Supercaracter Theories for Groups Defined by Involutions

Ana Neto, Carlos A. M. André, Pedro J. Freitas

CEAFEL – DM FCUL
Universidade de Lisboa

June 23, 2015

Outline

- Classic supercharacter theory
- Groups defined by involutions
- Supercharacter theories for groups defined by involutions
- Main examples

Supercharacter Theory

Main definition

G: a finite group.
$\mathrm{Irr}(G)$: the set of irreducible characters of G.

Definition (Diaconis & Isaacs)

Let $\mathcal{K}$ be a partition of G, and let $\mathcal{X}$ be a partition of $\mathrm{Irr}(G)$. The pair $(\mathcal{X}, \mathcal{K})$ is called a *supercharacter theory* for G if

- $|\mathcal{X}| = |\mathcal{K}|$.
- $\{1\} \in \mathcal{K}$.
- For each $X \in \mathcal{X}$, the character

$$\sum_{\psi \in X} \psi(1)\psi$$

  is constant on each element of $\mathcal{K}$.

Main examples

The unitriangular group with entries in a finite field F of characteristic $p \neq 2$.

$$UT_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} = I_n + \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}$$

More generally, we consider an algebra A and algebra groups of the form

$$P = 1 + J,$$

with $P, J \subseteq A$, J a nilpotent subalgebra of A.

The orbit method

Consider actions of P and $P \times P$ on J and $J^0 = \mathrm{Irr}(J^+)$.

For $x, y \in P$, $a \in J$ and $\lambda \in J^0$,

$$(x, y) \cdot a := x\, a\, y^{-1}$$

$$(\lambda x)(a) = \lambda(a x^{-1}) \qquad (x\lambda)(a) = \lambda(x^{-1} a)$$

$$(x, y) \cdot \lambda := x \lambda y^{-1}$$

So we can consider orbits $PaP \subseteq J$ and $P\lambda, P\lambda P \subseteq J^0$.

$$J = \bigsqcup_{a \in J} PaP \qquad J^0 = \bigsqcup_{\lambda \in J^0} P\lambda = \bigsqcup_{\lambda \in J^0} P\lambda P.$$

A supercharacter theory for P

Superclasses of $P = 1 + J$:

$$\{\widehat{K}_{1+a} = 1 + PaP : a \in J\}$$

Supercharacters of $P = 1 + J$:

$$\widehat{\xi}_\lambda(x) = \frac{|P\lambda|}{|P\lambda P|} \sum_{\mu \in P\lambda P} \mu(x - 1)$$

Another formula: denote by $\mathrm{Irr}_\lambda(P)$ the set of irreducible constituents of $\widehat{\xi}_\lambda$. Then

$$\widehat{\xi}_\lambda = \frac{|P\lambda|}{|P\lambda P|} \sum_{\widehat{\chi} \in \mathrm{Irr}_\lambda(P)} \widehat{\chi}(1)\widehat{\chi}$$

Note that supercharacters depend on $\lambda \in J^0$.

Groups defined by involutions

Involutions

Involution of A

We say that σ is an involution of the algebra A, with the following properties: for all $a, b \in A$,

- $\sigma^2(a) = a$
- $\sigma(a + b) = \sigma(a) + \sigma(b)$
- $\sigma(ab) = \sigma(b)\sigma(a)$
- $\sigma(F.1_A) = F.1_A$

Simple examples:

- $n \times n$ matrices with $\sigma(M) = M^T$.
- $M \mapsto JM^T J$ where J is the matrix with ones on the anti-diagonal.

Action of σ on P

Take $J \leq A$ to be a nilpotent σ-invariant subalgebra, and $P = 1 + J$.

The group P becomes invariant for the group action $x^\sigma := \sigma(x^{-1})$.

We want to construct a supercharacter theory for the centralizer of this action.

Groups defined by involutions

$$C_P(\sigma) = \{x \in P : x^\sigma = x\}$$

Some character theories for groups of this type are known:

- The symplectic groups
- The orthogonal groups
- The unitary groups

All these are subgroups of UT_n (definitions to appear later).

The Cayley transform

For $a \in J$, define $a^\sigma := -\sigma(a)$ and

$$C_J(\sigma) = \{a \in J : a^\sigma = a\}.$$

Recall $P = 1 + J$. The map

$$\begin{array}{ccc} J & \to & P \\ a & \mapsto & 1 + a \end{array}$$

doesn't map $C_J(\sigma)$ to $C_P(\sigma)$.

The Cayley transform

Remember $P = 1 + J$.

Cayley transform

$$\Phi : J \to P, \quad a \mapsto (1+a)(1-a)^{-1} \qquad \Psi : P \to J, \quad x \mapsto (x-1)(x+1)^{-1}$$

We have $\Phi(a^\sigma) = \Phi(a)^\sigma$, thefore the transform maps $C_J(\sigma)$ to $C_P(\sigma)$ and vice-versa.

Other actions of σ

Given $\widehat{\chi}$, a character of P, we define

$$\widehat{\chi}^{\sigma}(x) := \widehat{\chi}(x^{\sigma}) = \widehat{\chi}(\sigma(x^{-1}))$$

Denote $\mathrm{Irr}_{\sigma}(P) = \{\widehat{\chi} \in \mathrm{Irr}(P) : \widehat{\chi}^{\sigma} = \widehat{\chi}\}$ and $\mathrm{SCh}_{\sigma}(P)$ similarly.

Given $\lambda \in J^0$, we define

$$\lambda^{\sigma}(a) := \lambda(a^{\sigma}) = \lambda(-\sigma(a))$$

Other actions of σ

Given $\widehat{\chi}$, a character of P, we define

$$\widehat{\chi}^{\sigma}(x) := \widehat{\chi}(x^{\sigma}) = \widehat{\chi}(\sigma(x^{-1}))$$

Denote $\mathrm{Irr}_{\sigma}(P) = \{\widehat{\chi} \in \mathrm{Irr}(P) : \widehat{\chi}^{\sigma} = \widehat{\chi}\}$ and $\mathrm{SCh}_{\sigma}(P)$ similarly.

Given $\lambda \in J^0$, we define

$$\lambda^{\sigma}(a) := \lambda(a^{\sigma}) = \lambda(-\sigma(a))$$

Redefine:

$$\widehat{\xi}_{\lambda}(x) = \frac{|P\lambda|}{|P\lambda P|} \sum_{\mu \in P\lambda P} \mu(\Psi(x))$$

Main properties remain valid, and more:

$$\mathrm{SCh}_{\sigma}(P) = \{\widehat{\xi}_{\lambda} : \lambda^{\sigma} = \lambda\}$$

Supercharacter theories for groups defined by involutions

Superclasses

Conjugacy classes of $C_P(\sigma)$ are obtained by intersection of classes of P with $C_P(\sigma)$.

Let $\mathrm{SCl}_\sigma(P)$ be the set of σ-invariant superclasses of P.

For each $\widehat{K} \in \mathrm{SCl}_\sigma(P)$, we take $\widehat{K} \cap C_P(\sigma)$ as a superclass.

$$\begin{array}{rcl} \mathrm{SCl}_\sigma(P) & \to & \mathrm{Scl}(C_P(\sigma)) \\ \widehat{K} & \mapsto & \widehat{K} \cap C_P(\sigma) \end{array}$$

Supercharacters – the Glauberman correspondence

How to use the supercharacter theory of P to get one for $C_P(\sigma)$?

As with superclasses, we associate irreducible characters of P and $C_P(\sigma)$ using the Glauberman correspondence.

$$\pi_P : \mathrm{Irr}_\sigma(P) \to \mathrm{Irr}(C_P(\sigma))$$

where $\pi_P(\widehat{\xi})$ is the only constituent of $\widehat{\xi}|_{C_P(\sigma)}$ with odd multiplicity.

Supercharacters – definition

Let λ be a σ-invariant element of J^0.

Take $\widehat{\xi}_\lambda \in \mathsf{SCh}_\sigma(P)$.

Consider all irreducible characters $\widehat{\chi}$ that are components of $\widehat{\xi}_\lambda$, $< \widehat{\xi}_\lambda, \widehat{\chi} > \neq 0$.

Consider all their Glauberman correspondents $\chi := \pi_P(\widehat{\chi})$.

Supercharacters – definition

Let λ be a σ-invariant element of J^0.

Take $\widehat{\xi}_\lambda \in \mathsf{SCh}_\sigma(P)$.

Consider all irreducible characters $\widehat{\chi}$ that are components of $\widehat{\xi}_\lambda$, $< \widehat{\xi}_\lambda, \widehat{\chi} > \neq 0$.

Consider all their Glauberman correspondents $\chi := \pi_P(\widehat{\chi})$.

$$\lambda \to \widehat{\xi}_\lambda \to \widehat{\chi} \to \chi$$

$$\mathsf{Irr}_\lambda(C_P(\sigma)) = \{\chi \in \mathsf{Irr}(C_P(\sigma)) :< \widehat{\xi}_\lambda, \widehat{\chi} > \neq 0\}$$

Supercharacters – definition

Let λ be a σ-invariant element of J^0.

Take $\widehat{\xi}_\lambda \in \mathsf{SCh}_\sigma(P)$.

Consider all irreducible characters $\widehat{\chi}$ that are components of $\widehat{\xi}_\lambda$, $< \widehat{\xi}_\lambda, \widehat{\chi} > \neq 0$.

Consider all their Glauberman correspondents $\chi := \pi_P(\widehat{\chi})$.

$$\lambda \to \widehat{\xi}_\lambda \to \widehat{\chi} \to \chi$$

$$\mathsf{Irr}_\lambda(C_P(\sigma)) = \{\chi \in \mathsf{Irr}(C_P(\sigma)) :< \widehat{\xi}_\lambda, \widehat{\chi} > \neq 0\}$$

$$\rho_\lambda := \sum_{\chi \in \mathsf{Irr}_\lambda(C_p(\sigma))} \chi(1)\chi$$

Supercharacters – properties

$$\lambda \to \widehat{\xi}_\lambda \to \widehat{\chi} \to \chi$$

$$\mathrm{Irr}_\lambda(C_P(\sigma)) = \{\chi \in \mathrm{Irr}(C_P(\sigma)) : < \widehat{\xi}_\lambda, \widehat{\chi} > \neq 0\}$$

$$\rho_\lambda = \sum_{\chi \in \mathrm{Irr}_\lambda(C_p(\sigma))} \chi(1)\chi$$

The set $\{\mathrm{Irr}_\lambda(C_P(\sigma)) : \lambda \in J^0,\ \lambda = \lambda^\sigma\}$ is a partition of $\mathrm{Irr}(C_P(\sigma))$: each irreducible character of $C_P(\sigma)$ is a component of one and only one ρ_λ.

Supercharacters – properties

$$\rho_\lambda = \sum_{\chi \in \mathrm{Irr}_\lambda(C_p(\sigma))} \chi(1)\chi$$

Let $\Omega_P(\lambda) = \{\mu \in P\lambda P : \mu^\sigma = \mu\}$.

We have

$$\rho_\lambda(x) = \sum_{\mu \in \Omega_P(\lambda)} \mu(\Psi(x))$$

This proves ρ_λ is constant on superclasses.

The functions ρ_λ will be *the supercharacters of $C_P(\sigma)$*.

Supercharacters – main result

$$\rho_\lambda(x) = \sum_{\chi \in \mathrm{Irr}_\lambda(C_p(\sigma))} \chi(1)\chi(x) = \sum_{\mu \in \Omega_P(\lambda)} \mu(\Psi(x))$$

- Compare with $\widehat{\xi}_\lambda$:

$$\frac{|P\lambda P|}{|P\lambda|}\widehat{\xi}_\lambda = \sum_{\widehat{\chi} \in \mathrm{Irr}_\lambda(P)} \widehat{\chi}(1)\widehat{\chi}$$

  and reduce to the case where $\widehat{\xi}_\lambda$ has a linear constituent.
- Prove that, in this case, the second formula is a character induced from a linear character of a subgroup of $C_P(\sigma)$
- Gallagher's theorem identifies the components of the induced character.

Main Examples

In $GL_n(K)$

Let $q = |K^\sigma|$. For $a \in GL_n(K)$, define $(a^*)_{ij} = a_{ji}^q$.

We consider involutions $\sigma_u(a) = a \mapsto u^{-1} a^T u$ of two kinds:

- First kind: $K^\sigma = K$, $u^* = u^T$ and $u^T = \pm u$.
- Second kind: $K^\sigma < K$, $u^* = u$.

The symplectic, orthogonal and unitary groups can all be written as $C_{UT_n}(\sigma_u)$ for adequate involutions σ_u, with $\sigma_u(UT_n) = UT_n$.

Subgroups of UT_n defined by involutions

Recall that J_m is the matrix with ones on the anti-diagonal.

- For $Sp_{2m}(K)$, we choose $u = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$ (first kind, $u^T = -u$).
- For $O^+_{2m}(K)$ or $O_{2m+1}(K)$, we choose $u = J_n$ where, either $n = 2m$, or $n = 2m + 1$ (first kind, $u^T = u$).
- For $O^-_{2m+2}(K)$, we choose $u = \begin{pmatrix} 0 & 0 & J_m \\ 0 & c & 0 \\ J_m & 0 & 0 \end{pmatrix}$ where $c = \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon \end{pmatrix}$ for $\epsilon \in K^x \setminus (K^x)^2$ (first kind, $u^T = u$).
- For $U_n(K)$, we choose $u = J_n$ (second kind, $u^* = u$).

Basic pairs

The known character theory for $P = UT_n$ depends on *basic pairs* (D, φ).

- $D \subset \{(i,j) : 1 \leq i < j \leq n\}$ a basic subset.
- $\varphi : D \to K^x$.

Superclasses and supercharacters of P are indexed by basic pairs.

Basic pairs

The known character theory for $P = UT_n$ depends on *basic pairs* (D, φ).

- $D \subset \{(i, j) : 1 \leq i < j \leq n\}$ a basic subset.
- $\varphi : D \to K^x$.

Superclasses and supercharacters of P are indexed by basic pairs.

Superclasses and supercharacters of $C_P(\sigma)$ are indexed by σ-invariant basic pairs (need to define action of σ).

This generalises the known theories for these groups.

References

- A. M. Neto, C. A. M. André & P. J. Freitas, A Supercharacter Theory for Involutive Algebra Groups, *Journal of Algebra* (2015)
- C. A. M. André,
  - Basic characters of the unitriangular group (1995)
  - Irreducible characters of groups associated with finite nilpotent algebras with involution (2010)
- P. Diaconis & I. M. Isaacs, Supercharacters and superclasses for algebra groups (2008)
- C. A. M. André & A. M. Neto,
  - Super-characters of finite unipotent groups of types B_n, C_n and D_n (2006)
  - A supercharacter theory for the Sylow p-subgroups of the finite symplectic and orthogonal groups (2009)