

EIGENSTATE THERMALIZATION, RANDOM MATRICES AND BEHEMOTHS

Masud Haque

Maynooth University
Dept. Theoretical Physics



Maynooth, Ireland



Max Planck Institute for
Physics of Complex Systems
(MPI-PKS)

Dresden, Germany

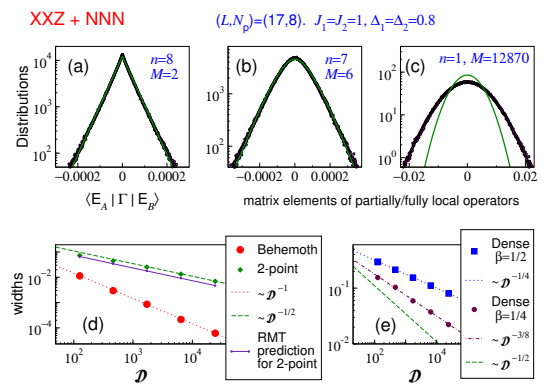
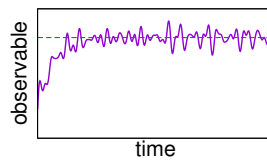
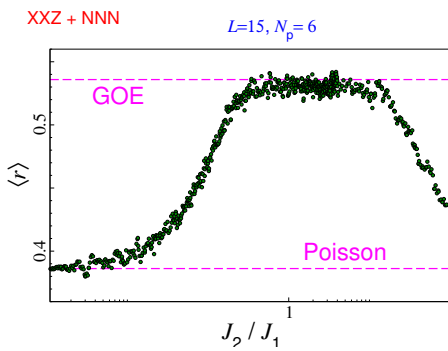


EIGENSTATE THERMALIZATION SCALING

Many-body Hamiltonians are random matrices (?!?)

Eigenstate thermalization hypothesis (which observables?)

E.T.H. Scaling — local, less local & Behemoth operators



REFERENCES

Beugeling, Moessner, Haque, P.R.E (2014)
Finite-size scaling of eigenstate thermalization

Beugeling, Andreanov, Haque, JSTAT (2015)
Entanglement & participation ratios, all eigenstates

Beugeling, Moessner, Haque, P.R.E (2015)
Off-diagonal matrix elements

Beugeling, Bäcker, Moessner, Haque, P.R.E (2018)
Eigenstate amplitudes (coefficients)

Haque and McClarty, P.R.B (2019)
ETH in Sachdev-Ye-Kitaev models

Khaymovich, Haque, McClarty, P.R.L. (2019)
Eigenstate Thermalization, Random Matrix Theory and Behemoths

Bäcker, Haque, Khaymovich, P.R.E (2019)
Multifractal dimensions

Khaymovich, McClarty, Haque, arXiv:2008
Entanglement in mid-spectrum states

REFERENCES



Beugeling, Moessner, Haque, P.R.E (2014)
Finite-size scaling of eigenstate thermalization

Beugeling, Andreanov, Haque, JSTAT (2015)
Entanglement & participation ratios, all eigenstates

Beugeling, Moessner, Haque, P.R.E (2015)
Off-diagonal matrix elements

Beugeling, Bäcker, Moessner, Haque, P.R.E (2018)
Eigenstate amplitudes (coefficients)

Haque and McClarty, P.R.B (2019)
ETH in Sachdev-Ye-Kitaev models

Khaymovich, Haque, McClarty, P.R.L. (2019)
Eigenstate Thermalization, Random Matrix Theory and Behemoths

Bäcker, Haque, Khaymovich, P.R.E (2019)
Multifractal dimensions

Khaymovich, McClarty, Haque, arXiv:2008
Entanglement in mid-spectrum states



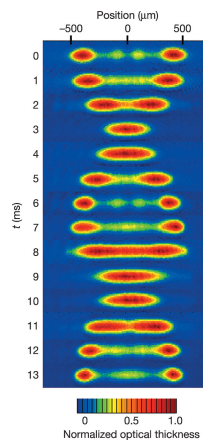
CONTEXT

Non-equilibrium dynamics of isolated quantum systems

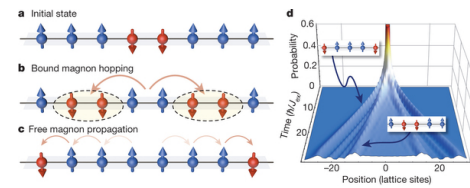
Increasing # of experiments in the limit of "isolation":

time of measurement \ll time scale of environment effects

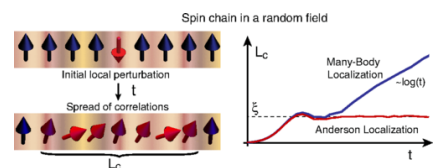
- Ultracold trapped atoms/ions
- NMR quantum computing
- Ultrafast pump-probe spectroscopy



Weiss group, Nature 2006



Bloch group, Nature Phys 2013



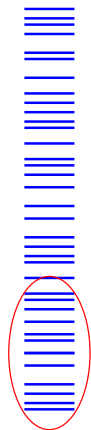
Wei, Ramanathan, Cappellaro, PRL 2018

THE MANY-BODY EIGENSPECTRUM

Standard many-body
quantum physics



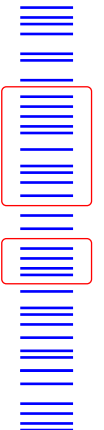
low-lying parts of
the many-body
spectrum.



Dynamics in isolation



No tendency toward ground state
Any part of spectrum can be important!



Motivates study of eigenstates in the **middle** of the many-body spectrum.

Mid-spectrum eigenstates are (often) somewhat 'random'

RANDOM MATRICES

Relevant classes:

GOE and GUE

GOE

Symmetric matrix

Real elements

Elements random & gaussian-distributed

GUE

Hermitian matrix

Complex elements

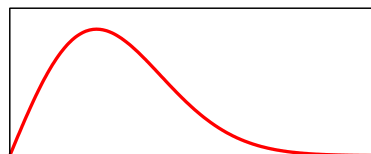
Real & imaginary parts are random and gaussian-distributed

Properties

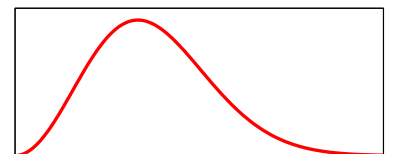
Eigenstates: coefficients are gaussian-distributed

Eigenvalues: level spacings ($s_i = E_{i+1} - E_i$) have Wigner-Dyson statistics

$$P_{\text{GOE}}(s) \propto s e^{-\alpha_1 s^2}$$



$$P_{\text{GUE}}(s) \propto s^2 e^{-\alpha_2 s^2}$$



Eigenvalues display level repulsion: $P(0) = 0$

MANY-BODY HAMILTONIANS...

This talk: \longrightarrow lattice systems, finite Hilbert space

Eigenspectrum has a bottom and a top

Typically N_p particles in L sites

Particles \longrightarrow fermions, bosons, or up-spins

Thermodynamic limit: $L \rightarrow \infty$, $N_p \rightarrow \infty$, constant N_p/L

MANY-BODY HAMILTONIANS: RANDOM-MATRIX BEHAVIOR?

Typical many-body Hamiltonian, with
few-body, local interactions:

Hamiltonian matrix is sparse

Elements are not random

MANY-BODY HAMILTONIANS: RANDOM-MATRIX BEHAVIOR?

Typical many-body Hamiltonian, with few-body, local interactions:

Hamiltonian matrix is sparse

Elements are not random

Example:

$$\begin{pmatrix} 0.4 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & -0.4 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.4 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & -0.4 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 & -0.4 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.4 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & -0.4 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & -0.4 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & -0.4 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.4 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & -0.4 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & -0.4 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & -0.4 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.4 \end{pmatrix}$$

XXZ chain,

$\Delta = 0.8$ ($L = 6$ sites, $N_p = 2$ \uparrow -spins)

MANY-BODY HAMILTONIANS: RANDOM-MATRIX BEHAVIOR?

Typical many-body Hamiltonian, with
few-body, local interactions:

Hamiltonian matrix is sparse

Elements are not random

Nevertheless

Eigenstate coefficients:

usually Gaussian-distributed

Energy eigenvalues:

usually Wigner-Dyson statistics

Exceptions: integrable systems
very large interactions
spectral edges

COEFFICIENTS OF MANY-BODY EIGENSTATES

$$|E_A\rangle = \sum_n c_n |\mathbf{n}\rangle$$

$|\mathbf{n}\rangle$'s \rightarrow many-body configurations

$$H = J_1 \sum_{i=1}^{L-1} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + \Delta_1 S_i^z S_{i+1}^z) + J_2 \sum_{i=2}^{L-2} (S_i^+ S_{i+2}^- + S_i^- S_{i+2}^+ + \Delta_2 S_i^z S_{i+2}^z)$$

$J_2 = 0 \rightarrow$ integrable XXZ chain

$J_2 \approx J_1 \rightarrow$ non-integrable
(‘chaotic’ or ‘ergodic’)

$|\mathbf{n}\rangle$'s \rightarrow

$|\downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\rangle$

$|\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\rangle$

$|\downarrow\downarrow\downarrow\uparrow\uparrow\downarrow\uparrow\uparrow\rangle$

\vdots

COEFFICIENTS OF MANY-BODY EIGENSTATES

$$|E_A\rangle = \sum_n c_n |\mathbf{n}\rangle$$

$$z = c_n \sqrt{\mathcal{D}},$$

$\mathcal{D} =$ Hilbert space dimension

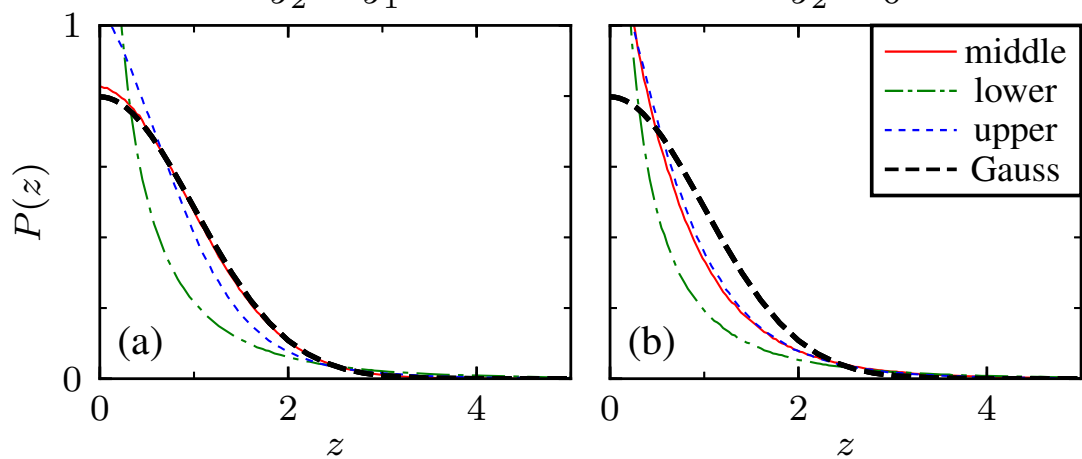
Beugeling, Moessner,
Bäcker, Haque,
P.R.E (2018)

NON-INTEGRABLE

INTEGRABLE

$J_2 = J_1$

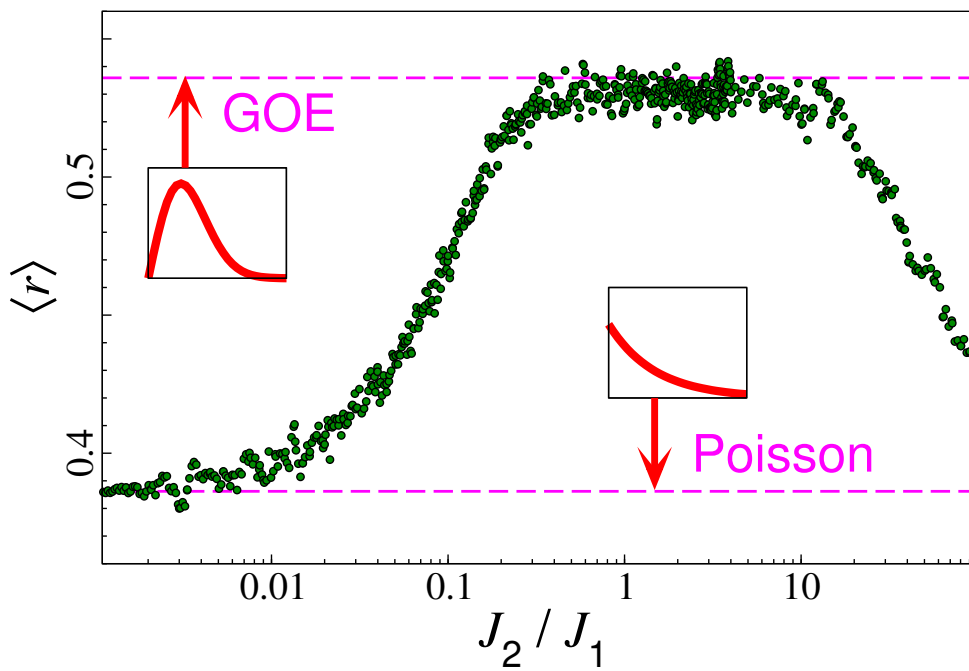
$J_2 = 0$



LEVEL STATISTICS OF MANY-BODY SPECTRA

XXZ + NNN

$L=15, N_p=6$



$\langle r \rangle$ distinguishes
GOE, GUE,
Poisson

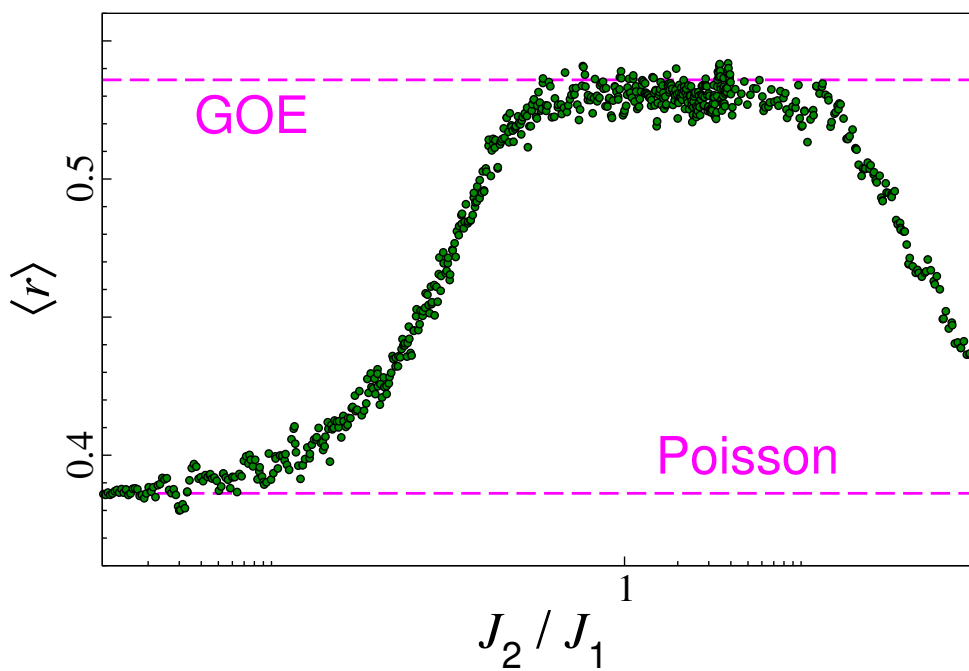
$$r_i = \min \left(\frac{s_{i+1}}{s_i}, \frac{s_i}{s_{i+1}} \right)$$

Integrable systems
usually have Poisson
statistics

LEVEL STATISTICS OF MANY-BODY SPECTRA

XXZ + NNN

$L=15, N_p=6$



$\langle r \rangle$ distinguishes
GOE, GUE,
Poisson

$$r_i = \min \left(\frac{s_{i+1}}{s_i}, \frac{s_i}{s_{i+1}} \right)$$

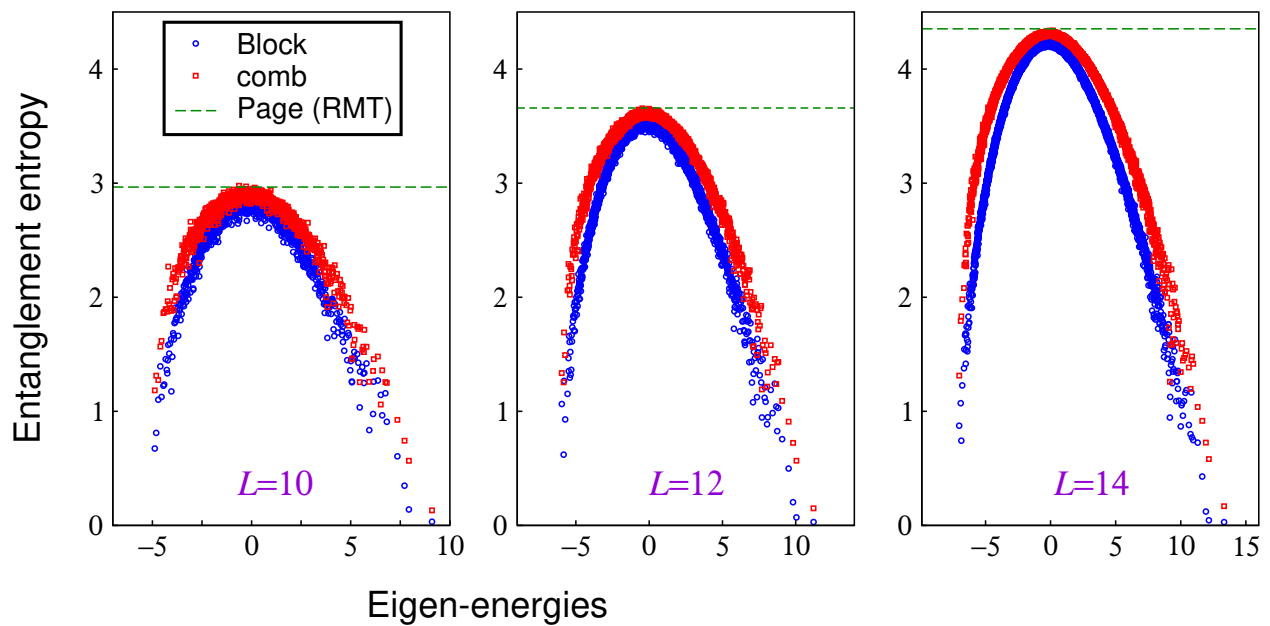
Integrable systems
usually have Poisson
statistics

ENTANGLEMENT ENTROPY OF MANY-BODY EIGENSTATES

XYZ + NNN

(both: $\eta = 0.5$, $\Delta = 0.9$)
+ h_x -field (0.8) + h_z -field (0.2)

Haque, McClarty,
Khaymovich,
arXiv:2008



ONLY THE MIDDLE OF THE SPECTRUM?

ONLY THE MIDDLE OF THE SPECTRUM?

Introduce temperature: (pretend: system is described by canonical $e^{-\beta H}$)

$$E_{\text{TOTAL}} = \frac{\text{tr}(e^{-\beta H} H)}{\text{tr}(e^{-\beta H})} = \frac{1}{Z(\beta)} \sum_A e^{-\beta E_A} E_A$$

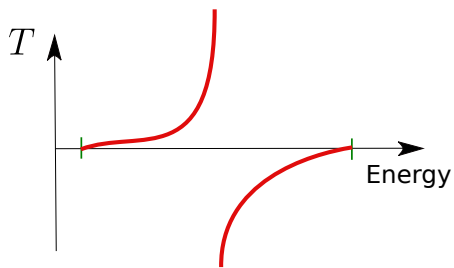
with $Z(\beta) = \text{tr}(e^{-\beta H}) = \sum_A e^{-\beta E_A}$

$|E_A\rangle, E_A \rightarrow$
eigenstates,
eigenenergies

\rightarrow provides a temperature \leftrightarrow energy map,
also for a finite system!

Based on eigenvalues alone.

NOT JUST THE MIDDLE OF THE SPECTRUM

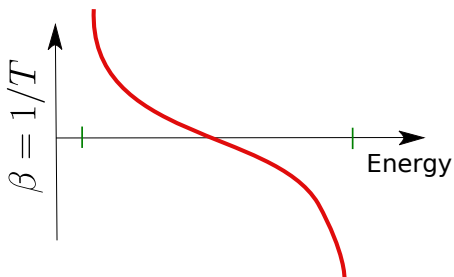


mid-spectrum
eigenstates

≡

infinite-
temperature
states

Negative temperatures!



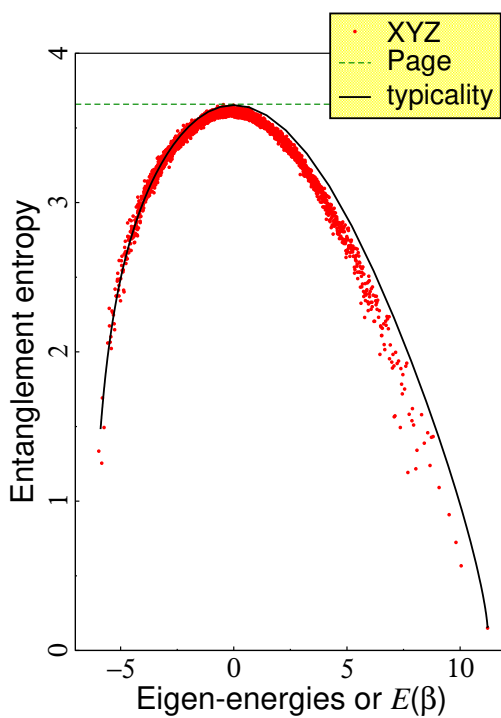
Mid-spectrum eigenstates

→ well-described by $|\psi_{\text{rand}}\rangle$

Finite-temperature eigenstates

→ well-described by $\exp\left[-\frac{\beta}{2}H\right] |\psi_{\text{rand}}\rangle$

NOT JUST THE MIDDLE OF THE SPECTRUM



mid-spectrum
eigenstates

≡

infinite-
temperature
states

Negative temperatures!

Mid-spectrum eigenstates

→ well-described by $|\psi_{\text{rand}}\rangle$

Finite-temperature eigenstates

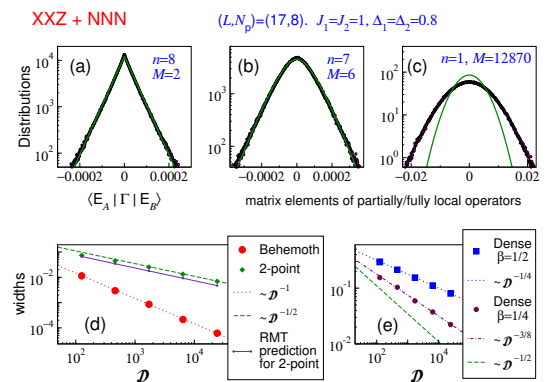
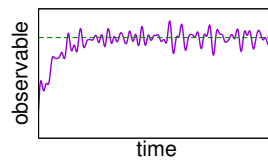
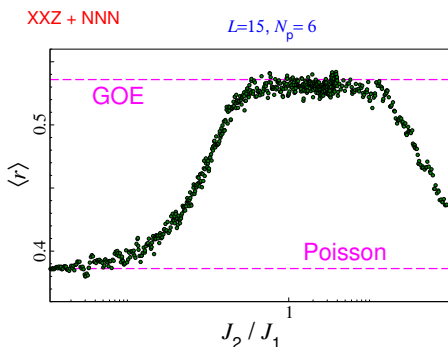
→ well-described by $\exp\left[-\frac{\beta}{2}H\right]|\psi_{\text{rand}}\rangle$

EIGENSTATE THERMALIZATION SCALING

Many-body Hamiltonians are random matrices (?!?)

Eigenstate thermalization hypothesis (which observables?)

E.T.H. Scaling — local, less local & Behemoth operators



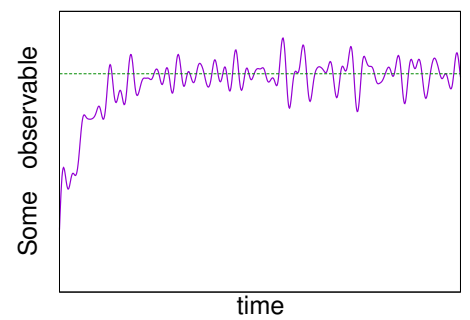
THERMALIZATION IN ISOLATED SYSTEMS

Generic (non-integrable) isolated system
driven out of equilibrium

⇒ many observables thermalize

An observable thermalizes

⇒ relaxes to value dictated by thermal ensemble



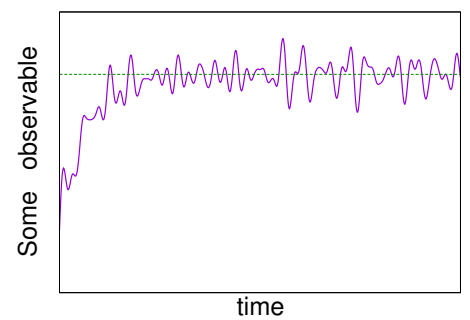
THERMALIZATION IN ISOLATED SYSTEMS

An observable thermalizes

\implies relaxes to value dictated by thermal ensemble

If initial state is $|\psi(0)\rangle = \sum_A c_A |E_A\rangle$, relaxes to

$$\langle O(t) \rangle = \langle \psi(t) | \hat{O} | \psi(t) \rangle \xrightarrow{t \rightarrow \infty, \langle \cdot \rangle} \sum_A |c_A|^2 \langle E_A | \hat{O} | E_A \rangle$$



$|E_A\rangle$, $E_A \rightarrow$ eigenstates,
eigenenergies

THERMALIZATION IN ISOLATED SYSTEMS

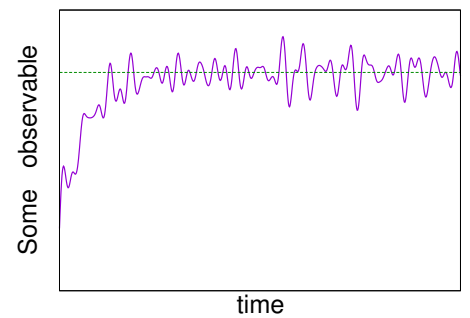
An observable thermalizes

⇒ relaxes to value dictated by thermal ensemble

$$\langle O \rangle_{\text{therm}} = \frac{1}{Z(\beta)} \sum_A e^{-\beta E_A} \langle E_A | \hat{O} | E_A \rangle$$

β defined by energy:

$$E_{\text{TOTAL}} = \frac{1}{Z(\beta)} \sum_A e^{-\beta E_A} E_A$$



$|E_A\rangle$, $E_A \rightarrow$ eigenstates,
eigenenergies

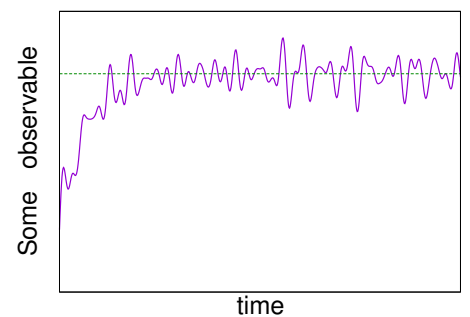
THERMALIZATION IN ISOLATED SYSTEMS

An observable thermalizes

⇒ relaxes to value dictated by thermal ensemble

$$\sum_A |c_A|^2 \langle E_A | \hat{O} | E_A \rangle = \frac{1}{Z(\beta)} \sum_A e^{-\beta E_A} \langle E_A | \hat{O} | E_A \rangle$$

Crucial: Eigenstate expectation values $\langle E_A | \hat{O} | E_A \rangle$



$|E_A\rangle$, $E_A \rightarrow$ eigenstates,
eigenenergies

THERMALIZATION IN ISOLATED SYSTEMS

Mechanism for thermalization: **Eigenstate Thermalization Hypothesis**

$\langle E_A | \hat{O} | E_A \rangle$'s are smooth functions of E_A

→ 'implies' thermalization

Deutsch, P.R.A (1991); Srednicki, P.R.E (1994)

Rigol, Dunjko, Olshanii, Nature (2008)

..... + many others

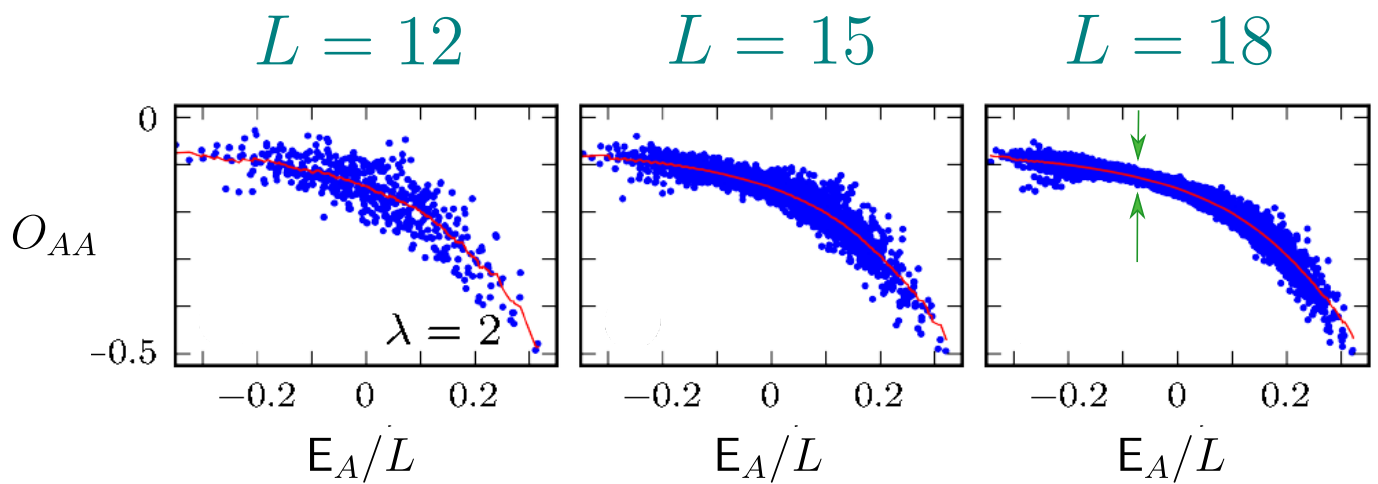
Beugeling, Moessner, Haque, P.R.E (2014)

Finite-Size Scaling of ETH

E.T.H. SCALING

$$H = H_{XXZ} + \lambda \sum_i (i - i_0)^2 S_j^z$$

$$O_{AA} = \langle E_A | \hat{O} | E_A \rangle = \langle E_A | S_{\text{middle}}^z | E_A \rangle$$



Scaling of E.T.H. fluctuations: $\sigma \sim \mathcal{D}^{-1/2} \sim e^{-\alpha L}$

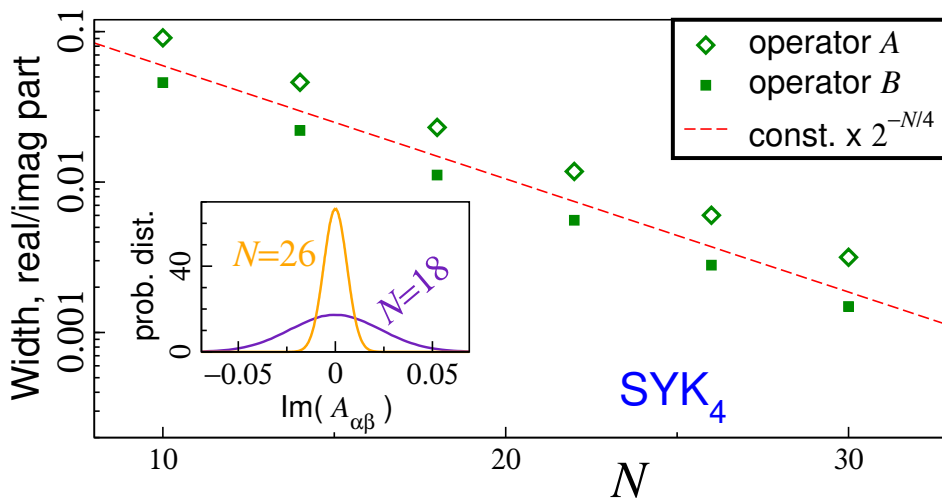
\mathcal{D} = dimension of Hilbert space

Beugeling,
Moessner, Haque,
P.R.E (2014)

**E.T.H. SCALING:
OFF-DIAGONAL MATRIX ELEMENTS**

$$O_{AB} = \langle E_A | \hat{O} | E_B \rangle$$

Sachdev-Ye-Kitaev model
(N Majorana fermions)



Haque & McClarty
P.R.B (2019)

$$\sigma \sim \mathcal{D}^{-1/2} \sim 2^{-N/4}$$

A STANDARD STATEMENT OF E.T.H.

Srednicki, 1999

$$\langle E_A | \hat{O} | E_B \rangle = \delta_{AB} f^{(1)}(\bar{E}) + e^{-S(\bar{E})/2} f^{(2)}(\bar{E}, \omega) R_{AB}$$

$$\bar{E} = (1/2)(E_A + E_B)$$

$$\omega = E_B - E_A$$

$R_{AB} \rightarrow$ a gaussian random variable; $f^{(1,2)} \rightarrow$ smooth functions.

$S \sim \log \mathcal{D}$ is the entropy \implies Distribution width $\sim \mathcal{D}^{-1/2}$

Local operators \rightarrow

diagonal matrix elements, off-diagonal matrix elements

both distributions have width $\sim \mathcal{D}^{-1/2}$

$\mathcal{D} \equiv$

Hilbert space
dimension

WHICH OPERATORS OBEY $\sim \mathcal{D}^{-1/2}$ SCALING?

Khaymovich, Haque, McClarty, P.R.L. (2019)

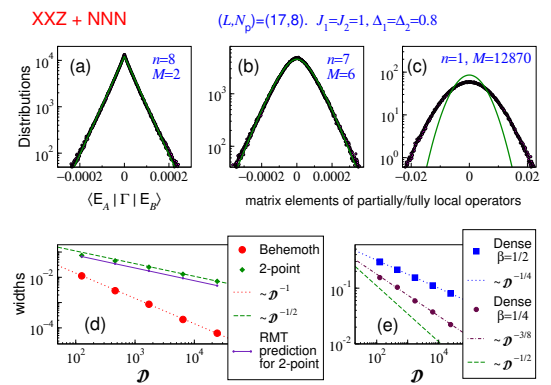
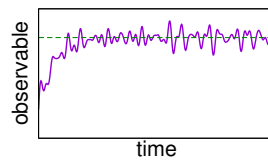
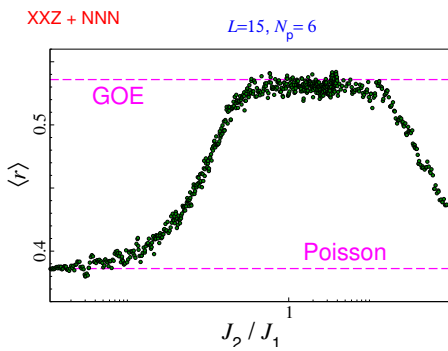
Eigenstate Thermalization,
Random Matrix Theory
and Behemoths

EIGENSTATE THERMALIZATION SCALING

Many-body Hamiltonians are random matrices (?!?)

Eigenstate thermalization hypothesis (which observables?)

E.T.H. Scaling — local, less local & Behemoth operators



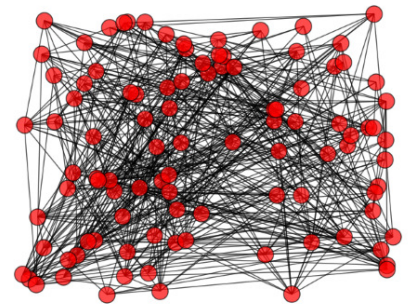
OPERATORS IN RMT AND MANY-BODY PHYSICS

Could interpret random matrix as:

Hamiltonian of a single particle
on a fully-connected graph
with random hoppings

$$H = \sum_{ij} h_{ij} \hat{d}_i^\dagger \hat{d}_j$$

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} & \dots & h_{1N} \\ h_{21} & h_{22} & h_{23} & h_{24} & \dots & h_{2N} \\ h_{31} & h_{32} & h_{33} & h_{34} & \dots & h_{3N} \\ h_{41} & h_{42} & h_{43} & h_{44} & \dots & h_{4N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h_{N1} & h_{N2} & h_{N3} & h_{N4} & \dots & h_{NN} \end{pmatrix}$$



OPERATORS IN RMT AND MANY-BODY PHYSICS

Hamiltonian of a single particle on a fully-connected graph

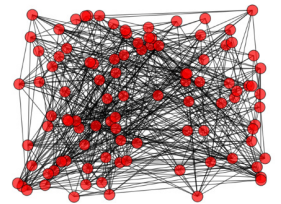
Not many interesting observables, except:

$$\hat{\omega}_{ij} \equiv \hat{d}_i^\dagger \hat{d}_j = |i\rangle\langle j|$$

Node-node correlation function

For $i = j$: node occupancy

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} & \dots & h_{1N} \\ h_{21} & h_{22} & h_{23} & h_{24} & \dots & h_{2N} \\ h_{31} & h_{32} & h_{33} & h_{34} & \dots & h_{3N} \\ h_{41} & h_{42} & h_{43} & h_{44} & \dots & h_{4N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h_{N1} & h_{N2} & h_{N3} & h_{N4} & \dots & h_{NN} \end{pmatrix}$$



$$H = \sum_{ij} h_{ij} \hat{d}_i^\dagger \hat{d}_j$$

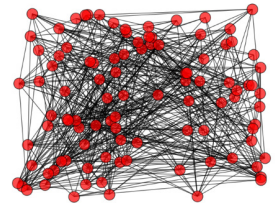
OPERATORS IN RMT AND MANY-BODY PHYSICS

$$\hat{\omega}_{ij} \equiv \hat{d}_i^\dagger \hat{d}_j = |i\rangle\langle j|$$

As a matrix? $\hat{\omega}_{ij}$ has a single nonzero element.

Hermitian version: $\hat{\gamma}_{ij} = \hat{\omega}_{ij} + \hat{\omega}_{ji}$

→ Matrix with two nonzero elements.



$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$H = \sum_{ij} h_{ij} \hat{d}_i^\dagger \hat{d}_j$$

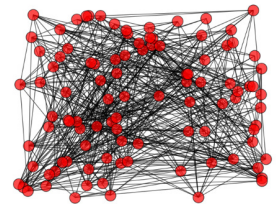
OPERATORS IN RMT AND MANY-BODY PHYSICS

$$\hat{\omega}_{ij} \equiv \hat{d}_i^\dagger \hat{d}_j = |i\rangle\langle j|$$

Many-body analogy \rightarrow

a node i \equiv a many-body configuration $|\mathbf{n}\rangle$

$$\hat{\Omega}_{nn'} \equiv |\mathbf{n}\rangle\langle \mathbf{n}'|$$



Changes one many-body configuration to another.

Highly nonlocal

Behemoth operators

OPERATORS IN RMT AND MANY-BODY PHYSICS

$$\hat{\Omega}_{nn'} \equiv |\mathbf{n}\rangle\langle\mathbf{n}'|$$

$\hat{\Omega}$ in configuration space \rightarrow

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Behemoths form a **basis** for operators.



Any operator is a sum of Behemoths.

(e.g. local observables, 2-point correlators)

FROM NONLOCAL TO LOCAL OPERATORS

(Using spinless-fermion language)

Series of operators

$$\hat{\Omega}_M \equiv \prod_{k=1}^n \hat{c}_{i_k}^\dagger \hat{c}_{j_k} = \sum_{\alpha=1}^M \hat{\Omega}_{nn'}^{(\alpha)}$$

A $(2n)$ -point correlator.

M nonzero terms in operator matrix.

Behemoth: $\hat{c}_{i_1}^\dagger \hat{c}_{i_1}^\dagger \dots \hat{c}_{i_{N_p}}^\dagger \hat{c}_{j_1} \hat{c}_{j_2} \dots \hat{c}_{j_{N_p}}$

$$n = N_p, \quad M = 1 \text{ or } 2$$

2-point correlator: $\hat{c}_{i_1}^\dagger \hat{c}_{j_1}$

$$n = 1, \quad M = \binom{L - 2n}{N_p - n} \sim O(\mathcal{D})$$

Can generalize \rightarrow spins, bosons, Hubbard, N_p -non-conserving systems, allow overlap between n, n' configurations....

Many types of operators covered in this framework.

WIDTH OF DISTRIBUTIONS: SCALING WITH \mathcal{D}

Behemoth distribution $\sim K_0(\mathcal{D}x) \rightarrow$ Width scales as $\sim \mathcal{D}^{-1} \rightarrow$ super-ETH scaling

Local operators are sums of $M \sim O(\mathcal{D})$ Behemoths.

Using central limit theorem, width $\sim \sqrt{M}\mathcal{D}^{-1} \sim \mathcal{D}^{-1/2} \rightarrow$ ETH scaling

A 'typical' operator is dense, $M > O(\mathcal{D})$

If $M \sim O(\mathcal{D}^{1+\beta})$, width (using CLT) $\sim \mathcal{D}^{-1/2+\beta/2} \rightarrow$ sub-ETH scaling

If $M \sim O(\mathcal{D}^2)$, width $\sim \mathcal{D}^0$

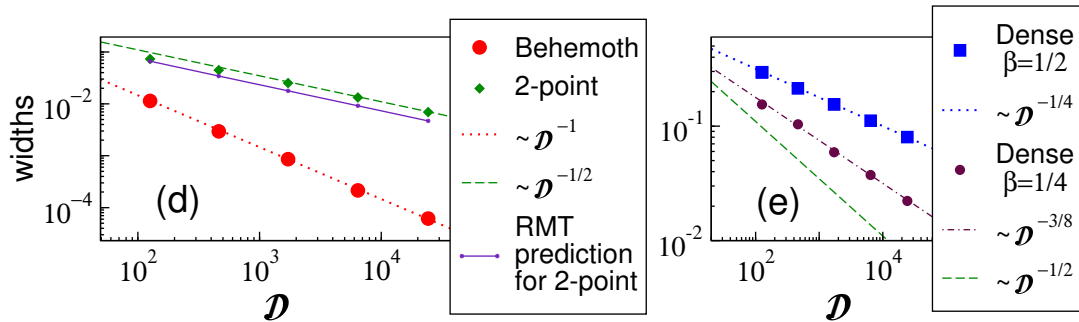
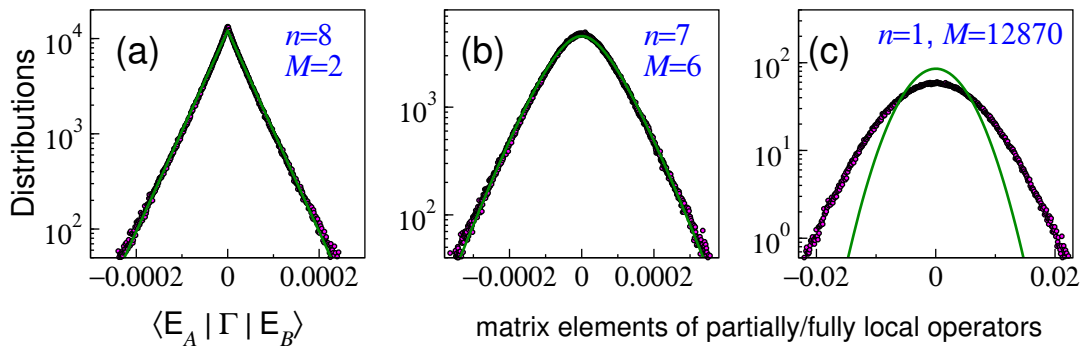
ETH works because physical operators are **sparse**.

$\mathcal{D}^{-1/2}$ scaling works because local operators have $M \sim O(\mathcal{D})$

NON-LOCAL TO LOCAL TO TYPICAL

XXZ + NNN

$(L, N_p) = (17, 8)$. $J_1 = J_2 = 1, \Delta_1 = \Delta_2 = 0.8$



EIGENSTATE THERMALIZATION SCALING

Many-body Hamiltonians are random matrices (?!?)

Eigenstate thermalization hypothesis (which observables?)

E.T.H. Scaling — local, less local & Behemoth operators

