmatrix}
\]

**XXZ chain,**
\(\Delta = 0.8\) \((L = 6\) sites, \(N_p = 2\) \(\uparrow\)-spins)
**Many-Body Hamiltonians: Random-Matrix Behavior?**

Typical many-body Hamiltonian, with few-body, local interactions:

- Hamiltonian matrix is **sparse**
- Elements are **not random**

Nevertheless:

- **Eigenstate coefficients:** usually Gaussian-distributed
- **Energy eigenvalues:** usually Wigner-Dyson statistics

Exceptions:

- Integrable systems
- Very large interactions
- Spectral edges
COEFFICIENTS OF MANY-BODY EIGENSTATES

\[ |E_A\rangle = \sum_n c_n |n\rangle \]

| \( |n\rangle \)'s \rightarrow many-body configurations

\[ H = J_1 \sum_{i=1}^{L-1} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + \Delta_1 S_i^z S_{i+1}^z) + J_2 \sum_{i=2}^{L-2} (S_i^+ S_{i+2}^- + S_i^- S_{i+2}^+ + \Delta_2 S_i^z S_{i+2}^z) \]

\[ J_2 = 0 \rightarrow integrable \ XXZ \ chain \]

\[ J_2 \approx J_1 \rightarrow non-integrable \] ('chaotic' or 'ergodic')
Coefficients of many-body eigenstates

\[ |E_A\rangle = \sum_n c_n |n\rangle \]

\[ z = c_n \sqrt{D}, \quad D = \text{Hilbert space dimension} \]

Beugeling, Moessner, Bäcker, Haque, P.R.E (2018)

Non-integrable

\[ J_2 = J_1 \]

Integrable

\[ J_2 = 0 \]

(a) \( J_2 = J_1 \)

(b) \( J_2 = 0 \)
LEVEL STATISTICS OF MANY-BODY SPECTRA

XXZ + NNN \[ L=15, N_p = 6 \]

\[ \langle r \rangle \] distinguishes GOE, GUE, Poisson

\[ r_i = \min \left( \frac{s_{i+1}}{s_i}, \frac{s_i}{s_{i+1}} \right) \]

Integrable systems usually have Poisson statistics
Level statistics of many-body spectra

XXZ + NNN

$L=15$, $N_p=6$

$\langle r \rangle$ distinguishes GOE, GUE, Poisson

$r_i = \min \left( \frac{s_{i+1}}{s_i} , \frac{s_i}{s_{i+1}} \right)$

Integrable systems usually have Poisson statistics
ENTANGLEMENT ENTROPY OF MANY-BODY EIGENSTATES

\[ XYZ + NNN \quad (\text{both: } \eta = 0.5, \Delta = 0.9) \]
\[ + h_x\text{-field (0.8)} + h_z\text{-field (0.2)} \]

Only the middle of the spectrum?
ONLY THE MIDDLE OF THE SPECTRUM?

Introduce temperature: 

\[
\text{pretend: system is described by canonical } e^{-\beta H}
\]

\[
E_{\text{TOTAL}} = \frac{\text{tr} \left( e^{-\beta H} H \right)}{\text{tr} \left( e^{-\beta H} \right)} = \frac{1}{Z(\beta)} \sum_A e^{-\beta E_A} E_A
\]

\[
Z(\beta) = \text{tr} \left( e^{-\beta H} \right) = \sum_A e^{-\beta E_A}
\]

\[
\langle E_A \rangle, E_A \rightarrow \text{eigenstates, eigenenergies}
\]

→ provides a temperature ↔ energy map, also for a finite system!

Based on eigenvalues alone.
Not just the middle of the spectrum

Mid-spectrum eigenstates $\rightarrow$ well-described by $|\psi_{\text{rand}}\rangle$

Finite-temperature eigenstates $\rightarrow$ well-described by $\exp\left[-\frac{\beta}{2}H\right]|\psi_{\text{rand}}\rangle$
NOT JUST THE MIDDLE OF THE SPECTRUM

\[ \text{mid-spectrum eigenstates} \equiv \text{infinite-temperature states} \]

Negative temperatures!

Mid-spectrum eigenstates \( \rightarrow \) well-described by \( |\psi_{\text{rand}}\rangle \)

Finite-temperature eigenstates \( \rightarrow \) well-described by \( \exp\left[-\frac{\beta}{2}H\right]|\psi_{\text{rand}}\rangle \)
**EigEnState Thermalization Scaling**

Many-body Hamiltonians are random matrices (?!?)

Eigenstate thermalization hypothesis (which observables?)

E.T.H. Scaling — local, less local & Behemoth operators

\[
\begin{align*}
\langle r \rangle & \approx \text{GOE} \\
\end{align*}
\]

- \( J_2 / J_1 \)
- \( \Delta_1 = \Delta_2 = 0.8 \)
- \( n = 8, M = 2 \)
- \( n = 7, M = 6 \)
- \( n = 1, M = 12870 \)

\( \langle E_A | \Gamma | E_B \rangle \)

\[
\begin{align*}
\text{Distributions} \\
\end{align*}
\]

- XXZ + NNN

\( (L,N_p) = (17,8) \)

\( J_1 = J_2 = 1 \)

\( \text{RMT prediction for 2-point } \sim c_l^{-1/4} \)

- Dense \( \beta = 1/4 \) 
- \( \sim c_l^{-3/8} \)
- \( \sim c_l^{-1/2} \)
Thermalization in isolated systems

Generic (non-integrable) isolated system

driven out of equilibrium

⇒ many observables thermalize

An observable thermalizes

⇒ relaxes to value dictated by thermal ensemble
An observable thermalizes

\[ \Rightarrow \text{relaxes to value dictated by thermal ensemble} \]

If initial state is

\[ |\psi(0)\rangle = \sum_A c_A |E_A\rangle \]

relaxes to

\[ \langle O(t) \rangle = \langle \psi(t) | \hat{O} | \psi(t) \rangle \xrightarrow{t \to \infty} \sum_A |c_A|^2 \langle E_A | \hat{O} | E_A \rangle \]

\[ |E_A\rangle, E_A \to \text{eigenstates, eigenenergies} \]
An observable thermalizes

\[ \langle O \rangle_{\text{therm}} = \frac{1}{Z(\beta)} \sum_A e^{-\beta E_A} \langle E_A | \hat{O} | E_A \rangle \]

\( \beta \) defined by energy:

\[ E_{\text{TOTAL}} = \frac{1}{Z(\beta)} \sum_A e^{-\beta E_A} E_A \]

\( |E_A\rangle, E_A \rightarrow \text{eigenstates, eigenenergies} \)
THERMALIZATION IN ISOLATED SYSTEMS

An observable thermalizes

$$\implies$$ relaxes to value dictated by thermal ensemble

$$\sum_A |c_A|^2 \langle E_A | \hat{O} | E_A \rangle = \frac{1}{Z(\beta)} \sum_A e^{-\beta E_A} \langle E_A | \hat{O} | E_A \rangle$$

Crucial: Eigenstate expectation values $$\langle E_A | \hat{O} | E_A \rangle$$

|$$E_A$$, $$E_A \rightarrow$$ eigenstates, eigenenergies|

|Some observable |

|time|
**THERMALIZATION IN ISOLATED SYSTEMS**

Mechanism for thermalization: **Eigenstate Thermalization Hypothesis**

$$\langle E_A | \hat{O} | E_A \rangle \text{'s are smooth functions of } E_A$$

$$\rightarrow \text{‘implies’ thermalization}$$

Deutsch, P.R.A (1991); Srednicki, P.R.E (1994)


............ + many others

Beugeling, Moessner, Haque, P.R.E (2014)

Finite-Size Scaling of ETH
E.T.H. Scaling

\[ H = H_{XXZ} + \lambda \sum_i (i - i_0)^2 S_j^z \]

\[ O_{AA} = \langle E_A | \hat{O} | E_A \rangle = \langle E_A | S_{\text{middle}}^z | E_A \rangle \]

Scaling of E.T.H. fluctuations:
\[ \sigma \sim D^{-1/2} \sim e^{-\alpha L} \]

\[ D = \text{dimension of Hilbert space} \]

Beugeling, Moessner, Haque, P.R.E (2014)
E.T.H. Scaling:
Off-diagonal matrix elements

$$O_{AB} = \langle E_A | \hat{O} | E_B \rangle$$

Sachdev-Ye-Kitaev model
($N$ Majorana fermions)

$\sigma \sim D^{-1/2} \sim 2^{-N/4}$

Haque & McClarty
P.R.B (2019)
A STANDARD STATEMENT OF E.T.H.  

\[ \langle E_A|\hat{O}|E_B \rangle = \delta_{AB} f^{(1)}(\bar{E}) + e^{-S(\bar{E})/2} f^{(2)}(\bar{E},\omega) R_{AB} \]

\[ \bar{E} = \frac{1}{2}(E_A + E_B) \]

\[ \omega = E_B - E_A \]

\[ R_{AB} \rightarrow \text{a gaussian random variable;} \quad f^{(1,2)} \rightarrow \text{smooth functions.} \]

\[ S \sim \log \mathcal{D} \text{ is the entropy } \implies \text{Distribution width } \sim \mathcal{D}^{-1/2} \]

\[ \mathcal{D} \equiv \text{Hilbert space dimension} \]

Local operators $\rightarrow$

- diagonal matrix elements, off-diagonal matrix elements
- both distributions have width $\sim \mathcal{D}^{-1/2}$
Which operators obey $\sim \mathcal{D}^{-1/2}$ scaling?

Khaymovich, Haque, McClarty, P.R.L. (2019)

Eigenstate Thermalization, Random Matrix Theory and Behemoths
**Eigenstate thermalization scaling**

Many-body Hamiltonians are random matrices (?!?)

Eigenstate thermalization hypothesis (which observables?)

E.T.H. Scaling — local, less local & Behemoth operators

---

![Graph showing the relationship between $J_2/J_1$ and the distribution of observables over time.](image)

**XXZ + NNN**

$L=15, N_p=6$

- **GOE**
- **Poisson**

![Graph showing the distribution of matrix elements of partially/fully local operators.](image)

**XXZ + NNN**

$(L,N_p)=(17,8), J_1=J_2=1, \Delta_1=\Delta_2=0.8$

- **n=8, M=2**
- **n=7, M=6**
- **n=1, M=12870**

(a) (b) (c) (d) (e)
Operators in RMT and many-body physics

Could interpret random matrix as:

Hamiltonian of a single particle

on a fully-connected graph

with random hoppings

\[ H = \sum_{ij} h_{ij} \hat{d}_i \hat{d}_j \]
Hamiltonian of a single particle on a fully-connected graph

Not many interesting observables, except:

\[ \tilde{\omega}_{ij} \equiv \hat{d}_i^\dagger \hat{d}_j = |i\rangle \langle j| \]

Node-node correlation function

For \( i = j \): node occupancy

\[ H = \sum_{ij} h_{ij} \hat{d}_i^\dagger \hat{d}_j \]
As a matrix? \( \hat{\omega}_{ij} \) has a single nonzero element.

Hermitian version: \( \hat{\gamma}_{ij} = \hat{\omega}_{ij} + \hat{\omega}_{ji} \)

\( \rightarrow \) Matrix with two nonzero elements.

\[ H = \sum_{ij} h_{ij} \hat{d}_i^\dagger \hat{d}_j \]
Operators in RMT and Many-body Physics

\[ \hat{\omega}_{ij} \equiv \hat{d}^\dagger_i \hat{d}_j = |i\rangle \langle j| \]

Many-body analogy \( \rightarrow \)

- a node \( i \) \( \equiv \) a many-body configuration \( |n\rangle \)

\[ \hat{\Omega}_{nn'} \equiv |n\rangle \langle n'| \]

Changes one many-body configuration to another.

Highly nonlocal

Behemoth operators
**Operators in RMT and Many-Body Physics**

\[ \hat{\Omega}_{nn'} \equiv |n\rangle \langle n'| \]

\( \hat{\Omega} \) in configuration space \( \rightarrow \)

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

Behemoths form a basis for operators.

\( \Downarrow \)

Any operator is a sum of Behemoths.

(e.g. local observables, 2-point correlators)
(Using spinless-fermion language)

Series of operators

\[ \hat{\Omega}_M = \prod_{k=1}^{n} \hat{c}^\dagger_i \hat{c}_j = \sum_{\alpha=1}^{M} \hat{\Omega}^{(\alpha)}_{nn'} \]

A \((2n)\)-point correlator. \(M \) nonzero terms in operator matrix.

Behemoth: \(\hat{c}^\dagger_{i_1} \hat{c}^\dagger_{i_2} \ldots \hat{c}^\dagger_{i_{N_p}} \hat{c}_{j_1} \hat{c}_{j_2} \ldots \hat{c}_{j_{N_p}}\)

\(n = N_p, \quad M = 1 \) or \(2 \)

2-point correlator: \(\hat{c}^\dagger_{i_1} \hat{c}_{j_1}\)

\(n = 1, \quad M = \left( \frac{L - 2n}{N_p - n} \right) \sim O(D) \)

Can generalize \(\rightarrow\) spins, bosons, Hubbard, \(N_p\)-non-conserving systems, allow overlap between \(n, n'\) configurations....

Many types of operators covered in this framework.
Width of distributions: scaling with $D$

Behemoth distribution $\sim K_0(Dx) \rightarrow$ Width scales as $\sim D^{-1} \rightarrow$ super-ETH scaling

Local operators are sums of $M \sim O(D)$ Behemoths.

Using central limit theorem, \[ \text{width} \sim \sqrt{MD^{-1}} \sim D^{-1/2} \rightarrow \text{ETH scaling} \]

A ‘typical’ operator is dense, $M > O(D)$

If $M \sim O(D^{1+\beta})$, width (using CLT) $\sim D^{-1/2+\beta/2} \rightarrow$ sub-ETH scaling

If $M \sim O(D^2)$, width $\sim D^0$

ETH works because physical operators are sparse.

$D^{-1/2}$ scaling works because local operators have $M \sim O(D)$
**Non-local to Local to Typical**

\[ \langle E_A | \Gamma | E_B \rangle \]

Distributions

\[ \langle E_A | \Gamma | E_B \rangle \]

matrix elements of partially/fully local operators

\[ (L, N_p) = (17, 8). \quad J_1 = J_2 = 1, \quad \Delta_1 = \Delta_2 = 0.8 \]

XXZ + NNN

(a) \( n=8 \quad M=2 \)

(b) \( n=7 \quad M=6 \)

(c) \( n=1, \quad M=12870 \)

(widths)

\[ \sim D^{-1} \quad \sim D^{-1/2} \]

RMT prediction for 2-point

\[ \sim D^{-1/4} \quad \sim D^{-1/2} \]

Dense \( \beta=1/2 \)

Dense \( \beta=1/4 \)

\[ \sim D^{-3/8} \quad \sim D^{-1/2} \]
**Eigenstate thermalization scaling**

Many-body Hamiltonians are random matrices (?!?)

Eigenstate thermalization hypothesis (which observables?)

E.T.H. Scaling — local, less local & Behemoth operators

![Graph showing GOE and Poisson distributions](image)

XXZ + NNN $(L,N_p) = (17,8)$. $J_1 = J_2 = 1$, $\Delta_1 = \Delta_2 = 0.8$

$\langle E^A | \Gamma | E^B \rangle$

Matrix elements of partially/fully local operators

**Distributions**

(a) (b) (c) (d) (e)