Lectures on étale groupoids, inverse semigroups and quantales

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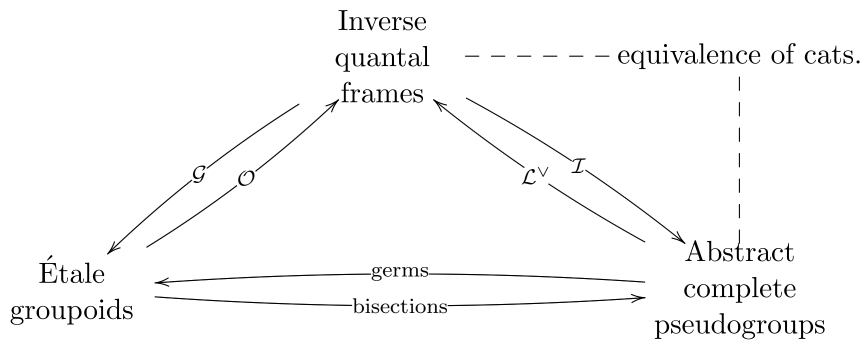
Preface

Groupoids and inverse semigroups are two generalizations of the notion of group. Both provide a handle on more general kinds of symmetry than groups do, in particular symmetries of a local nature, and applications of them crop up almost everywhere in mathematics — evidence of this is the number of related textbooks in analysis, geometry, topology, algebra, category theory, etc. [1, 2, 3, 5, 9, 14, 15, 16, 19, 29, 32]. The fact that inverse semigroups and étale groupoids are somewhat similar in spirit is made apparent by several known constructions that go back and forth between both. The main purpose of these notes is to deepen this correspondence in a way that to some extent subsumes all the others. The method in question has been introduced in [33], where it turns out that quantales are crucial because they play the role of mediating objects between inverse semigroups and étale groupoids. For this reason I shall also address quantales in these notes, in particular providing much more background material about them than in [33].

I also hope in this way to make available (at least the beginnings of) a useful reference textbook for those general algebraic aspects of quantales and their modules that keep being recalled in introductory sections of research papers but otherwise are scattered in the literature. I shall not attempt to be exhaustive, and in particular I shall not address spatial aspects of quantales such as those of [11]. An older monograph on quantales is that of Rosenthal [36], part of whose material I revisit in these notes, and the interested reader may also wish to look at two more recent survey papers, one by Mulvey [22], and the other by Paseka and Rosický [28].

A strong motivation for developing this material comes from noncommutative geometry in the sense of Connes [3]. Quantales have been originally proposed [20] as being generalized spaces in their own right and, in particular, spaces that can provide a notion of "spectrum" for noncommutative C*-algebras capable of classifying them up to isomorphism [12, 13, 21, 24, 25, 26]. This effort has been greeted with several difficulties (see, e.g., [12, 13]), and in order to make sense of the ideas involved it may be useful to focus on specific types of C*-algebra. The C*-algebras that are constructed out of groupoids and inverse semigroups play a major role in noncommutative geometry, and it is therefore natural to address them. More than that, it is natural to look at groupoids per se because they alone often play the role of generalized spaces. It is fortunate that, as we shall see, there is a rather perfect match between étale groupoids, complete and infinitely distributive inverse semigroups (herein called abstract complete pseudogroups), and the quantales known as inverse quantal frames. We can summarize this with the following diagram, where all the arrows are invertible maps up to isomorphism, and those on the right are also functors:



From a slightly different standpoint, C*-algebras are sometimes regarded as mediating objects between groupoids and inverse semigroups (e.g., this happens occasionally in [29]). The material presented in these notes has the effect, in this context, of replacing C*-algebras by quantales. To a large extent these are the right mediating objects because they are not subject to restrictions motivated by analysis, such as requiring unit spaces of groupoids to be locally compact Hausdorff, etc. In addition, an advantage of making the role of quantales explicit is the possibility of bringing the theory to bear on more general kinds of groupoids, such as open groupoids. More precisely, the equivalence of categories in the above diagram enables us to replace inverse semigroups by quantales when moving from étale groupoids to open groupoids. Apart from brief considerations about open groupoids, I shall not address this topic in these notes, and I also leave untouched important aspects such as algebraic topological tools for quantales themselves (e.g., cohomology), and the connections to C*-algebras, which need further elaboration.

I have tried to be as constructive as possible, in the sense of providing definitions and results that can be carried over to an arbitrary topos, although without being fussy about this. The main advantage of such an effort, at least in principle, is that in this way theorems are proved in their greatest generality; for instance, a theorem proved constructively yields automatically an equivariant version of itself, simply by interpreting the theorem in the topos of G-sets for a group G, or a version for sheaves by interpreting the theorem in the topos of sheaves (of sets) on a space or a site. The main results in these notes are indeed constructive, and occasionally readers will see explicit symptoms of this such as when a distinction is made between the powerset $\wp(1)$ of a singleton and its order dual $\wp(1)^{\mathrm{op}}$. Classically these two lattices are isomorphic, of course, but this is not so in a general topos. Another symptom is the use of locales instead of topological spaces, although this is also motivated by pragmatic reasons that do not have to do with constructivity. In any case there are places, for instance in many exercises, where constructivity is neglected. This tends to happen, for instance, when we deal with spatial aspects of locales. I have not placed any clear road signs in the text warning about constructivity or the lack of it because readers familiar with these things probably do not need to be warned, while others may comfortably assume that everything is taking place in the category of sets.

The notes are divided into three chapters. Chapter I addresses the more classical aspects of topological groupoids and inverse semigroups. Still, the general equivalence between abstract complete pseudogroups and topological étale groupoids that it presents does not seem to be found elsewhere in the literature. Chapter II develops general algebraic aspects of quantales, including the theory of supported quantales that has been introduced in [33]. Finally, Chapter III describes localic groupoids and in particular localic étale groupoids and their relation to quantales.

A large portion of these notes is based on [33]. Most of the rest has grown out of six lectures which I gave at the University of Antwerp in September of 2005 during a two week course organized by Freddy van Oystaeyen in the scope of the SOCRATES Intensive Program 103466-IC-1-2003-1-BE-ERASMUS-IPUC-3: GAMAP: Geometric and Algebraic Methods of Physics and Applications, and they have received an additional boost from a working seminar on groupoids and noncommutative geometry that I have been running in the scope of the FCT/POCI2010/FEDER grant POCI/MAT/55958/2004, jointly with Catarina Carvalho, Rui Loja Fernandes, and Radu Popescu, at the Department of Mathematics of Instituto Superior Técnico since October 2005.

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Contents

Chapter I

Groupoids and inverse semigroups

In this chapter we shall give basic definitions and results concerning groupoids and inverse semigroups, culminating with a general bijective (up to isomorphism) correspondence between topological étale groupoids with unit space X and abstract complete pseudogroups (i.e., complete and infinitely distributive inverse semigroups) that act on X in a suitable way. The construction of abstract complete pseudogroups from the groupoids is given in terms of local bisections (equivalently, open G-sets in the terminology of [32]), and, in the converse direction, from an abstract complete pseudogroup S we construct a groupoid whose arrows are the germs of the elements of S. This generalizes the well known germ groupoid of a pseudogroup, which we also recall, and it can be regarded also as a generalization of the classical correspondence between sheaves and local homeomorphisms.

1 Groupoids

Let us start by looking at groupoids. We start with discrete groupoids. Then, using these as motivation, we introduce internal groupoids in a category, and then topological groupoids, which are internal groupoids in *Top*; as an example we describe the germ groupoid of a pseudogroup. Of course, there would be no need to give the general definition of internal groupoid just for defining topological groupoids, but we do this because we shall later, in Chapter III, have to work with localic groupoids, which are internal groupoids in the category *Loc* of locales.

Discrete groupoids. A *(discrete) groupoid* is a small category all of whose arrows are invertible. Hence, a groupoid G can be explicitly described as consisting of a set of *arrows* (or *morphisms*) G_1, a set of *objects* G_0, and structure maps

$$G_2 \xrightarrow{m} G_1 \overset{i}{\circlearrowleft} \underset{d}{\overset{r}{\underset{u}{\rightleftarrows}}} G_0$$

where G_2 is the set $G_1 \times_{G_0} G_1$ of *composable pairs* of arrows:

$$G_2 = \{(x, y) \in G_1 \times G_1 \mid r(x) = d(y)\} \ .$$

The map m is called the *multiplication*, or *product*; d is the *domain* map; r is the *range* map, or *codomain*; i is the *inverse* map; and u is the *units* map. The axioms satisfied by these maps are, writing x^{-1} for $i(x)$ (the *inverse* of x), xy for $m(x, y)$ (the *product*, or *composition*, of x and y), and 1_x for $u(x)$ (the *identity arrow*, or *unit*, on x), the following.

Category axioms:

$$\begin{array}{rcll} d(1_x) & = & x & \text{(domain of a unit)} \\ r(1_x) & = & x & \text{(range of a unit)} \\ d(xy) & = & d(x) & \text{(domain of a product)} \\ r(xy) & = & r(y) & \text{(range of a product)} \\ (xy)z & = & x(yz) & \text{(associativity)} \\ 1_{d(x)}x & = & x & \text{(left unit law)} \\ x1_{r(x)} & = & x & \text{(right unit law)} \end{array}$$

Inverse axioms:

$$\begin{array}{rcll} d(x^{-1}) & = & r(x) & \text{(domain of inverse)} \\ r(x^{-1}) & = & d(x) & \text{(codomain of inverse)} \\ xx^{-1} & = & 1_{d(x)} & \text{(left inverse law)} \\ x^{-1}x & = & 1_{r(x)} & \text{(right inverse law)} \end{array}$$

We shall often write G for the set of arrows G_1 and regard G_0 as a subset of G by identifying it with its image $u(G_0)$, which makes sense because u is necessarily injective (cf. exercise I.1.5-2).

We shall usually write $z : x \to y$ to specify that an arrow z has domain $d(z) = x$ and range $r(z) = y$, and we denote by $G(x, y)$ the set of arrows $z : x \to y$. The set $G(x, x)$ is a group. It is called the *isotropy group* at x and we denote it by I_x.

For any groupoid G the binary relation on G_0 defined by $x \sim y$ if $z : x \to y$ for some arrow z is an equivalence relation whose equivalence classes are called the *orbits* of the groupoid. The isotropy groups are invariants of the orbits, i.e., $I_x \cong I_y$ if $x \sim y$. A groupoid is *connected*, or *transitive*, if there is exactly one equivalence class. In this case there is, up to isomorphism, only one isotropy group. We refer to it as the *isotropy* of G, and denote it by I_G.

A *map* of groupoids $f : G \to G'$ (G' has structure maps d', r', etc.) is a functor from G to G'. In other words, it consists of a pair of functions

$$f_1 : G_1 \to G'_1$$
$$f_0 : G_0 \to G'_0$$

that preserve the structure maps of the groupoids in the appropriate way; that is, for all $x \in G_0$ and $y, z \in G_1$ (we use the same names for the structure maps of both groupoids):

$$\begin{aligned} f_0(d(y)) &= d'(f_1(y)) \\ f_0(r(y)) &= r'(f_1(y)) \\ f_1(1_x) &= 1_{f_0(x)} \\ f_1(yz) &= f_1(y)f_1(z) \end{aligned}$$

It is clear that f_0 is completely determined by f_1, and we shall usually write just f instead of f_1. Maps automatically preserve inverses: $f(x^{-1}) = f(x)^{-1}$. An *isomorphism* of groupoids $f : G \to H$ is a map that has an inverse functor $f^{-1} : H \to G$. Two groupoids G and H are *isomorphic* if there is an isomorphism $f : G \to H$.

Example I.1.1 Groups. Any group G is a groupoid with $G_1 = G$, $G_2 = G \times G$, and $G_0 = \{1\}$. This is a transitive groupoid with $I_G \cong G$.

Example I.1.2 Group bundles. Consider a set X and a family (G_x) of groups indexed by X. We obtain a groupoid G, called a *group bundle over* X, by defining $G_0 = X$, $G_1 = \coprod_x G_x$, setting 1_x to be the unit of the group G_x for each $x \in X$, letting $d(g) = r(g) = x$ for each $g \in G_x$, defining the product of composable arrows to be group multiplication and the inverses to be group inverses. (In other words, a group bundle is just a groupoid such that $d = r$.) The isotropy groups are the groups G_x, and the orbits are singletons.

Example I.1.3 Equivalence relations and pair groupoids. A groupoid which has at most one arrow between any two units is the same thing as an equivalence relation on G_0, with $d(x,y) = x$ and $r(x,y) = y$ and multiplication given by $(x,y)(y,z) = (x,z)$. Equivalently, this is, up to isomorphism, a groupoid whose isotropy groups are all trivial; the orbits are precisely the equivalence classes. For any set X the obvious groupoid G with $G_0 = X$ and $G_1 = X \times X$ is called the *pair groupoid* of X and it is denoted by $\mathrm{Pair}(X)$. This is the same as a transitive groupoid with trivial isotropy.

Example I.1.4 Groupoids of "automorphisms" of bundles. Let X be a set, C a category, and consider a family $A = (A_x)_{x \in X}$ of objects of C. We define the groupoid $\mathrm{Aut}(A)$ whose set of objects is X and whose arrows from x to y are the triples (x,f,y) with $f : A_x \to A_y$ an isomorphism of C. The domain and range maps are given by $d(x,f,y) = x$ and $r(x,f,y) = y$, the multiplication is given by

$$(x,f,y)(y,g,z) = (x, g \circ f, z) \ ,$$

the inverses are given by

$$(x,f,y)^{-1} = (y, f^{-1}, x)$$

and the units are $1_x = (x, \mathrm{id}_{A_x}, x)$. Given a groupoid G, a *representation* of G on A is a map of groupoids $G \to \mathrm{Aut}(A)$. For instance, an *abelian representation* is a representation on a family of abelian groups (i.e., C is the category of abelian groups), a *linear representation* is a representation on a family of linear spaces, etc.

We shall see more examples in section 1.

Exercise I.1.5

1. Show that in a groupoid G the following conditions hold for all $x \in G_0$ and $y,z \in G_1$: $(1_x)^{-1} = 1_x$, $(y^{-1})^{-1} = y$ and $(yz)^{-1} = z^{-1}y^{-1}$. (Hence, any groupoid is an *involutive* category.)

2. (Alternative definition of groupoid.) Let G be a set, G_2 a subset of $G \times G$, $x \mapsto x^{-1}$ an endomap of G, and $(x,y) \mapsto xy$ a map from G_2 to G, satisfying the following relations:

   (a) $(x^{-1})^{-1} = x$

   (b) If (x,y), $(y,z) \in G_2$ then (xy,z), $(x,yz) \in G_2$ and $(xy)z = x(yz)$

   (c) $(x^{-1},x) \in G_2$ and if $(x,y) \in G_2$ then $x^{-1}(xy) = y$

   (d) $(x,x^{-1}) \in G_2$ and if $(z,x) \in G^2$ then $(zx)x^{-1} = z$

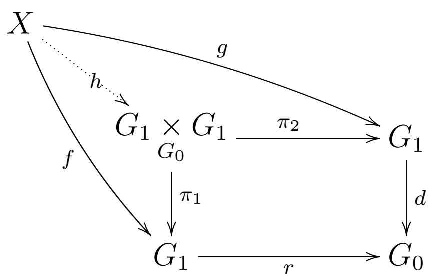
   Show that G is a groupoid with d defined by $x \mapsto xx^{-1}$ and $G_0 = d(G)$. Show that any groupoid is of this form, up to isomorphism.

3. Let G and H be discrete groupoids. Show that the product $G \times H$ as categories is a groupoid.

4. Let G and H be transitive discrete groupoids, with isotropy groups I_G and I_H, respectively. Show that the product groupoid $G \times H$ is a transitive groupoid with isotropy $I_G \times I_H$.

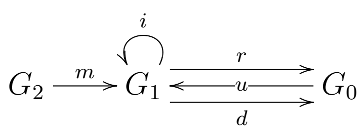
5. Let G be a transitive discrete groupoid with isotropy I. Show that G is isomorphic to the product groupoid $I \times \mathrm{Pair}(G_0)$.

Internal groupoids. Many of the groupoids that arise in practice, such as topological groupoids, algebraic groupoids, the localic groupoids that we shall see later in these notes, etc., are examples of internal groupoids in the sense of internal category theory (see [17, Ch. XII]), and it is useful to look at their axioms and elementary properties in this general context. In order to do this, first we observe that the set $G_2 = G_1 \times_{G_0} G_1$ in the definition of a discrete groupoid G is the pullback (in the category of sets) of the maps d and r (we write π_1 and π_2 for the first and second projections, respectively):



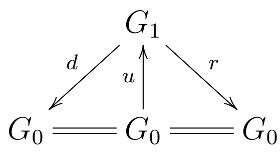
$$h = \langle f, g\rangle$$

Let C be any category with pullbacks (i.e., any pair of arrows $\bullet \to \bullet \leftarrow \bullet$ has a pullback in C). Then, generalizing the definition of discrete groupoid, we define an *internal groupoid G in C* to consist of a pair of objects (G_1, G_0) of C — G_1 is called the *object of arrows* and G_0 is called the *object of objects* — equipped with structure morphisms (in C)



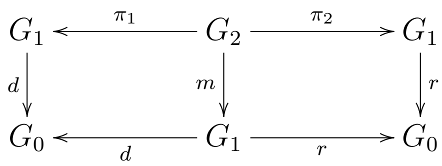
where G_2 is the pullback $G_1 \times_{G_0} G_1$ (we shall use for these maps the same names as in the definition of a discrete groupoid), all of which are required to satisfy the axioms that we now describe in the form of commutative diagrams. As in the previous section for discrete groupoids, we begin by listing the category axioms (i.e., the axioms that define an internal category in C), where for the associativity axiom we write G_3 for the pullback $G_2 \times_{G_0} G_1$ of the morphisms $r \circ \pi_2 : G_2 \to G_0$ and $d : G_1 \to G_0$.

Domain and codomain of a unit



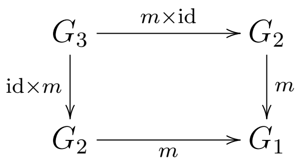
$$\begin{aligned} d \circ u &= \mathrm{id}_{G_0} \\ r \circ u &= \mathrm{id}_{G_0} \end{aligned}$$

Domain and codomain of a product



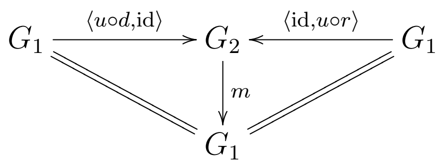
$$\begin{aligned} d \circ m &= d \circ \pi_1 \\ r \circ m &= r \circ \pi_2 \end{aligned}$$

Associativity (See also exercise I.1.61.)



$$m \circ (m \times \mathrm{id}_{G_1}) = m \circ (\mathrm{id}_{G_1} \times m)$$

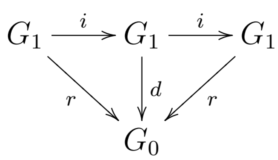
Left and right unit laws



$$\begin{aligned} m \circ \langle u \circ d, \mathrm{id}_{G_1} \rangle &= \mathrm{id}_{G_1} \\ m \circ \langle \mathrm{id}_{G_1}, u \circ r \rangle &= \mathrm{id}_{G_1} \end{aligned}$$

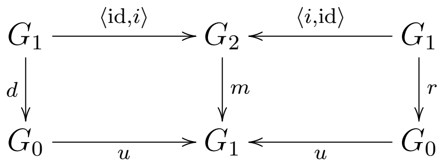
The remaining axioms of internal groupoids are now those for the inverse morphism i.

Domain and codomain of inverses



$$\begin{aligned} d \circ i &= r \\ r \circ i &= d \end{aligned}$$

Left and right inverse laws

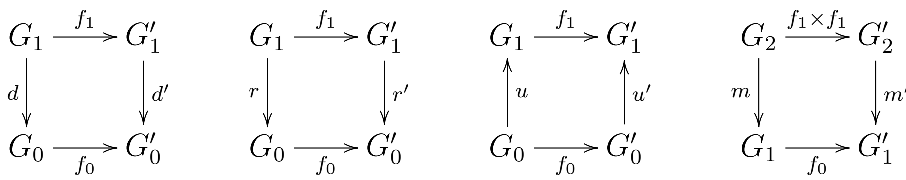


$$\begin{array}{rcl} m \circ \langle \mathrm{id}, i \rangle & = & u \circ d \\ m \circ \langle i, \mathrm{id} \rangle & = & u \circ r \end{array}$$

Let G and G' be internal groupoids in a category with pullbacks C. The definition of map of groupoids is a straightforward generalization of that of the discrete case: a *map* from G to G' is an *internal functor*, i.e., a pair of morphisms of C

$$f_1 : G_1 \to G'_1$$
$$f_0 : G_0 \to G'_0$$

such that the following diagrams are commutative:



Similarly to discrete groupoids, f_0 is completely determined by f_1, and we shall usually write just f instead of f_1. An *isomorphism* of groupoids is a map (f_1, f_0) such that both f_0 and f_1 are invertible in C.

Exercise I.1.6 1. Recall that we have defined G_3 to be the pullback $G_2 \times_{G_0} G_1$ of the morphisms $r \circ \pi_2 : G_2 \to G_0$ and $d : G_1 \to G_0$. The expression $m \times \mathrm{id}$ in the associativity axiom can then be explicitly identified with the pairing morphism $\langle m \circ \pi_1, \pi_2 \rangle$. (Warning: we use the same notation π_1 and π_2 for the projections of both pullbacks G_2 and G_3.) For the same definition of G_3 give a similar interpretation of $\mathrm{id} \times m$.

2. Let $G_0 \times_{G_0} G_1$ be an arbitrary pullback of $G_0 \xrightarrow{\mathrm{id}} G_0 \xleftarrow{d} G_1$. Show that $\pi_2 : G_0 \times_{G_0} G_1 \to G_1$ is an isomorphism with inverse $\langle d, \mathrm{id} \rangle$.

3. Show that for any internal groupoid we have $i \circ i = \mathrm{id}$.

4. Let $\chi : G_2 \to G_2$ be the pairing $\langle i \circ \pi_2, i \circ \pi_1 \rangle$. Show that

$$i \circ m = m \circ \chi$$

(this is the analogue of the condition $(xy)^{-1} = y^{-1}x^{-1}$ of discrete groupoids).

5. Show that χ as above is an isomorphism with $\chi^{-1} = \chi$.

6. Show that the following diagram (which is always commutative for any category G) is a pullback if G is a groupoid.

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 & \xrightarrow{\pi_1} & G_1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle d} \\ G_1 & \xrightarrow[d]{} & G_0 \end{array}$$

7. Show that the converse holds; that is, the above diagram is a pullback if and only if G is a groupoid.

8. Prove that maps of internal groupoids preserve inverses; that is, show that for any map $f : G \to G'$ the following diagram is commutative:

$$\begin{array}{ccc} G_1 & \xrightarrow{f} & G_1' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ G_1 & \xrightarrow[f]{} & G_1' \end{array}$$

Topological groupoids. A *topological groupoid* is an internal groupoid G in the category of topological spaces and continuous maps. In other words, it consists of a pair of topological spaces G_1 and G_0 equipped with continuous maps

$$G_2 \xrightarrow{m} G_1 \circlearrowleft^{i} \quad G_1 \;\underset{d}{\overset{r}{\rightrightarrows}}\; G_0, \qquad G_0 \xrightarrow{u} G_1$$

where now G_2 is the pullback in the category of topological spaces, namely the subspace $G_1 \times_{G_0} G_1$ (with the relative topology) of the space $G_1 \times G_1$ with the product topology. The axioms of internal groupoids can be written in the same way as those of discrete groupoids, of course, and we shall adopt the same notation and terminology as in the discrete case.

The topological groupoids that arise in practice usually satisfy properties that make them especially well behaved. As examples we mention two that will be relevant in these notes: a topological groupoid G is *open* if its domain map d is open; if furthermore d is a local homeomorphism the groupoid G is said to be *étale*.

Equivalently, an open groupoid is one whose topology is closed under pointwise multiplication of open sets (equivalently, m is an open map), and the étale groupoids are the open groupoids whose subspace of units is open. (For these and other facts about open or étale groupoids see the list of exercises below.) The latter condition is strong, in particular implying that the topology of the groupoid has a basis of "open G-sets" (see exercise I.1.8-6), but it does not imply that the groupoid in question is open, hence not necessarily étale, as the following example shows.

Example I.1.7 Let G be the topological groupoid defined by

$$\begin{aligned} G_0 &= \mathbb{R} \\ G_1 &= \mathbb{R} \amalg \{*\} \\ d(x) = r(x) &= x \text{ if } x \in \mathbb{R} \\ d(*) = r(*) &= 0 \\ m(*,*) &= 0 \end{aligned}$$

(Hence, this is a group bundle over $\mathbb{R}$ with trivial isotropy I_x for all $x \neq 0$ and $I_0 \cong \mathbb{Z}_2$.) G_0 is open in G_1, but the groupoid is not open because $\{*\}$ is open in G_1 but $d(\{*\}) = \{0\}$ is not open in $\mathbb{R}$.

Exercise I.1.8 In the exercises that follow, G is a fixed but arbitrary topological groupoid. We identify the space of objects G_0 with the space of units $u(G_0)$, we write G instead of G_1, for any subsets $U, V \subseteq G$ we write UV for the pointwise product

$$UV = \{xy \mid x \in U,\ y \in V,\ r(x) = d(y)\}$$

and we write U^{-1} for the pointwise inverse $\{x^{-1} \mid x \in U\}$.

1. Show that for any open set $U \subseteq G$ we have, denoting the topology of G by $\Omega(G)$,

$$\begin{aligned} (U \cap G_0)G &\subseteq \bigcup \{X \cap Y \mid X,\ Y \in \Omega(G),\ XY^{-1} \subseteq U\} \\ &\subseteq \bigcup \{V \in \Omega(G) \mid VV^{-1} \subseteq U\} \\ G(U \cap G_0) &\subseteq \bigcup \{X \cap Y \mid X,\ Y \in \Omega(G),\ X^{-1}Y \subseteq U\} \\ &\subseteq \bigcup \{V \in \Omega(G) \mid V^{-1}V \subseteq U\} \end{aligned}$$

2. Show that if m is open then so is d.

3. Prove the converse: if d is open then so is m (hint: show that open maps are stable under pullback and use exercise I.1.6). Conclude that G is open if and only if the pointwise product UV of any two open sets $U, V \subseteq G$ is itself an open set.

4. Show that G is étale if and only if m is a local homeomorphism (hint: local homeomorphisms are stable under pullback).

5. Prove that if f and g are local homeomorphisms and $g \circ h = f$ then h is a local homeomorphism. Based on this, show that if G is étale then the set of units G_0 is open in G_1.

6. By a *G-set* of G is meant a subset $U \subseteq G$ such that the restrictions $d|_U : U \to G_0$ and $r|_U : U \to G_0$ are both injective.

   (a) Show that U is a G-set if and only if $UU^{-1} \subseteq G_0$ and $U^{-1}U \subseteq G_0$.

   (b) Show that if G_0 is open in G then the open G-sets form a basis for the topology of G. (Hint: apply exercise 1 with $U = G_0$ and show that the open G-sets cover G.)

7. Show that G is étale if and only if it is open and G_0 is open in G.

8. By an *r-discrete* groupoid is meant a topological groupoid G whose r-fibers $r^{-1}(x)$ are discrete subspaces of G. Prove that any topological groupoid G whose unit subspace G_0 is open in G (in particular, any étale groupoid) is necessarily r-discrete. (Renault [32] defines r-discrete to mean that G_0 is open.[1])

Germ groupoids of pseudogroups. Let X be a topological space, and let $\mathcal{I}(X)$ be the set of all the partial homeomorphisms on X, by which are meant the homeomorphisms $h : U \to V$ with U and V open sets of X. This set has the structure of an involutive semigroup; the involution is inversion,

$$h : U \to V \quad \mapsto \quad h^{-1} : V \to U ,$$

and the multiplication is given by composition of partial homeomorphisms wherever this composition is defined: if $h : U \to V$ and $h' : U' \to V'$ are partial homeomorphisms then their product is the partial homeomorphism

$$hh' : h^{-1}(V \cap U') \to h'(V \cap U')$$

defined at each point of its domain by $(hh')(x) = h'(h(x))$. In these notes we shall use the following essentially standard terminology.

[1] In [32, I.2.8(iv)] it is further stated that a locally compact r-discrete Hausdorff groupoid is étale if and only if it has a basis of open G-sets. This is wrong, as exercise 6 and example I.1.7 show. However, this plays no role in [32] because the groupoids are assumed to have Haar measures, which implies that they are open.

Definition I.1.9 Let X be a topological space. By a *pseudogroup over* X will be meant any subset $P \subseteq \mathcal{I}(X)$ which is closed under the multiplication and the involution of $\mathcal{I}(X)$. A pseudogroup P over X is *full* if it is also closed under identities in the sense that $\mathrm{id}_U \in P$ for every open set $U \subseteq X$; and it is *complete* if it is full and for all $h \in \mathcal{I}(X)$ and every open cover (U_α) of $\mathrm{dom}(h)$ we have $h \in P$ if $h|_{U_\alpha} \in P$ for all α. The (complete) pseudogroup $\mathcal{I}(X)$ is called the *symmetric pseudogroup on* X.

There is a very natural construction of a topological étale groupoid from any full pseudogroup over X. First we define a small category whose set of objects is X and whose arrows are the pairs (x, h) such that $h \in P$ and $x \in \mathrm{dom}(h)$. We set $d(x, h) = x$ and $r(x, h) = h(x)$, the composition is defined by

$$(x, h)(h(x), k) = (x, k \circ h) \ ,$$

and the object inclusion map is defined by $x \mapsto (x, \mathrm{id}_X)$. Furthermore this category has an involution defined by

$$(x, h)^* = (h(x), h^{-1}) \ ,$$

and satisfying the property

$$(x, h)(x, h)^*(x, h) = (x, h) \ ;$$

This is an *inverse category*, meaning that for each arrow (x, h) there is exactly one other arrow (y, k) such that

$$(x, h)(y, k)(x, h) = (x, h) \quad \text{and} \quad (y, k)(x, h)(y, k) = (y, k) \ ;$$

of course, this unique arrow is $(x, h)^*$.

From this inverse category we define a groupoid as a quotient: define two arrows (x, h) and (x, k) to be equivalent, and write $(x, h) \sim (x, k)$, if $h|_U = k|_U$ for some open neighborhood $U \subseteq \mathrm{dom}(h) \cap \mathrm{dom}(k)$ of x (hence, in particular, $h(x) = k(x)$). The equivalence class of (x, h) is therefore identifed with the pair $(x, \mathrm{germ}_x h)$, where $\mathrm{germ}_x h$ is the *germ of* h *at* x, that is the set

$$\mathrm{germ}_x h = \{k \in P \mid x \in \mathrm{dom}\, k,\ k|_U = h|_U \text{ for some open } U \text{ with } x \in U\} \ .$$

The pairs $(x, \mathrm{germ}_x h)$ are the arrows of a groupoid whose set of units is X, and whose structure maps are as follows:

$$\begin{aligned}
d(x, \mathrm{germ}_x h) &= x \\
r(x, \mathrm{germ}_x h) &= h(x) \\
u(x) &= (x, \mathrm{germ}_x \mathrm{id}_X) \\
(x, \mathrm{germ}_x h)(h(x), \mathrm{germ}_{h(x)}, k) &= (x, \mathrm{germ}_x(k \circ h)) \\
(x, \mathrm{germ}_x h)^{-1} &= (h(x), \mathrm{germ}_{h(x)}(h^{-1})) \ .
\end{aligned}$$

This is called the *groupoid of germs of* P, and it is denoted by $\mathrm{Germs}(P)$. It can be given a topology by defining as a basic open set, for each $h \in P$, the set

$$U_h = \{(x, \mathrm{germ}_x h) \mid x \in \mathrm{dom}\, h\} \ .$$

It is clear that d is then a local homeomorphism, and thus $\mathrm{Germs}(P)$ is an étale groupoid.

2 Inverse semigroups

Now we present the basic theory of inverse semigroups and immediately prove that every topological étale groupoid G is the groupoid of germs of an inverse semigroup equipped with an action on G_0. In order to obtain a converse to this we shall have to study completeness and distributivity for inverse semigroups. In particular, the inverse semigroups of local bisections of étale groupoids will be shown to be precisely the complete and infinitely distributive inverse semigroups.

Basic definitions and examples. By an *inverse* of an element x of a semigroup is meant an element y in the semigroup such that

$$\begin{aligned} xyx &= x \\ yxy &= y \ . \end{aligned}$$

An *inverse semigroup* is a semigroup for which each element has a unique inverse. Equivalently, an inverse semigroup is a semigroup for which each element has an inverse (hence, a regular semigroup) and for which any two idempotents commute.

In an inverse semigroup the inverse operation defines an involution, and we shall always denote the inverse of an element x by x^{-1} or x^*. The set of idempotents of an inverse semigroup S is denoted by $E(S)$.

An *inverse monoid* is an inverse semigroup that has a multiplicative unit, which is usually denoted by e.

A semigroup homomorphism beween inverse semigroups automatically preserves inverses. The category whose objects are the inverse semigroups and whose arrows are the semigroup homomorphisms will be denoted by *InvSGrp*. The category whose objects are the inverse monoids and whose arrows are the monoid homomorphisms will be denoted by *InvMon*.

As examples of inverse semigroups we have:

1. Any pseudogroup on a topological space.

2. Any subsemigroup of operators on a Hilbert space consisting entirely of partial isometries and closed under adjoints.

3. The set $\mathcal{I}(G)$ of "G-sets" (in the sense of [32, p. 10]) of a discrete groupoid G, under pointwise multiplication and inverses, where by a *G-set* is meant a set U for which both the domain and range maps are injective when restricted to U. (We remark that this terminology is unfortunate because it collides with the standard usage of "G-set" for a set equipped with an action by a group G — see, e.g., [18].) This is an inverse monoid with unit G_0.

The Wagner-Preston theorem asserts that every inverse semigroup is isomorphic to a pseudogroup.

Exercise I.2.1 Give an example of an inverse semigroup that is not isomorphic to any full pseudogroup.

Étale groupoids as germ groupoids. Not every étale groupoid is of the form Germs(P) for a pseudogroup P, but every étale groupoid is obtained by taking germs of a more general notion of "pseudogroup". First we need the following notion:

Definition I.2.2 Let S be an inverse semigroup. By a *representation* of S on a topological space X will be meant a homomorphism of semigroups $\rho : S \to \mathcal{I}(X)$. The representation is *full* if ρ restricts to an isomorphism $E(S) \to E(\mathcal{I}(X)) \cong \Omega(X)$. By an *inverse semigroup over* X will be meant a pair (S, ρ) consisting of an inverse semigroup S equipped with a representation $\rho : S \to \mathcal{I}(X)$. If ρ is full (in this case S is necessarily a monoid) then (S, ρ) is said to be a *full inverse semigroup over* X.

Of course, any pseudogroup P over a space X is an inverse semigroup over X if we take the underlying representation to be the inclusion $\iota : P \to \mathcal{I}(X)$, and P is a full pseudogroup if and only if (P, ι) is a full inverse semigroup over X. The construction of the germ groupoid of a full pseudogroup (see section 1) can be generalized in a straightforward way. As we shall see, the étale groupoids are precisely the topological groupoids which can be thus obtained, up to isomorphism.

For each element $s \in S$ of a full inverse semigroup (S, ρ) over X we shall think of the open set $\mathrm{dom}(\rho(s))$ as being the *domain* of s, hence defining a map $S \to \Omega(X)$. By an *element of* S *over* U will be meant an element $s \in S$ whose domain is U.

Theorem I.2.3 *Let (S, ρ) be a full inverse semigroup over X, and for each open set $U \in \Omega(X)$ let $\mathcal{D}(U)$ be the set of elements over U:*

$$\mathcal{D}(U) = \{s \in S \mid \mathrm{dom}(\rho(s)) = U\} .$$

The assignment

$$\mathcal{D} : \Omega(X) \to \wp(S)$$

defines a presheaf of sets on the space X, with each restriction map

$$\mathcal{D}(U) \to \mathcal{D}(V)$$

for $V \subseteq U$ being defined by

$$s \mapsto fs ,$$

where $f \in E(S)$ is the (unique) idempotent such that $\rho(f) = \mathrm{id}_V$ (equivalently, $\mathrm{dom}(\rho(f)) = V$). The local homeomorphism of germs of the presheaf $\mathcal{D}$ is the domain map of an étale groupoid $\mathrm{Germs}(S, \rho)$ *whose unit space is X.*

Proof. The restriction maps are obviously functorial, and thus $\mathcal{D}$ is a presheaf. Its germs are concretely described as pairs $(x, \mathrm{germ}_x s)$ where $x \in X$ and $\mathrm{germ}_x s$ is the set of all $t \in S$ such that $x \in \mathrm{dom}(\rho(t))$ and $ft = fs$ for some $f \in E(S)$ such that $x \in \mathrm{dom}(\rho(f))$. The set Λ of all the germs can be equipped with the usual "sheaf topology", namely that which is obtained from a basis of open sets of the form, for each $s \in S$,

$$U_s = \{(x, \mathrm{germ}_x s) \mid x \in \mathrm{dom}(\rho(s))\} ,$$

making the projection $d : \Lambda \to X$ defined by $(x, \mathrm{germ}_x s) \mapsto x$ a local homeomorphism. The groupoid structure is similar to that of the germ groupoid of a full pseudogroup:

$$\begin{aligned}
d(x, \mathrm{germ}_x s) &= x \\
r(x, \mathrm{germ}_x s) &= \rho(s)(x) \\
u(x) &= (x, \mathrm{germ}_x e) \\
(x, \mathrm{germ}_x s)(\rho(s)(x), \mathrm{germ}_{\rho(s)(x)} t) &= (x, \mathrm{germ}_x(st)) \\
(x, \mathrm{germ}_x s)^{-1} &= (\rho(s)(x), \mathrm{germ}_{\rho(s)(x)}(s^{-1})) .
\end{aligned}$$

Verifying that this is a topological (hence, étale) groupoid is now straightforward. ∎

Now we see that every étale groupoid can be obtained in this way from a suitable inverse semigroup over its unit space. Recall from section 2 that the G-sets of a discrete groupoid G form an inverse semigroup $\mathcal{I}(G)$. We can make this more general:

Definition I.2.4 For an étale groupoid G we define the inverse semigroup $\mathcal{I}(G)$ to consist of the open G-sets of G.

Exercise I.2.5 Let G be an étale groupoid.

1. Show that $\mathcal{I}(G)$ is a sub inverse monoid of the inverse monoid of all the G-sets.

2. Show that each open G-set is the image of a unique *local bisection* of G (by a local bisection is meant a continuous local section $s : U \to G_1$ of the domain map d, defined on an open set $U \subseteq G_0$, and such that $r \circ s$ is injective and open).

3. Show that the assignment $\rho_G : \mathcal{I}(G) \to \mathcal{I}(G_0)$ defined by

$$U \mapsto r \circ s,$$

where $s : d(U) \to G_1$ is the local bisection whose image is U, makes $\mathcal{I}(G)$ a full inverse monoid over G_0.

Theorem I.2.6 *Let G be an étale groupoid, and let $\rho_G : \mathcal{I}(G) \to \mathcal{I}(G_0)$ be its full representation as in the previous exercise. Then $G \cong \mathrm{Germs}(\mathcal{I}(G), \rho_G)$.*

Proof. Part of the proof is a consequence of the equivalence between sheaves and local homeomorphisms: from a local homeomorphism $p : E \to X$ we obtain the sheaf of continuous local sections of p, whose space of germs with the sheaf topology is homeomorphic to E. The only unusual aspect to be checked is that, despite the fact that in moving from G to $\mathcal{I}(G)$ we are not producing the sheaf of all the local sections of d, the space Λ of germs of local bisections is nevertheless homeomorphic to G. This is because r is a local homeomorphism and thus every continuous local section is, once restricted to a suitably small open set, a local bisection. This shows not only that the germs are the same but also that the local bisections are enough to give us a basis for the sheaf topology. It remains to be seen that the remaining structure maps of G are preserved in the passage to $\mathrm{Germs}(\mathcal{I}(G), \rho_G)$. This is done by straightforward verification, and we check the multiplication only. Consider two arrows $x, y \in G$ with $r(x) = d(y)$. Let U and V with $x \in U$ and $y \in V$ be open G-sets. The previous discussion shows that such G-sets necessarily exist and their germs at $d(x)$ and $d(y)$ correspond to x and y, respectively, via the homeomorphism $h : G \xrightarrow{\cong} \Lambda$. Then we have

$$h(x)h(y) = (\mathrm{germ}_{d(x)} U)(\mathrm{germ}_{d(y)} V) = \mathrm{germ}_{d(x)}(UV) = h(xy) ,$$

showing that the groupoid multiplication is preserved. ∎

Distributivity and completeness. The *natural order* of an inverse semigroup S is a partial order, defined as follows:

$$x \le y \iff x = fy \text{ for some } f \in E(S) .$$

The product of any two idempotents f and g is their meet, $fg = f \wedge g$, and S is an inverse monoid if and only if the set of idempotents has a join, in which case we have $e = \bigvee E(S)$. In the case of a pseudogroup an idempotent f is the identity map on an open set U, and thus the natural order becomes

$$x \le y \iff x = y|_U \text{ for some open set } U \subseteq \mathrm{dom}(y) \text{ such that } \mathrm{id}_U \in S .$$

Hence, for pseudogroups the natural order is just the restriction order on partial maps.

Exercise I.2.7 Let S be an inverse semigroup.

1. Show that $s \le t$ if and only if $s = ss^{-1}t$.
2. Show that $s \le t$ if and only if $s = tg$ for some $g \in E(S)$.
3. Let X be a subset of S such that the join $\bigvee X$ exists. Show that the join $\bigvee X^{-1}$ exists and $\bigvee X^{-1} = (\bigvee X)^{-1}$.
4. (This is [14, p. 27, Prop. 17].) Let $X \subseteq S$ be a subset. Show that:
   (a) if $\bigvee X$ exists in S then $\bigvee_{x\in X} xx^{-1}$ exists in $E(S)$ and
   $$(\bigvee X)(\bigvee X)^{-1} = \bigvee_{x\in X} xx^{-1} ;$$
   (b) if $\bigvee X$ exists in S then $\bigvee_{x\in X} x^{-1}x$ exists in $E(S)$ and
   $$(\bigvee X)^{-1}(\bigvee X) = \bigvee_{x\in X} x^{-1}x .$$

   Hint: the solution is similar to the final part of the proof of theorem I.2.8 below.

5. Let $X \subseteq E(S)$ be such that X has a join in S. Show that $\bigvee X \in E(S)$.

The following is a useful property of homomorphisms with respect to the natural order. We shall say that a homomorphism $h : S \to T$ of inverse semigroups *preserves joins* if, for all subsets $X \subseteq S$, if the join $\bigvee X$ exists in S (with respect to the natural order) then the join $\bigvee h(X)$ exists in T and we have $h(\bigvee X) = \bigvee h(X)$.

Theorem I.2.8 *Let S and T be inverse semigroups and $h : S \to T$ a homomorphism of semigroups. The following are equivalent:*

1. *$h|_{E(S)} : E(S) \to E(T)$ preserves joins;*

2. *$h : S \to T$ preserves joins.*

Proof. The implication $2 \Rightarrow 1$ is trivial (cf. exercise I.2.7-5 above). For the other implication assume that $h|_{E(S)}$ preserves joins, and let $X \subseteq S$ be a subset for which the join $\bigvee X$ exists. We have $\bigvee X = (\bigvee X)(\bigvee X)^{-1}(\bigvee X) = (\bigvee_x xx^{-1})(\bigvee X)$. Hence,

$$\begin{aligned} h\left(\bigvee X\right) &= h\left(\left(\bigvee_x (xx^{-1})\right)\bigvee X\right) = h\left(\bigvee_x (xx^{-1})\right) h\left(\bigvee X\right) \\ &= \left(\bigvee_x h(xx^{-1})\right) h\left(\bigvee X\right) = \left(\bigvee_x h(x)h(x)^{-1}\right) h\left(\bigvee X\right) . \end{aligned}$$

Let us write ξ for the last expression on the right. It is clear that for all $x \in X$ we have $h(x) = h(x)h(x)^{-1}h(x) \leq \xi$, and thus ξ is an upper bound of the set $h(X)$. Let z be another upper bound. Then for all $x \in X$ we have $h(x)h(x)^{-1} \leq zh((\bigvee X)^{-1}) = z(h(\bigvee X))^{-1}$. Hence, we have $\bigvee_x h(x)h(x)^{-1} \leq z(h(\bigvee X))^{-1}$, and thus $\xi \leq z(h(\bigvee X))^{-1}h(\bigvee X) \leq z$, showing that ξ is the least upper bound of $h(X)$. ∎

An important property of any symmetric pseudogroup $\mathcal{I}(X)$ is that it is distributive in the sense that the multiplication distributes over all the joins that exist; that is, for all $s \in P$ and all $X \subseteq P$ such that $\bigvee X$ exists in P we have that both $\bigvee_{x\in X} sx$ and $\bigvee_{x\in X} xs$ exist in P and

$$s(\bigvee X) = \bigvee_{x\in X} sx \quad \text{and} \quad (\bigvee X)s = \bigvee_{x\in X} xs .$$

Accordingly, we adopt the following definition:

Definition I.2.9 An inverse semigroup S is said to be *infinitely distributive* if, for all $s \in S$ and all subsets $X \subseteq S$ for which $\bigvee X$ exists in S, the following conditions hold:

1. $\bigvee_{x \in X} sx$ exists in S;
2. $\bigvee_{x \in X} xs$ exists in S;
3. $s(\bigvee X) = \bigvee_{x \in X} sx$;
4. $(\bigvee X)s = \bigvee_{x \in X} xs$.

A very important property of inverse semigroups is the following:

Theorem I.2.10 *Let S be an inverse semigroup. The following conditions are equivalent:*

1. *S is infinitely distributive;*
2. *$E(S)$ is infinitely distributive.*

Proof. This is proved in [14] for infinite distributivity with respect to joins of non-empty sets, but the proof applies equally to joins of any subsets. ∎

Corollary I.2.11 *If (S, ρ) is a full inverse semigroup over a space X then S is infinitely distributive. In particular, any full pseudogroup is infinitely distributive.*

An analogous and related property of inverse semigroups concerns distributivity of binary meets over joins:

Theorem I.2.12 ([35]) *Let S be an infinitely distributive inverse semigroup, let $x \in S$, and let (y_i) be a family of elements of S. Assume that the join $\bigvee_i y_i$ exists, and that the meet $x \wedge \bigvee_i y_i$ exists. Then, for all i the meet $x \wedge y_i$ exists, the join $\bigvee_i (x \wedge y_i)$ exists, and we have*

$$x \wedge \bigvee_i y_i = \bigvee_i (x \wedge y_i) \ .$$

Hence, the distributivity in $E(S)$ determines the distributivity in the whole of S, both with respect to multiplication and to binary meets (which in $E(S)$ are the same, of course).

Finally, we notice that a full pseudogroup P over a topological space X is complete (cf. definition I.1.9) if and only if any set $Z \subseteq P$ of pairwise compatible elements has a join $\bigvee Z$ (see exercise I.2.14 below), where two partial homeomorphisms h and k are *compatible* if they coincide on the intersection of their domains and their inverses h^{-1} and k^{-1}, too, coincide on the intersection of their domains — in other words h and k have a join $h \vee k$ in $\mathcal{I}(X)$. Noticing that h and k are compatible if and only if both hk^{-1} and $h^{-1}k$ are idempotents in P, we are led to the following definitions:

Definition I.2.13 Let S be an inverse semigroup. Two elements $s, t \in S$ are said to be *compatible* if both st^{-1} and $s^{-1}t$ are idempotents. A subset $Z \subseteq S$ is *compatible* if any two elements in Z are compatible. Then S is said to be *complete* if every compatible subset Z has a join $\bigvee Z$ in S (hence, S is necessarily a monoid with $e = \bigvee E(S)$). By a *complete inverse semigroup over a space* X is meant a complete (and necessarily infinitely distributive) inverse semigroup equipped with a full representation $S \to \mathcal{I}(X)$.

We have defined completeness with respect to arbitrary compatible subsets (instead of just non-empty ones as in [14]). Hence, a complete inverse semigroup necessarily has a least element 0.

Exercise I.2.14

1. Let S be an inverse semigroup, and let $s, t \in S$.
   (a) Show that $s^{-1}t \in E(S)$ if and only if $fs = ft$ for $f = ss^{-1}tt^{-1}$ (i.e., s and t coincide on the intersection of their domains).
   (b) Show that $st^{-1} \in E(S)$ if and only if $gs^{-1} = gt^{-1}$ for $g = s^{-1}st^{-1}t$ (i.e., s^{-1} and t^{-1} coincide on the intersection of their domains).
2. Let S be an inverse semigroup. Show that any two elements of S for which an upper bound exists in S are necessarily compatible.
3. Show that a full pseudogroup is complete if and only if it is complete as an inverse semigroup.
4. Any full representation $\rho : S \to \mathcal{I}(X)$ preserves all the joins that exist in S. Why?

Representation theorem. We are now ready to give a characterization of the inverse semigroup actions that arise from étale groupoids.

Theorem I.2.15 *Let G be an étale groupoid with unit space X. Then $(\mathcal{I}(G), \rho_G)$ is a complete inverse semigroup over X. Any complete inverse semigroup over X arises in a similar way from an étale groupoid.*

Proof. It is easy to see that $\mathcal{I}(G)$ is a complete inverse semigroup. For the converse, let then (S,ρ) be an arbitrary complete inverse semigroup over X, and let $G = \mathrm{Germs}(S,\rho)$. We shall show that S and $\mathcal{I}(G)$ are isomorphic, much in the same way in which one shows that a sheaf is isomorphic to the sheaf of local sections of its local homeomorphism. First let us define a map $\widehat{(-)} : S \to \mathcal{I}(G)$ as follows, for each $s \in S$:

$$\widehat{s} = \{(x, \mathrm{germ}_x s) \mid x \in \mathrm{dom}(\rho(s))\} \ .$$

This assignment clearly is a semigroup homomorphism, and it is injective due to distributivity, for if $\mathrm{dom}(\rho(s)) = \mathrm{dom}(\rho(t))$ (equivalently, $ss^{-1} = tt^{-1}$) then the condition $\mathrm{germ}_x s = \mathrm{germ}_x t$ for all $x \in \mathrm{dom}(\rho(s))$ implies that there is a cover (f_x) of ss^{-1} such that for each $x \in \mathrm{dom}(\rho(s))$ we have $f_x s = f_x t$, and thus, due to infinite distributivity,

$$s = ss^{-1}s = (\bigvee_x f_x)s = \bigvee_x (f_x s) = \bigvee_x (f_x t) = (\bigvee_x f_x)t = tt^{-1}t = t \ .$$

(In other words, the presheaf $\mathcal{D}$ of theorem I.2.3 is necessarily separated.) Now let U be an open G-set of G. Then, by definition of the topology of G, U is a union of G-sets of the form $U_s = \{(x, \mathrm{germ}_x s) \mid x \in \mathrm{dom}(\rho(s))\}$. Let U_s and U_t be two such G-sets. For all $x \in \mathrm{dom}(\rho(s)) \cap \mathrm{dom}(\rho(t))$ we must have, since U is a G-set, a unique arrow of G in U with domain x. But $U_s \cup U_t \subseteq U$, and thus both $(x, \mathrm{germ}_x s)$ and $(x, \mathrm{germ}_x t)$ belong to U, therefore implying that $\mathrm{germ}_x s = \mathrm{germ}_x t$; that is, there is an idempotent $f_x \le ss^{-1}tt^{-1}$ such that $x \in \mathrm{dom}(\rho(f_x))$ and $f_x s = f_x t$, and thus

$$(ss^{-1}tt^{-1})s = (\bigvee_x f_x)s = \bigvee_x (f_x s) = \bigvee_x (f_x t) = (ss^{-1}tt^{-1})t \ .$$

Hence, by exercise I.2.14, we have $s^{-1}t \in E(S)$. Similarly, considering any point $x \in \mathrm{cod}(\rho(s)) \cap \mathrm{cod}(\rho(t))$ we conclude, because there must be a unique element in U with codomain x, that $s(s^{-1}st^{-1}t) = t(s^{-1}st^{-1}t)$ [this is immediate from the previous argument because $x \in \mathrm{dom}(\rho(s^{-1})) \cap \mathrm{dom}(\rho(t^{-1}))$], and thus $st^{-1} \in E(S)$ again by exercise I.2.14. We have thus proved that the set X that indexes the cover $U = \bigcup_{s\in X} U_s$ is compatible. Since S is complete, we have a join $\bigvee X$ in S, and it is now clear that $\widehat{\bigvee X} = U$, for

$$\widehat{\bigvee X} = \bigcup_{s\in X} \widehat{s} = \bigcup_{s\in X} U_s = U \ ,$$

where we have used the fact that $\widehat{(-)}$ preserves all the existing joins, which is a consequence (cf. theorem I.2.8) of the fact that its restriction to the idempotents does, since that restriction is an isomorphism. Finally, we obviously have $\rho(s) = \rho_G(\widehat{s})$ for all $s \in S$; that is, $\widehat{(-)}$ commutes with the representations ρ and ρ_G, and thus (S, ρ) and $(\mathcal{I}(G), \rho_G)$ are the same up to isomorphism. ∎

We shall conclude this section by establishing some terminology, which will be used later.

Definition I.2.16 By an *abstract full pseudogroup* will be meant an infinitely distributive inverse semigroup whose idempotents form a complete lattice. And by an *abstract complete pseudogroup* will be meant a complete and infinitely distributive inverse semigroup. The *category of abstract full pseudogroups AFPGrp* has the abstract full pseudogroups as objects and the monoid homomorphisms that preserve the joins of sets of idempotents as arrows. The *category of abstract complete pseudogroups* is the full subcategory of *AFPGrp* whose objects are the abstract complete pseudogroups.

By an abstract complete (resp. full) inverse semigroup *over a topological space* X is meant an abstract complete (resp. full) inverse semigroup S equipped with an order isomorphism $E(S) \cong \Omega(X)$.

Exercise I.2.17

1. Let S and T be abstract full pseudogroups. Show that
   (a) S is meet-complete; that is, if $X \neq \emptyset$ then $\bigwedge X$ exists in S;
   (b) A homomorphism $h : S \to T$ of abstract full pseudogroups necessarily preserves all the joins that exist in S; that is, if $X \subseteq S$ has a join $\bigvee X$ then so does $h(X)$, and we have $\bigvee h(X) = h(\bigvee X)$.
2. Show that a sheaf of abelian groups on a topological space X is "the same" as a commutative abstract complete pseudogroup over X.

Chapter II

Quantales

Quantales have much in common with rings. Just as rings are semigroups in the tensor category of abelian groups, so quantales are semigroups in the tensor category of sup-lattices. Accordingly, we shall begin by studying general properties of sup-lattices, including all the basic constructions such as direct sums, quotients, tensor products, etc. Then we address the theory of quantales themselves, including involutive quantales and quantale modules, along with several basic examples. To conclude this chapter we study supported quantales and inverse quantales, as well as adjunctions between their categories and the categories of inverse semigroups and abstract complete pseudogroups.

1 Sup-lattices

Sup-lattices are just complete lattices, but the former name suggests that only the suprema are relevant algebraic operations. All the material of this section follows closely the presentation given in [10], where the term "sup-lattice" has been introduced.

Basic definitions and properties. By a *sup-lattice* is meant a partially ordered set L each of whose subsets X has a join, or supremum, $\bigvee X$ in L. We shall also write $\bigvee_i x_i$ for the join of a family (x_i) of elements of L. The join of the empty set, $\bigvee \emptyset$, is the minimum of L and we denote it by 0_L, or simply 0. Similarly, the join of L itself, $\bigvee L$, is the maximum of L, and we denote it by 1_L or just 1.

A *homomorphism* of sup-lattices $h : L \to M$ is a function that preserves arbitrary joins; that is, such that for each $X \subseteq L$ we have

$$h\left(\bigvee X\right) = \bigvee h(X) .$$

In particular any homomorphism preserves 0 (but in general not 1):

$$h(0_L) = 0_M \ .$$

The category whose objects are the sup-lattices and whose arrows are the homomorphisms is denoted by SL.

Remark II.1.1 Any sup-lattice L is of course a complete lattice: given any subset $X \subseteq L$, the infimum, or meet, $\bigwedge X$ coincides with $\bigvee X^\ell$, where X^ℓ is the set of lower bounds of X:

$$X^\ell = \{y \in L \mid y \leq x \text{ for all } x \in X\} \ .$$

However, the structure defined by the joins is not "the same" as that which is given by the meets because the homomorphisms of sup-lattices do not preserve meets in general, and we adopt the terminology "sup-lattice" whenever we think of a complete lattice as being an object of SL.

Example II.1.2 The following are sup-lattices:

1. The powerset $\wp(X)$ of a set X, with the inclusion order.

2. The set of projections of a Hilbert space, with the order given by $P \leq Q$ if $PQ = QP = P$.

3. The topology of a topological space, under inclusion of open sets. The joins are unions, and the meet of a family (U_i) of open sets is the interior of their intersection:

$$\bigwedge_i U_i = \operatorname{int}\left(\bigcap_i U_i\right) \ .$$

4. The set of closed sets of a topological space. The join of a family (X_i) of closed sets is the closure of their union (meets are just intersections):

$$\bigvee_i X_i = \overline{\bigcup_i X_i} \ .$$

5. The set of subgroups of a group. Meets are intersections, and the join of a family (H_i) of subgroups is the subgroup generated by the union $\bigcup_i H_i$.

6. The set of right ideals of a ring, with meets and joins as in the previous example.

7. The completed real line $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.

We shall see that there are many ways in which the category *SL* can be regarded as being "similar" to the category of abelian groups. As a first example, we remark that just as the set of homomorphisms $\hom(A, B)$ from an abelian group A to another abelian group B is itself an abelian group with sums computed pointwise, so the set of homomorphisms $\hom(L, M)$ from a sup-lattice L to another sup-lattice M is itself a sup-lattice; given two homomorphisms $f, g : L \to M$ we define $f \leq g$ if $f(x) \leq g(x)$ for all $x \in L$, and the joins are calculated pointwise:

$$\left(\bigvee_i f_i\right)(x) = \bigvee_i f_i(x) .$$

The operation of composition of homomorphisms

$$- \circ - : \hom(M, N) \times \hom(L, M) \to \hom(L, N)$$

is then, extending the analogy with abelian groups, a "bilinear" operation:

$$\begin{aligned} g \circ \left(\bigvee_i f_i\right) &= \bigvee_i (g \circ f_i) \\ \left(\bigvee_i g_i\right) \circ f &= \bigvee_i (g_i \circ f) . \end{aligned}$$

Duality. Let L be a sup-lattice. The *dual* of L is L with the order reversed, and we denote it by L^{op}.

Now let $f : L \to M$ be a homomorphism of sup-lattices. This has a right adjoint (in the sense of category theory — see [17, Ch. IV])

$$f_* : M \to L ,$$

which is defined by

$$f_*(y) = \bigvee \{x \in L \mid f(x) \leq y\} .$$

Since right adjoints preserve meets, it follows that f_* defines a sup-lattice homomorphism

$$f^{\mathrm{op}} : M^{\mathrm{op}} \to L^{\mathrm{op}} ,$$

which is called the *dual of* f.

Hence, we obtain a contravariant endofunctor on SL

$$(-)^{\mathrm{op}} : SL^{\mathrm{op}} \to SL\ .$$

It is immediate that $(L^{\mathrm{op}})^{\mathrm{op}} = L$ for any sup-lattice L, and it is also easy to see that $(f^{\mathrm{op}})^{\mathrm{op}} = f$ for any sup-lattice homomorphism (the second dual $(f^{\mathrm{op}})^{\mathrm{op}}$ is defined as the right adjoint of $f^{\mathrm{op}} : M^{\mathrm{op}} \to L^{\mathrm{op}}$, which coincides with the left adjoint of $f_* : M \to L$, i.e., it is f itself). Hence, the functor $(-)^{\mathrm{op}}$ is an isomorphism of categories and it is its own inverse. We say that it is a *strong self-duality* on SL.

Products and coproducts. Let L and M be sup-lattices. Their cartesian product $L \times M$ is a sup-lattice with the order defined so that $(x, y) \leq (z, w)$ if and only if $x \leq z$ and $y \leq w$; the joins are given by

$$\bigvee_i (x_i, y_i) = \left(\bigvee_i x_i, \bigvee_i y_i \right)\ .$$

This is a product in the categorical sense because the projections

$$L \xleftarrow{\pi_1} L \times M \xrightarrow{\pi_2} M$$

have the universal property of a product: for any pair of sup-lattice homomorphisms

$$L \xleftarrow{h_1} N \xrightarrow{h_2} M$$

there is a unique homomorphism $h : N \to L \times M$ such that $\pi_1 \circ h = h_1$ and $\pi_2 \circ h = h_2$:

$$\begin{array}{ccccc} & & N & & \\ & \swarrow^{h_1} & \downarrow{\scriptstyle h} & \searrow^{h_2} & \\ L & \xleftarrow{\pi_1} & L \times M & \xrightarrow{\pi_2} & M \end{array}$$

Of course, h is the pairing $\langle h_1, h_2 \rangle$ defined by $\langle h_1, h_2 \rangle(n) = (h_1(n), h_2(n))$. This construction extends in the obvious way to any family (L_i) of sup-lattices, with the product $\prod_i L_i$ having coordinatewise order and joins.

Another analogy with the category of abelian groups is that products and coproducts in SL coincide (for sup-lattices this is even true for infinitary products and coproducts, as we shall see). In order to see this, let (L_i) be a family of sup-lattices. Define, for each i, the homomorphism

$$\iota_i : L_i \to \prod_i L_i$$

to be the dual $(\pi_i)^{\mathrm{op}}$ of the projection

$$\pi_i : \prod_i {L_i}^{\mathrm{op}} \to {L_i}^{\mathrm{op}} ,$$

where of course $\prod_i {L_i}^{\mathrm{op}} = (\prod_i L_i)^{\mathrm{op}}$. It is easy to see that the homomorphisms ι_i are the coprojections of a coproduct.

We call the cartesian product $\prod_i L_i$ thus defined the *direct sum* of the family L_i, and denote it by $\bigoplus_i L_i$. The direct sum of two sup-lattices L and M is denoted by $L \oplus M$.

Exercise II.1.3 1. Verify that the coprojections ι_i indeed define a coproduct. Do this directly in terms of the definition by duality of the coprojections and the universal property of the product.

2. Show that the coprojections ι_1 and ι_2 of the direct sum $L \oplus M$ are given explicitly by $\iota_1(x) = (x, 0)$ and $\iota_2(y) = (0, y)$.

The last exercise can be generalized in the obvious way to an arbitrary coproduct $\bigoplus_{i \in I} L_i$, by means of the formula $\iota_i(x) = (y_j)_{j \in I}$ with $y_j = x$ if $j = i$ and $y_j = 0$ if $j \neq i$.[1]

Free sup-lattices. Let X be a set, and consider the mapping $x \mapsto \{x\}$ of X into $\wp(X)$. It is easy to see that for any other mapping $f : X \to L$, where L is a sup-lattice, there is a unique homomorphism of sup-lattices

$$f^\sharp : \wp(X) \to L$$

such that $f^\sharp(\{x\}) = f(x)$ for all $x \in X$. In other words, $\wp(X)$, as a sup-lattice, is freely generated by the set X. Concretely, the homomorphism $f^\sharp$ is explicitly defined by, for all $U \subseteq X$,

$$f^\sharp(U) = \bigvee f(U) .$$

As an application, consider a terminal set 1, and let L be a sup-lattice. The maps $1 \to L^{\mathrm{op}}$ are in bijective correspondence with the homomorphisms from the free sup-lattice $\wp(1)$ to L^{op}, and thus we have an isomorphism (of sup-lattices)

$$L^{\mathrm{op}} \cong \hom(\wp(1), L^{\mathrm{op}}) .$$

[1]This requires, when working in an arbitrary topos, that the indexing set I be decidable. In particular, this formula can always be used in "standard" mathematics, i.e., in the topos of sets.

Hence, by duality we also have

$$L^{\rm op} \cong \hom(L, \wp(1)^{\rm op}) \ .$$

It follows that $(-)^{\rm op}$ is a representable functor, with representing object $\wp(1)^{\rm op}$ (the "dualizing object" in SL). We shall denote this sup-lattice by $\mho$. In order to simplify our notation we shall denote $\hom(L, \mho)$ by L^*, and, accordingly, the hom-dual of a sup-lattice homomorphism $f : L \to M$, which is given by composition with φ,

$$\varphi \mapsto \varphi \circ f \ ,$$

is denoted by

$$f^* : M^* \to L^* \ .$$

Quotients. Although quotients of sup-lattices can be described in terms of the standard tools of (infinitary) universal algebra, for sup-lattices there is a better way. For motivation we remark that if L is a sup-lattice and $\theta \subseteq L \times L$ is a *congruence* on L (i.e., a sub-sup-lattice of $L \oplus L$) then any $x \in L$ is congruent to the join $\bigvee [x]_\theta$ of its equivalence class, and the subposet of $L_\theta \subseteq L$ defined by

$$L_\theta = \{x \in L \mid x = \bigvee [x]_\theta\}$$

is isomorphic to the quotient sup-lattice L/θ. In other words, quotients of L can be represented by certain canonical subsets of L. As we shall see, these are just the subsets of L which are closed under arbitrary meets.

Let L be a sup-lattice. A *closure operator* on L is a monotone map $j : L \to L$ such that for all $x \in L$ we have

1. $x \leq j(x)$,
2. $j(j(x)) \leq j(x)$ (hence, $j(j(x)) = j(x)$).

We shall refer to the fixed-points of a closure operator j on L as the j*-closed* elements (or simply *closed*), and we shall denote the set of j-closed elements of L by

$$L_j = \{x \in L \mid x = j(x)\} \ .$$

(Equivalently, this is the image of j in L.) It is easy to see that the set L_j is closed under arbitrary meets in L (including $1 \in L_j$).

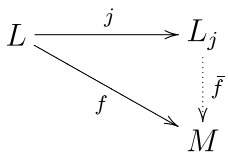
Conversely, if $S \subseteq L$ is a subset closed under arbitrary meets we define a closure operator $j_S : L \to L$ by

$$j_S(x) = \bigwedge \{y \in S \mid x \leq y\} \ ,$$

and we have $j_{L_j} = j$ and $L_{j_S} = S$.

Now let us see that the quotients correspond precisely to closure operators (and hence to subsets closed under meets). Let $f : L \to M$ be a homomorphism of sup-lattices, and let $j : L \to L$ be the composition of f with its right adjoint: $j = f_* \circ f$. From the properties of adjoints it follows that j is a closure operator, and the restriction $j|_{L_j} : L_j \to M$ defines an order isomorphism onto the image $f(L) \subseteq M$. In particular, the following universal property enables us to think of L_j as the *quotient of* L *by* j:

Proposition II.1.4 *Let* L *be a sup-lattice and* j *a closure operator on* L*. Let also* $f : L \to M$ *be a homomorphism such that* $j(x) = j(y) \Rightarrow f(x) = f(y)$ *for all* $x, y \in L$ *(equivalently, such that* $j \leq f_* \circ f$*). Then there is a unique homomorphism* $\bar{f} : L_j \to M$ *such that the following triangle commutes:*



The poset of closure operators on a sup-lattice L is a complete lattice and for any binary relation $R \subseteq L \times L$ there is a least closure operator j_R such that $j(x) = j(y)$ for all $(x, y) \in R$. The set of closed elements L_{j_R} coincides with the set

$$L_R = \{x \in L \mid y \leq x \iff z \leq x \text{ for all } (y, z) \in R\} ,$$

and it is the *quotient of* L *by the congruence generated by* R (quotient of L by R, for short) in the sense that any homomorphism $f : L \to M$ such that $(y, z) \in R \Rightarrow f(y) = f(z)$ factors uniquely through $j_R : L \to L_R$.

Exercise II.1.5

1. Let L be a sup-lattice and j a closure operator on L. Show that the join of a subset $X \subseteq L_j$ in L_j is $j(\bigvee X)$, where $\bigvee X$ is the join in L.

2. Show that any closure operator on a sup-lattice L is of the form $j = f_* \circ f$ for a suitable homomorphism f.

3. Let $f : L \to M$ be a sup-lattice homomorphism. Show that $f(x) = f(y)$ if and only if $f_*(f(x)) = f_*(f(y))$, for all $x, y \in L$.

4. Let L be a sup-lattice and let j, k two closure operators on L. Show that the following conditions are equivalent:

(a) $j(x) = j(y) \Rightarrow k(x) = k(y)$ for all $x, y \in L$;
(b) $j \leq k$;
(c) $j \circ k = k \circ j = k$;
(d) $L_k \subseteq L_j$.

Generators and relations. The previous results provide an easy way of presenting sup-lattices by generators and relations. If X is a set and

$$R \subseteq \wp(X) \times \wp(X)$$

is a binary relation on $\wp(X)$ then $\wp(X)_R$ is the *sup-lattice generated by X subject to the set of relations R*, where each pair $(U, V) \in R$ can be thought of as a formal equation

$$\bigvee U = \bigvee V \ .$$

The universal property of this construction is easy to describe: for any map $f : X \to L$ into a sup-lattice L such that

$$\bigvee f(U) = \bigvee f(V)$$

for all $(U, V) \in R$ there is a unique homomorphism

$$f^\sharp : \wp(X)_R \to L$$

that extends f in the obvious way (i.e., $f^\sharp(j_R(\{x\})) = f(x)$ for all $x \in X$).

In order to simplify notation we shall often denote the injection of generators $X \to \wp(X)_R$ by $[.]$ and the defining relations by the conditions with respect to which $[.]$ is universal; for instance, a pair $(U, V) \in R$ may be represented by the equation

$$\bigvee_{x \in U} [x] = \bigvee_{y \in V} [y] \ .$$

Tensor products. Let L, M and N be sup-lattices. A *bimorphism* from $L \times M$ to N is a function

$$f : L \times M \to N$$

that preserves joins in each variable separately:

$$f\left(\bigvee X, y\right) = \bigvee_{x \in X} f(x, y)$$

$$f\left(x, \bigvee Y\right) = \bigvee_{y \in Y} f(x, y) \ .$$

The *tensor product* of L and M is by definition the codomain of a universal bimorphism

$$(x,y) \mapsto x \otimes y : L \times M \to L \otimes M \;,$$

of which it is easy to give an explicit description by generators and relations: the set of generators is $L \times M$ and the relations are described by the equations

$$\begin{aligned} \left(\bigvee X\right) \otimes y &= \bigvee_{x \in X} x \otimes y \\ x \otimes \left(\bigvee Y\right) &= \bigvee_{y \in Y} x \otimes y \;. \end{aligned}$$

(In the notation of the previous section we would have $x \otimes y = [(x,y)]$.)

SL is a closed monoidal category with respect to this tensor product, with $\wp(1)$ as the tensor unit. Similarly to the category of abelian groups, the right adjoint to $- \otimes N$ is $\hom(N,-)$; that is, for each N we have the familiar isomorphism

$$\hom(M \otimes N, L) \cong \hom(M, \hom(N,L)) \;,$$

natural in the variables M and L, which in fact is an order isomorphism.

Exercise II.1.6 1. Show that $x \otimes y$ can be explicitly described as the following subset of $L \times M$, for each $(x,y) \in L \times M$:

$$\downarrow(0,1) \cup \downarrow(x,y) \cup \downarrow(1,0) \;.$$

2. Show, for all sup-lattices M and N, that we have order isomorphisms

$$\begin{aligned} \hom(M,N) &\cong \hom(N^* \otimes M)^* \\ M \otimes N &\cong \hom(M, N^*)^* \;. \end{aligned}$$

3. Show, for all sets X and all sup-lattices M, that we have order isomorphisms

$$\begin{aligned} \wp(X) \otimes M &\cong \bigoplus_{x \in X} M \\ \wp(X) \otimes \wp(Y) &\cong \wp(X \times Y) \;. \end{aligned}$$

2 Quantales and modules

In this section we shall study basic properties of quantales and their modules, and we shall describe several simple examples related to C*-algebras, groupoids, etc. We shall focus entirely on algebraic aspects. In particular, although locales are examples of quantales (and we mention this here), we shall wait until Chapter III in order to provide a more thorough treatment of locale theory, in particular one that includes the view of locales as being generalized (point free) spaces.

Unital quantales. In the previous sections we have seen how we can obtain, from an étale groupoid G, an inverse semigroup whose elements are the open G-sets of G. But in fact it is not only the open G-sets that can be multiplied; for any open groupoid G the pointwise multiplication of any two open sets is an open set (in other words, the multiplication map is open, cf. exercise I.1.8), and thus the topology $\Omega(G_1)$ is itself a semigroup. If furthermore G is étale then $u(G_0)$ is open in G_1, and thus this semigroup is a monoid. The structure thus obtained is the following:

Definition II.2.1 By a *quantale* Q is meant a sup-lattice together with an associative product $(a, b) \mapsto ab$ satisfying

$$a\left(\bigvee_i b_i\right) = \bigvee_i (ab_i)$$

and

$$\left(\bigvee_i a_i\right) b = \bigvee_i (a_i b)$$

for all $a, b, a_i, b_i \in Q$. The quantale Q is said to be *unital* provided that there exists an element $e \in Q$ for which

$$ea = a = ae$$

for all $a \in Q$. An *idempotent* quantale is one whose multiplication is idempotent, and a *commutative* quantale is one whose multiplication is commutative.

By a *quantale homomorphism* $f : Q \to Q'$ is meant a sup-lattice homomorphism which is also a homomorphism of semigroups. A homomorphism of unital quantales is *unital* if it is a homomorphism of monoids. We shall denote by Qu the category whose objects are the quantales and whose morphisms are the quantale homomorphisms. The category of unital quantales and unital homomorphisms will be denotd by Qu_e.

Example II.2.2 Quantales as generalized topologies. The topology $\Omega(X)$ of any topological space X is a unital quantale with multiplication given by intersection of open sets and $e = X$. More generally, let L be a locale, by which is meant a sup-lattice satisfying the distributivity law

$$a \wedge \left(\bigvee_i b_i \right) = \bigvee_i (ab_i) \ .$$

(Locales will be studied in section 1.) Then L is a quantale with unit $e = 1$ and multiplication given by $ab = a \wedge b$. Such quantales are idempotent and commutative. Conversely, any idempotent unital quantale with $e = 1$ is a locale (and therefore commutative), as will be seen in section 1.

Example II.2.3 Groupoid quantales. The topology of any open groupoid G is a quantale under pointwise composition of arrows, and it is unital if G is étale. In fact the latter condition characterizes topological étale groupoids; that is, G is étale if and only if its quantale is unital, as will be seen later.

Example II.2.4 Quantales of binary relations. Let X be a set. The set of binary relations

$$\mathrm{Rel}(X) = \wp(X \times X)$$

is the topology of the discrete pair groupoid $\mathrm{Pair}(X)$ and thus it is a unital quantale. The multiplication is the composition of binary relations, which extends the usual composition of functions. We shall in general take the multiplication in the forward direction: $RS = S \circ R$.

Example II.2.5 Endomorphism quantales. Let L be a sup-lattice. The set of sup-lattice endomorphisms of L

$$\mathrm{End}(L) = \mathrm{hom}(L, L)$$

is a unital quantale with multiplication $fg = g \circ f$ and unit $e = \mathrm{id}_L$ (cf. section 1). If $L = \wp(X)$ then $\mathrm{End}(L) \cong \mathrm{Rel}(X)$: each endomorphism

$$f : \wp(X) \to \wp(X)$$

can be identified with the relation

$$\{(x, y) \in X \times X \mid y \in f(\{x\})\} \ .$$

Example II.2.6 Free quantales on semigroups. Let M be a monoid with unit 1. Then the powerset $\wp(M)$ is a unital quantale with $e = \{1\}$ and multiplication computed pointwise:

$$UV = \{xy \mid x \in U, \ y \in V\} \ .$$

This is actually the free unital quantale on M, in the sense that if Q is a unital quantale and $f : M \to Q$ is a homomorphism of monoids then there is a unique homomorphism $f^\sharp : \wp(M) \to Q$ of unital quantales such that $f^\sharp(\{x\}) = f(x)$ for all $x \in M$ ($f^\sharp$, of course, coincides with the unique sup-lattice extension of f — cf. section 1). This establishes an adjunction between the category of monoids and the category of unital quantales Qu_e, where the left adjoint is, on objects, the assignment $M \to \wp(M)$, and the right adjoint in the forgetful functor.

It is often useful to think of a quantale as being a kind of ring, with the difference that the underlying additive abelian group has been replaced by a sup-lattice, and the bilinearity of the multiplication has become distributivity of the multiplication with respect to joins (instead of sums) in each variable. We can make this idea precise in terms of the tensor category structure of *SL*: a quantale Q is just a semigroup in *SL* (in the sense of tensor categories), just as a ring is a semigroup in the category of abelian groups. Concretely, then, a quantale is a sup-lattice Q equipped with a sup-lattice homomorphism (the multiplication)

$$\gamma : Q \otimes Q \to Q$$

satisfying associativity in the sense that the following diagram is commutative:

$$\begin{array}{ccccc}
Q \otimes (Q \otimes Q) & \xrightarrow{\cong} & (Q \otimes Q) \otimes Q & \xrightarrow{\gamma \otimes \mathrm{id}} & Q \otimes Q \\
{\scriptstyle \mathrm{id}\otimes\gamma}\downarrow & & & & \downarrow{\scriptstyle \gamma} \\
Q \otimes Q & & \xrightarrow[\gamma]{\hspace{6em}} & & Q
\end{array}$$

Similarly, a unital quantale is the same as a monoid in *SL*; that is, in addition to the multiplication γ there is a unit homomorphism

$$\upsilon : \wp(1) \to Q$$

such that the following diagram is commutative:

$$\begin{array}{ccccc}
\wp(1) \otimes Q & \xrightarrow{\upsilon \otimes \mathrm{id}} & Q \otimes Q & \xleftarrow{\mathrm{id} \otimes \upsilon} & Q \otimes \wp(1) \\
& {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \gamma} & \swarrow{\scriptstyle \cong} & \\
& & Q & &
\end{array}$$

Example II.2.7 Tensor quantales. Unital quantales can be defined from sup-lattices by the usual "tensor algebra" construction: if L is a sup-lattice, its "tensor quantale" is

$$\mathcal{T}(L) = \bigoplus_{n \in \mathbb{N}_0} L^{\otimes n} = \wp(1) \oplus L \oplus (L \otimes L) \oplus (L \otimes L \otimes L) \oplus \ldots$$

with multiplication defined by concatenation of pure tensors:

$$(a_1 \otimes \ldots \otimes a_n)(b_1 \otimes \ldots \otimes b_m) = a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_m \ .$$

This has a universal property: if Q is a unital quantale and $h : L \to Q$ is a sup-lattice homomorphism there is a unique homomorphism of unital quantales $h^\sharp : \mathcal{T}(L) \to Q$ that extends h (we identify L with a sub-sup-lattice of $\mathcal{T}(L)$). This shows that the forgetful functor from unital quantales to sup-lattices has a left adjoint.

Exercise II.2.8 1. Let Q be a quantale, and $a \in Q$. Since the multiplication preserves joins in each variable we have two sup-lattice endomorphisms $Q \to Q$, namely multiplication by a on the left and on the right. The right adjoints to these are called *residuations*, they are denoted by $a\backslash -$ and $-/a$,[2] and they are defined by the formulas

$$\begin{aligned} ax \le y &\iff x \le a\backslash y \\ xa \le y &\iff x \le y/a \ . \end{aligned}$$

(a) Show that the following properties hold:

i. $a(a\backslash y) \le y$

ii. $x \le a\backslash(ax)$

iii. $a(a\backslash(ax)) = ax$

iv. $a\backslash(a(a\backslash y)) = a\backslash y$

(b) Write and prove analogous formulas for $/$.

2. Let C be a small category with set of objects $C_0 \subseteq C_1$. Show that the powerset $\wp(C_1)$ is a quantale under pointwise multiplication, and that it is unital with unit C_0.

3. The previous exercise generalizes the example of II.2.6. Can you find an analogous universal property for the unital quantale $\wp(C_1)$ of a small category C?

[2]Another common notation for a/x and $x\backslash a$ is $x \to_l a$ and $x \to_r a$, respectively — see [11]. This and other arrow based notations are motivated by intuitionistic logic and Heyting algebras, where the implication operation is right adjoint to meet (conjunction): $x \wedge a \le y \iff x \le a \to y$. More generally, for any commutative quantale the two residuations coincide and this provides the grounds for interpreting the implication connective of the linear logic of [4] — see also [36] and [37].

Involutive quantales. An *involutive semigroup* is a semigroup S equipped with an *involution*

$$(-)^* : S \to S ,$$

by which is meant a map satisfying, for all $x, y \in S$, the conditions

$$\begin{aligned} x^{**} &= x \\ (xy)^* &= y^* x^* . \end{aligned}$$

If S is a monoid with unit 1 then we necessarily have $1^* = 1$; that is, 1 is a *self-adjoint*, or *hermitian*, element.

Definition II.2.9 By an *involutive quantale* is meant an involutive semigroup in *SL*; in other words, this is a quantale Q equipped with a sup-lattice endomorphism

$$(-)^* : Q \to Q$$

which is also a semigroup involution. By a *homomorphism of involutive quantales* $f : Q \to Q'$ is meant a quantale homomorphism that commutes with the involution:

$$f(a^*) = f(a)^* .$$

The category of involutive quantales and involutive homomorphisms will be denoted by Qu^*, and the category of unital involutive quantales and unital involutive homomorphisms will be denoted by Qu^*_e.

Example II.2.10 Commutative quantales. Any commutative quantale Q can be made an involutive quantale by equipping it with the *trivial* involution $a^* = a$. Conversely, an involutive quantale whose involution is trivial is necessarily commutative.

Example II.2.11 Groupoid quantales. Let G be an open topological groupoid. The topology $\Omega(G_1)$ is an involutive quantale with the involution given by pointwise inversion:

$$U^* = U^{-1} .$$

In particular, the quantale $\mathrm{Rel}(X)$ of binary relations on a set X is involutive with involution

$$R^* = \{(y, x) \mid (x, y) \in R\} .$$

Example II.2.12 Endomorphism quantales. If L is a self-dual sup-lattice, by which one means a sup-lattice equipped with an antitone order automorphism $(-)' : L \to L$, then $\mathrm{End}(L)$ is involutive with

$$f^*(y) = \left(\bigvee\{x \in L \mid f(x) \le y'\}\right)' .$$

This agrees with the involution of $\mathrm{Rel}(X)$ when $L = \wp(X)$ and we take the duality to be complementation: $Y' = X \setminus Y$.

Example II.2.13 Symmetric sup-lattice 2-forms. Let L be a sup-lattice. A *symmetric 2-form* on L [34] is a sup-lattice bimorphism

$$\varphi : L \times L \to \mho$$

satisfying $\varphi(x, y) = \varphi(y, x)$ for all $x, y \in L$.[3] A *continuous endomap* of φ is a pair of sup-lattice homomorphisms $f, g : L \to L$ such that

$$\varphi(x, g(y)) = \varphi(f(x), y) .$$

The set $\mathrm{End}(\varphi)$ of continuous endomaps of φ is a unital involutive quantale under the pointwise order, with unit $(\mathrm{id}_L, \mathrm{id}_L)$, multiplication

$$(f, g)(f', g') = (f' \circ f, g \circ g')$$

and involution $(f, g)^* = (g, f)$. If the 2-form φ is *faithful*, by which is meant that if $\varphi(x, y) = \varphi(x, z)$ for all $x \in L$ then $y = z$, then any continuous endomap is of the form (f, f^*) with f^* as in the previous example for the duality $x \mapsto x' = \bigvee\{y \in L \mid \varphi(x, y) = 0_\mho\}$, and thus $\mathrm{End}(\varphi) \cong \mathrm{End}(L)$ as unital involutive quantales.

Example II.2.14 Free involutive quantales. Example II.2.6 can be extended to involutive quantales. If S is an involutive semigroup then $\wp(S)$ is an involutive quantale with the pointwise involution. Furthermore it is a free involutive quantale on S; in other words, the assignment $S \mapsto \wp(S)$ is the object part of a functor which is left adjoint to the forgetful functor from involutive quantales to involutive semigroups, and similar remarks apply to involutive monoids and unital involutive quantales. A different adjunction holds for groups and unital involutive quantales. First, a functor from unital involutive quantales to groups can be defined by mapping each unital involutive quantale Q to its group of *units*

$$Q^\times = \{a \in Q \mid aa^* = a^*a = e\} ,$$

[3]Readers not interested in constructivity may assume that $\mho$ is the chain $2 = \{0, 1\}$ with $0 < 1$, as in [34].

since any homomorphism of unital involutive quantales $h : Q \to K$ restricts to a group homomorphism $h|_{Q^\times} : Q^\times \to K^\times$. Then it is easy to see that this functor has a left adjoint that to each group G assigns $\wp(G)$; in other words, if G is a group, Q is a unital involutive quantale, and $h : G \to Q^\times$ is a group homomorphism then there is a unique homomorphism of unital involutive quantales $h^\sharp : \wp(G) \to Q$ such that $h^\sharp(\{x\}) = h(x)$ for all $x \in G$.

Exercise II.2.15 1. Provide a suitable definition of involutive semigroup in an arbitrary tensor category. Prove, in this general setting and in the case of a monoid, that the unit is self-adjoint.

2. Let G be a discrete groupoid, and let Ω^G be the set of maps $G_1 \to \Omega$, where $\Omega = \wp(1)$. Show that Ω^G is a unital involutive quantale under the following convolution multiplication,

$$(f * g)(x) = \bigvee_{x=yz} f(y) \wedge g(z) ,$$

with unit being the map $e : G_1 \to \Omega$ such that $e(x) = 1$ if and only if $x \in u(G_0)$, and with involution

$$f^*(x) = f(x^{-1}) .$$

3. Show that we have an isomorphism of unital involutive quantales

$$\wp(G_1) \cong \Omega^G .$$

4. Show, for an open (resp. étale) topological groupoid G, that the involutive quantale (resp. unital involutive quantale) $\Omega(G_1)$ is isomorphic to $\$^G$, by which is meant the set of continuous maps $G_1 \to \$$ equipped with a quantale structure given by formulas as above, where $\$ = \{0, 1\}$ is Sierpiński space, whose topology is $\Omega(\$) = \{\emptyset, \{1\}, \{0, 1\}\}$.

5. Let Q and R be unital involutive quantales.

   (a) Show that the sup-lattice tensor product $Q \otimes R$ has a natural structure of unital involutive quantale with

$$\begin{aligned} e &= e_Q \otimes e_R \\ (x \otimes y)(x' \otimes y') &= (xx') \otimes (yy') \\ (x \otimes y)^* &= x^* \otimes y^* . \end{aligned}$$

(b) Show that $Q \otimes R$ is the *commuting coproduct* of Q and R, by which is meant that the *coprojections*

$$\begin{aligned} x &\mapsto x \otimes e : Q \to Q \otimes R \\ y &\mapsto e \otimes y : R \to Q \otimes R \end{aligned}$$

are universal among the pairs of homomorphisms of unital involutive quantales

$$Q \xrightarrow{f} K \xleftarrow{g} R$$

such that $f(x)g(y) = g(y)f(x)$ for all $x \in Q$ and $y \in R$; that is, for any such pair there is a unique homomorphism

$$h : Q \otimes R \to K$$

such that the following diagram commutes:

$$\begin{array}{ccccc} & & K & & \\ & \overset{f}{\nearrow} & \uparrow{\scriptstyle h} & \overset{g}{\nwarrow} & \\ Q & \xrightarrow[-\otimes e]{} & Q \otimes R & \xleftarrow[e\otimes -]{} & R \end{array}$$

(c) Conclude that the coproduct in the category of unital involutive quantales Qu_e coincides with the sup-lattice tensor product.

6. Let G and H be discrete groupoids. Show that the unital involutive quantale $\wp(G \times H)$ is isomorphic to $\wp(G) \otimes \wp(H)$.

7. Let G be a transitive discrete groupoid with isotropy I. Show that we have an isomorphism of unital involutive quantales:

$$\wp(G) \cong \wp(I) \otimes \mathrm{Rel}(G_0) \ .$$

8. Let G be a topological groupoid whose decomposition into transitive components is

$$G = \coprod_i G_i \ .$$

Show that we have an isomorphism of unital involutive quantales:

$$\Omega(G) \cong \bigoplus_i \Omega(G_i) \ .$$

Nuclei and quotients. Quotients of quantales and of involutive quantales can be handled in terms of a suitable kind of closure operator, called a *nucleus*. These generalize the nuclei for locales (cf. Chapter III) and have been introduced under the name *quantic nucleus* in [27] (cf. [36]).

Definition II.2.16 Let Q be a quantale. A *nucleus* (or *quantic nucleus*) on Q is a closure operator

$$j : Q \to Q$$

such that for all $x, y \in Q$ we have

$$j(x)j(y) \leq j(xy) \ .$$

Moreover, if Q is involutive then the nucleus j is said to be *involutive* if it preserves the involution:

$$j(x^*) = j(x)^* \ .$$

(Equivalently, $j(x)^* \leq j(x^*)$, see II.2.23.)

Nuclei define quotients of quantales just as closure operators define quotients of sup-lattices:

Theorem II.2.17 *Let Q be a quantale and let j be a nucleus on Q.*

1. *The sup-lattice Q_j is a quantale with multiplication defined by*

$$(x, y) \mapsto j(xy)$$

and $j : Q \to Q_j$ is a (surjective) homomorphism of quantales.

2. *If Q is unital then so is Q_j, with unit $j(e)$.*

3. *If Q is involutive and j is an involutive nucleus then Q_j is involutive and the involution is the same as in Q (hence, we have $Q_j^* = Q_j$).*

4. *Every quotient of quantales arises like this: if $h : Q \to R$ is a surjective homomorphism of quantales and h_* is its right adjoint then the closure operator $k = h_* \circ h$ is a nucleus and the restriction*

$$f|_{Q_k} : Q_j \to R$$

is an isomorphism of quantales. Similar remarks apply to unital quantales and involutive quantales.

Example II.2.18 Let R be a ring. The multiplicative semigroup structure of R makes $\wp(R)$ a quantale, and the closure operator j that assigns to each subset $X \subseteq R$ the additive subgroup generated by X is a nucleus. The quotient quantale $\wp(R)_j$ is the set $\mathrm{Sub}(R)$ of all the additive subgroups of R. The product of two subgroups A and B is the subgroup generated by the pointwise product of and A and B, and it can be described explicitly by the formula

$$AB = \{a_1b_1 + \ldots + a_nb_n \mid a_i \in A,\ b_i \in B\ (n = 1, 2, \ldots)\} . \tag{II.2.1}$$

Moreover, if R is a unital ring with unit 1_R then $\mathrm{Sub}(R)$ is a unital quantale with the unit e being the additive subgroup generated by 1_R. And if R is an involutive ring (i.e., R is equipped with an additive operation $x \mapsto x^*$ which is an involution for the multiplicative semigroup of R) then $\mathrm{Sub}(R)$ is an involutive quantale with the involution computed pointwise.

Example II.2.19 Let S be a topological semigroup with topology $\Omega(S)$. The topological closure on $\wp(S)$ is a nucleus, and thus the sup-lattice $\mathsf{c}(S)$ of closed sets of S is a quantale quotient of $\wp(S)$: the product of two closed sets X and Y is the topological closure of the pointwise product:

$$XY = \overline{\{xy \mid x \in X,\ y \in Y\}} .$$

If S is involutive then so is $\mathsf{c}(S)$, under pointwise involution.

Example II.2.20 The two previous examples can be combined into various kinds of topological algebras. For instance, if A is a topological $\mathbb{C}$-algebra we have several quotients: $\mathrm{Sub}(A)$ as in II.2.18; the subset $\mathrm{Sub}_{\mathbb{C}}(A) \subseteq \mathrm{Sub}\,A$ consisting of all the linear subspaces of A; the subset $\mathsf{c}(A) \subseteq \wp(A)$ of topologically closed subsets; the subset $\mathrm{Sub}^{\mathsf{c}}(A) = \mathrm{Sub}(A) \cap \mathsf{c}(A)$ of topologically closed additive subgroups; and the subset $\mathrm{Sub}^{\mathsf{c}}_{\mathbb{C}}(A) = \mathrm{Sub}_{\mathbb{C}}(A) \cap \mathsf{c}(A)$ of topologically closed linear subspaces. These form the following commutative diagram of quotient maps:

$$\begin{array}{ccccc}
\wp(A) & \longrightarrow & \mathrm{Sub}(A) & \longrightarrow & \mathrm{Sub}_{\mathbb{C}}(A) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{c}(A) & \longrightarrow & \mathrm{Sub}^{\mathsf{c}}(A) & \longrightarrow & \mathrm{Sub}^{\mathsf{c}}_{\mathbb{C}}(A)
\end{array}$$

The multiplication in $\mathrm{Sub}_{\mathbb{C}}(A)$ is given by the same formula (II.2.1) as in $\mathrm{Sub}(A)$, and in $\mathrm{Sub}^{\mathsf{c}}(A)$ and $\mathrm{Sub}^{\mathsf{c}}_{\mathbb{C}}(A)$ we have

$$UV = \overline{\{a_1b_1 + \ldots + a_nb_n \mid a_i \in U,\ b_i \in V\ (n = 1, 2, \ldots)\}} .$$

If A is an involutive algebra these sets define quotients of involutive quantales.

Example II.2.21 Self-adjoint operator algebras. If A is a C*-algebra the involutive quantale $\mathrm{Sub}^{\mathsf{c}}_{\mathbb{C}}(A)$ of II.2.20 plays the role of a *spectrum* of A [21, 24, 25, 12, 13], and it is usually denoted by $\mathrm{Max}\,A$. Similarly, the set of weakly closed linear subspaces of a Von Neumann algebra B is a unital involutive quantale $\mathrm{Max}_w(B)$ [30, 24, 25].

The assignment $A \mapsto \mathrm{Max}\,A$ of the above example extends to a functor from the category of C*-algebras and $*$-isomorphisms to the category of involutive quantales and involutive homomorphisms. It has been shown that this functor is a complete invariant for unital C*-algebras:

Theorem II.2.22 ([13]) *Let A and B be unital C*-algebras. Then A and B are $*$-isomorphic if and only if* $\mathrm{Max}\,A$ *and* $\mathrm{Max}\,B$ *are isomorphic as unital involutive quuantales.*

Exercise II.2.23 1. Show that the following conditions on a nucleus j of an involutive quantale Q are equivalent:

(a) j is involutive;

(b) $j(x)^* \leq j(x^*)$ for all $x \in Q$;

(c) Q_j is a self-adjoint subset.

2. Recall the notion of residuation of Exercise II.2.8-1. Show that for an involutive quantale Q the following properties hold for all $a, x \in Q$:

$$\begin{aligned} (a\backslash x)^* &= x^*/a^* \\ (x/a)^* &= a^*\backslash x^* \,. \end{aligned}$$

3. Show, for an arbitrary quantale Q, that the subsets of the form Q_j for a nucleus j on Q are precisely those (necessarily meet-closed) subsets S of Q such that $x/a \in S$ and $a\backslash x \in S$ for all $x \in S$ and $a \in Q$.

Generators and relations. Let Q be a quantale and $R \subseteq Q \times Q$ a subset. Similarly to what we did for sup-lattices in section 1, we want to find the universal solution, now for quantales, for identifying y and z for all $(y, z) \in R$. Noticing that the nuclei on Q form a complete lattice it is easy to show that this quotient coincides with Q_{ν_R} where ν_R is the least nucleus ν that equalizes R in the sense that $\nu(y) = \nu(z)$ for all $(y, z) \in R$.

Alternatively, we may begin by showing that if R is closed under multiplication then the quantale quotient coincides with the sup-lattice quotient:

Lemma II.2.24 *Let Q be a quantale and R a subset of $Q \times Q$ such that $(ayb, azb) \in R$ for all $(y,z) \in R$ and $a, b \in Q$ (for instance, R could be a sub-semigroup of $Q \times Q$). Then the closure operator j_R defined after Proposition II.1.4 is a nucleus (that is, $j_R = \nu_R$).*

Hence, for general R the quantale quotient can be computed in two steps. First we generate a new subset $R' \subseteq Q \times Q$ by closing R under multiplication:

$$R' = \{(ayb, azb) \in Q \times Q \mid (y,z) \in R,\ a, b \in Q\}\ .$$

(R' is the least subset of $Q \times Q$ that contains R and for which we have $(y,z) \in R' \Rightarrow (ayb, azb) \in R'$ for all $a, b \in Q$.) The second step is to define the sup-lattice quotient with respect to R'. In other words, $\nu_R = j_{R'}$.

In fact we may amalgamate the two steps into a single one, thus describing the quotient in one gulp, as follows:

Theorem II.2.25 *Let Q be a unital quantale, and $R \subseteq Q \times Q$ a set. Then Q_{ν_R} coincides with the set Q' of those elements $x \in Q$ such that for all $(y,z) \in R$ and all $a, b \in Q$ we have $ayb \le x \iff azb \le x$.*

Proof. First, it is easy to see that Q' is closed under meets and residuations (cf. Exercise II.2.23), and thus it defines a nucleus k on Q. Furthermore, for all $(y,z) \in R$ and $x \in Q$ we have $y \le k(x)$ if and only if $z \le k(x)$, and thus from $y \le k(y)$ and $z \le k(z)$ we conclude $z \le k(y)$ and $y \le k(z)$, i.e., $k(y) = k(z)$. Hence, $\nu_R \le k$, i.e., $Q' \subseteq Q_{\nu_R}$. Conversely, let us prove that $Q_{\nu_R} \subseteq Q'$. Let $x, a, b \in Q$ and $(y,z) \in R$. Then,

$$\begin{aligned} ayb \le \nu_R(x) &\iff \nu_R(ayb) \le \nu_R(x) \Rightarrow \nu_R(a)\nu_R(y)\nu_R(b) \le \nu_R(x) \\ &\iff \nu_R(a)\nu_R(z)\nu_R(b) \le \nu_R(x) \Rightarrow azb \le \nu_R(x)\ . \end{aligned}$$

In a similar way we conclude that $azb \le \nu_R(x) \Rightarrow ayb \le \nu_R(x)$, and thus $\nu_R(x) \in Q'$. ∎

Now we can present quantales easily by generators and relations: given a set of generators X and a set of relations $R \subseteq \wp(X^+) \times \wp(X^+)$, the quantale presented by X and R is $\wp(X^+)_{\nu_R}$. Similarly to what we did in the case of sup-lattices, we may replace the explicit description of the relations in R by the properties with respect to which the injection of generators

$$[.] : X \to \wp(X^+)_{\nu_R}$$

is universal.

Example II.2.26 Tensor quantales. Let L be a sup-lattice. The tensor quantale

$$\mathcal{T}(L) = \bigoplus_{n\in\mathbb{N}_0} L^{\otimes n}$$

can be presented by generators and relations by taking L to be the set of generators, with the following relations, for all $X \subseteq L$:

$$\left[\bigvee X\right] = \bigvee_{x\in X} [x] \; .$$

Exercise II.2.27

1. Obtain similar descriptions for unital quantales and involutive quantales presented by generators and relations.
2. State and prove a theorem similar to II.2.25 but with each $(y,z) \in R$ being interpreted as an inequality $y \leq z$.
3. Show that the unital involutive quantale $\mathrm{Max}\, A$ of a unital C*-algebra A (cf. Example II.2.21) has the following presentation by generators and relations:
   - The set of generators is A.
   - The relations are as follows:

$$\begin{array}{rcl} e & = & [1_A] \\ 0 & = & [0_A] \\ [a][b] & = & [ab] \\ [a] & \leq & \bigvee_i [a_i] \quad (\text{if } a \in \overline{\sum_i a_i}) \; . \end{array}$$

Right-sided elements. Let Q be a quantale. An element $a \in Q$ is *right-sided* if $a1 \leq a$, and *strictly right-sided* if $a1 = a$. Similarly, a is *left-sided* (resp. *strictly left-sided*) if $1a \leq a$ (resp. $1a = a$). An element a is *two-sided* if it is both right- and left-sided. The sets of right-sided, left-sided, and two-sided elements of Q are denoted respectively by $\mathrm{R}(Q)$, $\mathrm{L}(Q)$, $\mathrm{I}(Q)$.

Example II.2.28

1. Let R be a ring. Then $\mathrm{R}(\mathrm{Sub}(R))$ is the set of right ideals of R.
2. Let A be a C*-algebra. Then $\mathrm{R}(\mathrm{Max}\, A)$ is the set of norm-closed right ideals of A.
3. Let L be a sup-lattice. Then $\mathrm{R}(\mathrm{End}(L)) \cong L^{\mathrm{op}}$, $\mathrm{L}(\mathrm{End}(L)) \cong L$, and $\mathrm{I}(\mathrm{End}(L))$ has only two elements. For the first isomorphism, $L \to$

R(End(L)), we map each x to its annihilator $a_x : L \to L$, which is right-sided in End(L):

$$a_x(y) = \begin{cases} 0 & \text{if } y \leq x \\ 1 & \text{otherwise.} \end{cases}$$

The second isomorphism, $L \to \mathrm{L}(\mathrm{End}(L))$, is given by $x \mapsto c_x$ where

$$c_x(y) = \begin{cases} 0 & \text{if } y = 0 \\ x & \text{otherwise.} \end{cases}$$

4. In particular, let X be a set. Then $\mathrm{R}(\mathrm{Rel}(X)) \cong \mathrm{L}(\mathrm{Rel}(X)) \cong \wp(X)$.
5. More generally, let G be a discrete groupoid. Then

$$\mathrm{R}(\wp(G)) \cong \mathrm{L}(\wp(G)) \cong \wp(G_0) \ .$$

Modules and representations. Just as a quantale is a semigroup in the monoidal category *SL* so a *quantale module* is an action in *SL*. More precisely, let Q be a quantale with multiplication

$$\gamma : Q \otimes Q \to Q \ .$$

Then by a left Q-module is meant a sup-lattice M equipped with a left *action*

$$\alpha : Q \otimes M \to M$$

satisfying *associativity with respect to the quantale multiplication*; that is, such that the following diagram is commutative:

$$\begin{array}{ccccc} Q \otimes (Q \otimes M) & \xrightarrow{\cong} & (Q \otimes Q) \otimes M & \xrightarrow{\gamma \otimes \mathrm{id}} & Q \otimes M \\ \downarrow{\scriptstyle \mathrm{id} \otimes \alpha} & & & & \downarrow{\scriptstyle \alpha} \\ Q \otimes M & & \xrightarrow[\alpha]{} & & M \end{array}$$

We shall usually write ax for $\alpha(a \otimes x)$, and with this notation associativity becomes the equation

$$a(bx) = (ab)x \ .$$

We shall also in general assume that Q-modules are *unitary* whenever Q is unital, meaning that the action satisfies, for all $x \in M$,

$$ex = x \ .$$

A *right module* is defined in a similar way. Of course, a Q-module structure on M can be equivalently defined as a *representation* of Q on M, by which is meant a homomorphism

$$Q \to \mathrm{End}(M) .$$

Nuclei and quotients can be handled in an analogous way to that of quantales. A *nucleus* on M is a closure operator

$$j : M \to M$$

satisfying $aj(x) \leq j(ax)$ for all $a \in Q$ and $x \in M$, and the subsets M_j of fixed-points of a nucleus on M are precisely the quotients of M.

If Q is a unital quantale and X is a set then the free (unitary) left (or right) Q-module generated by X is the *function module* Q^X of all maps $X \to Q$ with the obvious action given by pointwise multiplication. The injection of generators

$$X \to Q^X$$

sends each $x \in X$ to the corresponding "unit vector" $\mathbf{e}_x$:

$$\mathbf{e}_x(y) = \begin{cases} e & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

We conclude this section with a note on tensor products of modules. Let Q be a quantale, and let M_Q and ${}_QN$ be a right and a left Q-module, respectively. A *module bimorphism*

$$f : M \times N \to L$$

into a sup-lattice L is a sup-lattice bimorphism such that

$$f(ma, n) = f(m, an)$$

for all $m \in M$, $n \in N$, and $a \in Q$. There is a universal module bimorphism

$$(m, n) \mapsto m \otimes n : M \times N \to M \otimes_Q N ,$$

and $M \otimes_Q N$ is called the *tensor product (over Q)* of the modules M and N. The usual formulas of commutative algebra and noncommutative algebra apply to quantale modules.

Exercise II.2.29 1. Let Q be a quantale. Show that the sup-lattice of right-sided elements $\mathrm{R}(Q)$ is a left Q-module under multiplication by elements of Q on the left, and that $\mathrm{L}(Q)$ is a right Q-module under multiplication by elements of Q on the right. Verify that if Q is unital then so are the modules $\mathrm{R}(Q)$ and $\mathrm{L}(Q)$.

2. By analogy with what was done above for quantales, fill in the details, for quantale modules, of the theory of nuclei, quotients, residuations, generators and relations.

3. Show that, concretely as a subset of $\wp(M \times N)$, the tensor product $M \otimes_Q N$ consists of those sets $X \in M \otimes N$ satisfying the additional property that, for all $m \in M$, $n \in N$, and $a \in Q$,

$$(ma, n) \in X \iff (m, an) \in X \ .$$

4. List "typical" formulas involving tensor products of ring modules that apply equally to quantale modules.

3 Supported quantales

In this section we study a special kind of unital involutive quantale for which there is an additional operation, called a support, satisfying a few simple properties whose consequences are important. Moreover, the supports that arise in our examples are of a kind known as stable. Such supports are uniquely defined and they are automatically preserved by unital involutive homomorphisms. Hence, possessing a stable support is a property rather than extra structure. In addition, we shall see that these quantales are closely related to inverse semigroups, and as we shall see in Chapter III these facts will ultimately help us characterize in a precise (and yet simple) way the quantales that arise as topologies of étale groupoids. Although in this chapter we do not go that far, we shall nevertheless establish adjunctions between inverse semigroups and stably supported quantales which will later, in Chapter III, provide the grounds for an equivalence of categories between abstract complete pseudogroups and inverse quantal frames.

Supports. Let G be a discrete groupoid, and let $\varsigma : \wp(G) \to \wp(G)$ be the direct image of the domain map followed by the direct image of the inclusion $G_0 \to G$: that is, for all $U \subseteq G$,

$$\varsigma(U) = u(d(U)) \ .$$

This operation satisfies the simple properties of the following definition:

Definition II.3.1 Let Q be a unital involutive quantale. A *support* on Q is a sup-lattice endomorphism

$$\varsigma : Q \to Q$$

satisfying, for all $a \in Q$,

(II.3.2) $$\varsigma a \leq e$$

(II.3.3) $$\varsigma a \leq aa^*$$

(II.3.4) $$a \leq \varsigma a a$$

A *supported quantale* is a unital involutive quantale equipped with a specified support.

Example II.3.2 1. The only support of a locale (cf. Example II.2.2) is the identity.

2. As already mentioned above, a support for the powerset $\wp(G)$ of a discrete groupoid G is obtained from the domain map of G:

$$\varsigma U = \{u(d(x)) \mid x \in U\} .$$

As we shall see ahead, this is the only possible support for such a quantale.

3. In particular, the support of the quantale $\mathrm{Rel}(X)$ of binary relations on a set X is given by $\varsigma R = \{(x, x) \mid (x, y) \in R \text{ for some } y\}$.

Now we address general properties of supported quantales.

Lemma II.3.3 *Let Q be a supported quantale. The following conditions hold:*

(II.3.4) $$\varsigma a = a, \text{ for all } a \leq e$$

(II.3.5) $$\varsigma\varsigma a = \varsigma a$$

(II.3.6) $$a = \varsigma b a, \text{ if } \varsigma a \leq \varsigma b$$

(II.3.7) $$a = \varsigma a a$$

(II.3.8) $$(\varsigma a)^* = \varsigma a$$

(II.3.9) $$a = a\varsigma(a^*)$$

(II.3.10) $$\varsigma a = 0 \Leftrightarrow a = 0$$

(II.3.11) $$\varsigma a \leq \varsigma(aa^*)$$

(II.3.12) $$\varsigma a 1 = a1$$

(II.3.13) $$a1 = aa^*1$$

(II.3.14) $$\varsigma a = \varsigma a \varsigma a$$

(II.3.15) $$a \leq aa^*a$$

(II.3.16) $$\varsigma(a1)b = a1 \wedge b$$

(II.3.17) $$\varsigma(a1) = a1 \wedge e$$

(II.3.18) $$\varsigma(a \wedge b) \leq ab^*$$

Furthermore,

- *the subquantale* $\downarrow e$ *coincides with* ςQ *and it is a locale with* $ab = a \wedge b$*;*
- *all the elements of* ςQ *are projections, and* ςQ *is a unital involutive subquantale with trivial involution;*
- *the sup-lattice homomorphism* $\varsigma Q \to \mathrm{R}(Q)$ *defined by* $a \mapsto a1$ *is a retraction split by the map* $\mathrm{R}(Q) \to \varsigma Q$ *which is defined by* $a \mapsto \varsigma a$*;*
- *the map* $\varsigma : \mathrm{R}(Q) \to \varsigma Q$ *is an order embedding.*

Proof. First we prove properties (II.3.4)–(II.3.18).

(II.3.4): From (II.3.4) and (II.3.2), if $a \le e$ we have $a \le \varsigma aa \le \varsigma ae = \varsigma a$, and from (II.3.3) and (II.3.2) we have $\varsigma a \le aa^* \le ae^* = ae = a$.

(II.3.5): Immediate from the previous one because $\varsigma a \le e$.

(II.3.6,II.3.7): From (II.3.4) and (II.3.2): if $\varsigma a \le \varsigma b$ we have $a \le \varsigma aa \le \varsigma ba \le ea = a$.

(II.3.8): We have $\varsigma a = \varsigma\varsigma a \le (\varsigma a)(\varsigma a)^* \le e(\varsigma a)^*$, and thus $\varsigma a \le (\varsigma a)^*$, i.e., $\varsigma a = (\varsigma a)^*$.

(II.3.9): From (II.3.7) and (II.3.8) we have $a = a^{**} = (\varsigma(a^*)a^*)^* = a\varsigma(a^*)^* = a\varsigma(a^*)$.

(II.3.10): If $\varsigma a = 0$ then $a = 0$ because $a \le \varsigma aa$. The converse, i.e., $\varsigma 0 = 0$, is trivial because ς preserves joins (but we remark that the axiom $\varsigma a \le aa^*$ would also imply $\varsigma 0 = 0$ for more general maps ς).

(II.3.11): Follows from (II.3.3) and (II.3.5).

(II.3.12,II.3.13): Follows from (II.3.3) and (II.3.4): $\varsigma a1 \le aa^*1 \le a1 \le \varsigma aa1 \le \varsigma a1$.

(II.3.14): Follows from (II.3.4) and (II.3.5): $\varsigma a \le \varsigma\varsigma a\varsigma a = \varsigma a\varsigma a \le \varsigma a$.

(II.3.15): Follows from (II.3.4) and (II.3.3): $a \le \varsigma aa \le aa^*a$.

(II.3.16): From (II.3.12) we have $\varsigma(a1)b \le \varsigma(a1)1 = a11 = a1$. Since $\varsigma(a1)b \le eb = b$, we obtain the inequality

$$\varsigma(a1)b \le a1 \wedge b\ .$$

The converse inequality follows from (II.3.4):

$$a1 \wedge b \le \varsigma(a1 \wedge b)(a1 \wedge b) \le \varsigma(a1)b\ .$$

(II.3.17): Follows from the previous one with $b = e$.

(II.3.18): Follows from (II.3.3): $\varsigma(a \wedge b) \leq (a \wedge b)(a \wedge b)^* \leq ab^*$.

The downsegment $\downarrow e$ coincides with ςQ due to (II.3.4). It is of course a unital subquantale, and it is idempotent due to (II.3.14). Therefore it is an idempotent quantale whose unit is the top, in other words a locale with $ab = a \wedge b$ (cf. section 1).

We have already seen that the elements $a \leq e$ are idempotent and, by (II.3.8), self-adjoint, i.e., projections. Hence, the locale ςQ, with the trivial involution, is an involutive subquantale of Q.

Now we verify that the support splits the map $(-)1 : \varsigma Q \to \mathrm{R}(Q)$. Let $a \in \mathrm{R}(Q)$. Then, by (II.3.12), $\varsigma a1 = a1 = a$.

It follows that $\varsigma : \mathrm{R}(Q) \to \varsigma Q$ is an order embedding because it is a section. ∎

Exercise II.3.4 1. Let Q be a unital involutive quantale such that $a \leq aa^*a$ for all $a \in Q$. Show that Q is a *Gelfand quantale* in the sense of [23] (i.e., such that $a = aa^*a$ for all right-sided elements a). Conclude that every supported quantale is a Gelfand quantale.

2. By a *locally Gelfand quantale* [31] is meant an involutive quantale Q such that for all projections $p = p^2 = p^* \in Q$ the involutive subquantale $\downarrow p$ is Gelfand. Show that any supported quantale is a locally Gelfand quantale.

Stable supports. Now we shall study the particularly well behaved supported quantales whose supports are stable.

Lemma II.3.5 *Let Q be a supported quantale. The following conditions are equivalent:*

1. *for all $a, b \in Q$, $\varsigma(ab) = \varsigma(a\varsigma b)$;*
2. *for all $a, b \in Q$, $\varsigma(ab) \leq \varsigma a$;*
3. *for all $a \in Q$, $\varsigma(a1) = \varsigma a$;*
4. *for all $a \in Q$, $a1 \wedge e = \varsigma a$;*
5. *for all $a, b \in Q$, $a1 \wedge b = \varsigma ab$;*
6. *the map $(-)1 : \varsigma Q \to \mathrm{R}(Q)$ is an order isomorphism whose inverse is ς restricted to $\mathrm{R}(Q)$ (in particular, $\mathrm{R}(Q)$ is a locale with $a1 \wedge b1 = \varsigma ab1$);*

7. *for all* $a, b \in Q$*,* $a \le \varsigma b a$ *if and only if* $\varsigma a \le \varsigma b$*;*

8. *for all* $a \in Q$*,* $\varsigma a \le b1$ *if and only if* $\varsigma a \le \varsigma b$*.*

Proof. First we show that the first five conditions are equivalent. First, assuming 1, we have $\varsigma(ab) = \varsigma(a\varsigma b) \le \varsigma(ae) = \varsigma a$, which proves 2. Conversely, if 2 holds then

$$\varsigma(ab) \le \varsigma(a\varsigma bb) \le \varsigma(a\varsigma b) \le \varsigma(abb^*) \le \varsigma(ab) \ ,$$

and thus 1 holds. It is obvious that 3 is equivalent to 2, and now let us show that 3, 4, and 5 are equivalent. First, (II.3.16) tells us that $\varsigma(a1)b = a1 \wedge b$, and thus if 3 holds we obtain $\varsigma ab = a1 \wedge b$. Hence, 3 implies 5, which trivially implies 4. Finally, if 4 holds we have $\varsigma(a1) = a11 \wedge e = a1 \wedge e = \varsigma a$, and thus 3 holds.

Now we deal with the remaining conditions.

(1⇒6) Assume 1, and let $a \in Q$. Then $\varsigma(\varsigma a1) = \varsigma(\varsigma a \varsigma 1) = \varsigma(\varsigma a e) = \varsigma\varsigma a = \varsigma a$. This shows that the map $(-)1 : \varsigma Q \to \mathrm{R}(Q)$ is an order isomorphism with ς as its inverse, because we have already seen that it is a retraction split by ς.

(6⇒3) Let $a \in Q$. We have $a1 = \varsigma a1$ for any support, and thus assuming 6 we have $\varsigma(a1) = \varsigma(\varsigma a1) = \varsigma a$.

(2⇒7) Assume $a \le \varsigma b a$ and that 2 holds. Then $\varsigma a \le \varsigma(\varsigma b a) \le \varsigma\varsigma b = \varsigma b$. The converse is the condition that $a \le \varsigma b a$ follows from $\varsigma a \le \varsigma b$, which coincides with (II.3.6).

(7⇒2) From the condition $a \le \varsigma a a$ we obtain, multiplying by b on the right, $(ab) \le \varsigma a(ab)$, and thus assuming 7 we obtain $\varsigma(ab) \le \varsigma a$.

(3⇒8) Assume that 3 holds and that $\varsigma a \le b1$. Then $\varsigma a = \varsigma\varsigma a \le \varsigma(b1) = \varsigma b$. Conversely, if $\varsigma a \le \varsigma b$ then $\varsigma a \le \varsigma b e \le \varsigma b 1 = b1$.

(8⇒2) We have $\varsigma(ab) \le abb^*a^* \le a1$, and thus assuming 8 we conclude $\varsigma(ab) \le \varsigma a$. ∎

Definition II.3.6 A support is *stable* if it satisfies the equivalent conditions of II.3.5. A quantale equipped with a specified stable support is *stably supported.*

Example II.3.7 All the examples of supports discussed so far are stable. A simple example of a supported quantale whose support is not stable is the four element unital involutive quantale that, besides the elements 0, e, and 1, contains an element a such that

$$a^2 = a^* = a < e \ ,$$
$$a1 = 1 \ .$$

This quantale has a unique support, defined by $\varsigma a = a$, which is not stable: $\varsigma(a1) = \varsigma 1 = e \not\leq a = \varsigma a$.

Lemma II.3.8 *Let Q be a stably supported quantale.*

1. *Let $a, b \in Q$, and assume that the following three conditions hold:*

$$\begin{aligned} b &\leq e \\ b &\leq aa^* \\ a &\leq ba \,. \end{aligned}$$

   Then $b = \varsigma a$.

2. *$\varsigma(ab) = a\varsigma b$ for all $a, b \in Q$ with $a \leq e$.*

Proof. 1. Assume $b \leq e$. Then, from II.3.5-7, the condition $a \leq ba$ implies $\varsigma a \leq b$. And the condition $b \leq aa^*$ implies $b \leq a1$, which, by II.3.5-8, is equivalent to $b \leq \varsigma a$. Hence, if all the three conditions hold we conclude that $b = \varsigma a$.

2. If $a \leq e$ we have $a\varsigma b \leq e$, and thus $a\varsigma b = \varsigma(a\varsigma b) = \varsigma(ab)$. ∎

Theorem II.3.9 1. *If Q has a stable support then that is the only support of Q, and the following equation holds:*

$$\varsigma a = e \wedge aa^* \,. \tag{II.3.10}$$

2. *If Q has a support and K has a stable support then any homomorphism of unital involutive quantales from Q to K preserves the support. (In particular, the relational representations $Q \to \mathrm{Rel}(X)$ of Q are exactly the same as the support preserving relational representations.)*

Proof. 1. Let $b = e \wedge aa^*$. Then by (II.3.2) and (II.3.3) we have $\varsigma a \leq b$, and thus $a \leq ba$, by II.3.5-7. Hence, by II.3.8, we conclude that $b = \varsigma a$, which justifies the equation.

2. Let $h : Q \to K$ be a homomorphism of unital involutive quantales, let $a \in Q$, and let $b = h(\varsigma a)$. Then we have:

$$\begin{aligned} &b = h(\varsigma a) \leq h(e) = e \,, \\ &b = h(\varsigma a) \leq h(aa^*) = h(a)h(a)^* \,, \\ &h(a) \leq h(\varsigma a a) = h(\varsigma a)h(a) = ba \,. \end{aligned}$$

Hence, by II.3.8 we conclude that $h(\varsigma a) = b = \varsigma h(a)$; that is, the support is preserved by h. ∎

This theorem justifies the assertion that having a stable support is a property of a unital involutive quantale, rather than extra structure on it, and it motivates the following definition for the category of stably supported quantales (whose morphisms necessarily preserve the supports):

Definition II.3.10 The *category of stably supported quantales*, $StabQu$, is the full subcategory of the category of unital involutive quantales Qu_e^* whose objects are the stably supported quantales.

In addition, the following theorem implies that each unital involutive quantale has an idempotent stably supported completion (idempotence meaning that any stably supported quantale is isomorphic to its completion, which is a consequence of $StabQu$ being a full subcategory):

Theorem II.3.11 *$StabQu$ is a full reflective subcategory of Qu_e^*.*

Proof. It is straightforward to see that the limits in $StabQu$ are calculated in Qu_e^*, and thus the proof of the theorem will follow from verifying that the solution set condition of Freyd's adjoint functor theorem [17, Ch. V] holds.

In order to see this, first consider the category whose objects are the involutive monoids M equipped with an additional operation $\varsigma : M \to M$, of which we require no special properties, and whose morphisms are the homomorphisms of involutive monoids that also preserve the operation ς. Let us refer to such monoids as *ς-monoids*. From standard universal algebra it follows that there exist free ς-monoids.

Now let X be a set and let us denote by $F(X)$ the corresponding free ς-monoid. Let also $f : X \to K$ be a map, where K is a supported quantale. Since the support makes K an ς-monoid, f has a unique homomorphic extension $f' : F(X) \to K$, of which there is then a unique join preserving extension $f'' : \wp(F(X)) \to K$. Besides being a sup-lattice, $\wp(F(X))$ is itself an ς-monoid whose operations are computed pointwise from those of $F(X)$ (hence preserving joins in each variable). Furthermore, each of these operations is preserved by f''.

Now let Q be a unital involutive quantale, and let

$$h : Q \to K$$

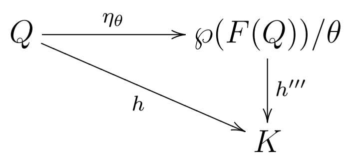
be a homomorphism of unital involutive quantales, where K is stably supported. As above, there is a factorization:

$$\begin{array}{ccc} Q & \longrightarrow & \wp(F(Q)) \\ & \searrow^{h} & \downarrow h'' \\ & & K. \end{array}$$

Hence, the (necessarily stable) supported subquantale $S \subseteq K$ generated by the image $h(Q)$ is a surjective image of $\wp(F(Q))$, where the surjection $\wp(F(Q)) \to S$ is both a sup-lattice homomorphism and a homomorphism of ς-monoids. This surjection determines an equivalence relation θ on $\wp(F(Q))$ such that:

- θ is a congruence with respect to joins, ς, and the involutive monoid structure;
- the injection of generators $Q \to \wp(F(Q))/\theta$ is a homomorphism of unital involutive quantales;
- the quotient $\wp(F(Q))/\theta$ is stably supported.

Therefore we obtain a factorization



in Qu_e^* (which implies that h''' is in fact a homomorphism of supported quantales, by II.3.9). Since K and h have been chosen arbitrarily, the solution set condition now follows from the observation that the set of congruences which satisfy the above three conditions is small.

(One may also observe that the set of congruences is closed under intersections, and that the desired reflection is $\wp(F(Q))/\Theta$, where Θ is the least congruence.) ∎

Exercise II.3.12 1. Show that the following conditions are equivalent, for any supported quantale Q:

(a) The support of Q is stable;

(b) For all $a \in Q$, $\varsigma a = \bigwedge\{b \in \varsigma Q \mid a \leq ba\}$;

(c) For all $a \in Q$, $\varsigma a = \bigvee\{b \in \varsigma Q \mid b \leq a1\}$;

(d) The sup-lattice bimorphism $Q \times \varsigma Q \to \varsigma Q$ defined by $(a, f) \mapsto \varsigma(af)$ makes ςQ a left Q-module.

2. Let Q be a supported quantale.

(a) Show that the following conditions are equivalent:

i. For all $a \in Q$, if $\varsigma a \leq bb^*$ then $\varsigma a \leq \varsigma b$;

ii. For all $a \in Q$, $\varsigma a = \bigvee\{b \in \varsigma Q \mid b \leq aa^*\}$.

(b) Calling a support that satisfies these equivalent conditions *weakly stable*, show that any stable support is weakly stable. Does the converse hold?

3. Let Q be a stably supported quantale. Show that the sup-lattice isomorphism $\varsigma Q \cong \mathrm{R}(Q)$ of Lemma II.3.5-6 is an isomorphism of left Q-modules with respect to the left module structure of ςQ of the previous exercise and the left module structure of $\mathrm{R}(Q)$ which is given by left multiplication (cf. Exercise II.2.29).

4. Provide an explicit description of the stably supported reflection of the unital involutive quantale $\mathrm{End}(\mathrm{Sub}_{\mathbb{C}}(\mathbb{C}^2))$ of sup-lattice endomorphisms on the lattice of complex linear subspaces of $\mathbb{C}^2$.

Inverse quantales. Now we shall see that supported quantales are closely related to inverse semigroups.

Definition II.3.13 Let Q be a unital involutive quantale. A *partial unit* in Q is an element $a \in Q$ such that the following two conditions hold:

$$\begin{aligned} aa^* &\leq e \\ a^*a &\leq e\,. \end{aligned}$$

The set of partial units of Q is denoted by $\mathcal{I}(Q)$.

Example II.3.14 Let X be a set, and $Q = \mathrm{Rel}(X)$ the quantale of binary relations on X. Then $\mathcal{I}(Q)$ is the set $\mathcal{I}(X)$ of partial bijections on X. Hence, $\mathcal{I}(Q)$ is an inverse monoid and, as we shall see below, this is a consequence of the fact that Q is a supported quantale. More generally, if $Q = \wp(G)$ for a discrete groupoid G, a partial unit is the same as a G-set, and thus $\mathcal{I}(Q) = \mathcal{I}(G)$.

Lemma II.3.15 *Let Q be a unital involutive quantale. Then $\mathcal{I}(Q)$ is an involutive submonoid of Q.*

Proof. The set $\mathcal{I}(Q)$ is clearly closed under involution, and $e \in \mathcal{I}(Q)$. It is also closed under multiplication, for if a and b are partial units then $(ab)(ab)^* = abb^*a^* \leq aea^* = aa^* \leq e$, and in the same way $(ab)^*(ab) \leq e$. Hence, $\mathcal{I}(Q)$ is an involutive submonoid of Q. ∎

Lemma II.3.16 *Let Q be a supported quantale, and let $a \in \mathcal{I}(Q)$. Then,*

1. *$\varsigma a = aa^*$,*

2. *$a = aa^*a$,*

3. *$a^2 = a$ if and only if $a \leq e$,*

4. *$b \leq a$ if and only if $b = \varsigma ba$, for all $b \in Q$.*

Proof. 1. We have $\varsigma a \leq aa^* \leq \varsigma aaa^* \leq \varsigma ae = \varsigma a$.

2. This is a consequence of the above and of the equality $a = \varsigma aa$.

3. If $a \leq e$ then a is an idempotent because ςQ is a locale. Now assume that $a^2 = a$. Then $aa^* = aaa^* \leq ae = a$. Hence, $aa^* \leq a$, and, since aa^* is self-adjoint, also $aa^* \leq a^*$. Finally, from here we conclude that $a \leq e$ because $a = aa^*a \leq a^*a \leq e$.

4. Let $b \in Q$ such that $b \leq a$ (in fact, then $b \in \mathcal{I}(Q)$). Then,

$$\varsigma ba \leq bb^*a \leq ba^*a \leq be = b \leq \varsigma bb \leq \varsigma ba\ .$$

This shows that $b \leq a$ implies $b = \varsigma ba$. The converse is trivial. ∎

In the following theorem the category of inverse quantales should be naturally assumed to be the category whose objects are the supported quantales and whose morphisms are the unital and involutive homomorphisms that preserve the supports, although the theorem is true even if we consider as morphisms all the homomorphisms of unital involutive quantales (in any case the distinction disappears once we restrict to stable supports).

Theorem II.3.17 *1. Let Q be a supported quantale. Then $\mathcal{I}(Q)$ is an inverse monoid whose natural order coincides with the order inherited from Q, and whose set of idempotents $E(\mathcal{I}(Q))$ coincides with ςQ.*

2. The assignment $Q \mapsto \mathcal{I}(Q)$ extends to a functor $\mathcal{I}$ from the category of supported quantales to the category of inverse monoids InvMon.

Proof. 1. $\mathcal{I}(Q)$ is an involutive submonoid of Q, and in particular it is a regular monoid because for each partial unit a we have both $aa^*a = a$ and $a^*aa^* = a^*$. Hence, in order to have an inverse monoid it suffices to show that all the idempotents commute, and this follows from II.3.16-3, which implies that the set of idempotents of $\mathcal{I}(Q)$ is the same as ςQ, which is a locale. Furthermore, the natural order of $\mathcal{I}(Q)$ is defined by $a \leq b \Leftrightarrow \varsigma ab = a$, and thus it coincides with the order of Q, by II.3.16-4.

2. $\mathcal{I}$ is a functor because if $h : Q \to K$ is any homomorphism of unital involutive quantales and $a \in Q$ is a partial unit then $h(a)$ is a partial unit: $h(a)h(a)^* = h(aa^*) \leq h(e) = e$, and, similarly, $h(a)^*h(a) \leq e$. Hence, h restricts to a homomorphism of monoids $\mathcal{I}(Q) \to \mathcal{I}(K)$. ∎

We shall be particularly interested in supported quantales with the additional property that each element is a join of partial units. As we shall see, such quantales are necessarily stably supported.

Definition II.3.18 By an *inverse quantale* will be meant a supported quantale Q such that every element $a \in Q$ is a join of partial units:

$$a = \bigvee \{b \in \mathcal{I}(Q) \mid b \leq a\} .$$

The *category of inverse quantales*, *InvQu*, is the full subcategory of Qu_e^* whose objects are the inverse quantales.

From II.3.14 it follows that $\wp(G)$ is an inverse quantale for any discrete groupoid G. In particular, the quantale of binary relations $\mathrm{Rel}(X)$ on a set X is an inverse quantale.

An alternative definition of the concept of inverse quantale can be found in Exercise II.3.22.

Lemma II.3.19 *Any inverse quantale is stably supported (hence, InvQu is a full subcategory of StabQu).*

Proof. Let Q be an inverse quantale, and let $a, b \in Q$, such that

$$\begin{aligned} a &= \bigvee_i s_i \\ b &= \bigvee_j t_j , \end{aligned}$$

where the s_i and t_j are partial units. Then,

$$\begin{aligned} \varsigma(ab) &= \varsigma\left(\bigvee_{ij} s_i t_j\right) = \bigvee_{ij} \varsigma(s_i t_j) = \bigvee_{ij} s_i t_j (s_i t_j)^* = \bigvee_{ij} s_i t_j t_j^* s_i^* \\ &\leq \bigvee_i s_i s_i^* = \bigvee_i \varsigma(s_i) = \varsigma\left(\bigvee_i s_i\right) = \varsigma a . \ \blacksquare \end{aligned}$$

The converse is not true; that is, not every stably supported quantale is an inverse quantale (see III.2.21 in section 2).

Lemma II.3.20 *Let S be an inverse semigroup. The set $\mathcal{L}(S)$ of subsets of S that are downwards closed in the natural order of S is an inverse quantale.*

The unit is the set of idempotents $E(S)$ (which, if S is a monoid, is just $\downarrow e$), multiplication is computed pointwise,

$$XY = \{xy \mid x \in X,\ y \in Y\}\ ,$$

the involution is pointwise inversion, $X^* = X^{-1}$, and the (necessarily unique) support is given by the formula

$$\varsigma X = \{xx^{-1} \mid x \in X\}\ .$$

Proof. Consider an inverse semigroup S. It is straightforward to verify that the sup-lattice $\mathcal{L}(S)$ is an involutive quantale, with the multiplication and involution defined above. In particular, for multiplication this means that downwards closed sets are closed under pointwise multiplication (this would not be true for an arbitrary partially ordered involutive semigroup, for which downwards closure would be required after taking the pointwise multiplication), which is a consequence of the fact that we are dealing with the natural order of an inverse semigroup: if $z \le w \in XY$ for $X, Y \in \mathcal{L}(S)$, then $w = xy$ with $x \in X$ and $y \in Y$, and $z = zz^{-1}w = zz^{-1}(xy) = (zz^{-1}x)y \in XY$ because $zz^{-1}x \le x \in X$, and X is downwards closed. For the involution it is similar, but more immediate.

In order to see that the set of idempotents $E(S)$ is the multiplicative unit, consider $X \in \mathcal{L}(S)$. The set $E(S)$ contains the idempotent $x^{-1}x$ for each $x \in X$, and thus the pointwise product $XE(S)$ contains all the elements of the form $xx^{-1}x = x$. Hence, $X \subseteq XE(S)$. The other elements of $XE(S)$ are of the form $xy^{-1}y$, with $x \in X$, and we have $xy^{-1}y \le x$ in the natural order of S, implying that $xy^{-1}y \in X$ because X is downwards closed. Hence, $XE(S) = X$. Similarly we show that $E(S)X = X$.

In order to see that $\mathcal{L}(S)$ is an inverse quantale we apply II.3.22. First, each $X \in \mathcal{L}(S)$ is of course a union of partial units:

$$X = \bigcup\{\downarrow x \mid x \in X\}\ .$$

Secondly, let $X \in \mathcal{L}(S)$. Then

$$XX^*X = \{xy^{-1}z \mid x, y, z \in X\} \supseteq \{xx^{-1}x \mid x \in X\} = \{x \mid x \in X\} = X\ .$$

To conclude, we show that the operation $\varsigma : \mathcal{L}(S) \to \mathcal{L}(S)$ defined by

$$\varsigma X = \bigcup\{UU^* \mid U \in \mathcal{I}(\mathcal{L}(S)) \text{ and } U \subseteq X\}$$

coincides with the (clearly join-preserving) operation

$$\varsigma X = \{xx^{-1} \mid x \in X\}$$

of the statement of this lemma, thus showing that $\mathcal{L}(S)$ is an inverse quantale. If $U \in \mathcal{I}(\mathcal{L}(S))$ then $UU^* \subseteq E(S)$, i.e., $xy^{-1} \in E(S)$ for all $x, y \in U$. But then

$$xy^{-1} = xy^{-1}(xy^{-1})^{-1} = xy^{-1}yx^{-1} = xy^{-1}yy^{-1}yx^{-1} = (xy^{-1}y)(xy^{-1}y)^{-1} ,$$

so we see that xy^{-1} is of the form zz^{-1} with $z = xy^{-1}y \leq x$, where $z \in U$ because U is downwards closed. We therefore conclude that UU^* coincides with the set $\{xx^{-1} \mid x \in U\}$, and from this the required formula for ςX follows. ∎

Theorem II.3.21 *The functor $\mathcal{I}$, restricted to the category of stably supported quantales StabQu, has a left adjoint from the category of inverse monoids InvMon to StabQu, which to each inverse monoid S assigns the quantale $\mathcal{L}(S)$.*

Proof. First we remark that the embedding $S \to \mathcal{L}(S)$ actually defines a homomorphism of monoids $S \to \mathcal{I}(\mathcal{L}(S))$, which provides the unit of the adjunction. Now let Q be a stably supported quantale, and $h : S \mapsto \mathcal{I}(Q)$ a homomorphism of monoids. Then h preserves the natural order, and thus it defines a homomorphism of ordered involutive monoids $S \to Q$ because, as we have seen, the natural order of $\mathcal{I}(Q)$ is just the order of Q restricted to $\mathcal{I}(Q)$. It follows that h extends (uniquely) to a homomorphism of unital involutive quantales $\overline{h} : \mathcal{L}(S) \to Q$, namely the sup-lattice extension $\overline{h}(U) = \bigvee h(U)$. ∎

Exercise II.3.22

1. Let Q be a unital involutive quantale such that every element in Q is a join of partial units. Show that Q is a supported quantale (hence, an inverse quantale) if and only if the following conditions hold:

   (a) $a \leq aa^*a$ for all $a \in Q$ (equivalently, $a = aa^*a$ for all $a \in \mathcal{I}(Q)$);

   (b) the operation $\varsigma : Q \to Q$ defined by

   (II.3.11) $$\varsigma a = \bigvee \{bb^* \mid b \in \mathcal{I}(Q) \text{ and } b \leq a\}$$

   is a sup-lattice homomorphism.

   Show that when these conditions hold the operation ς is the support.

2. Give an example of a unital involutive quantale Q satisfying $a \leq aa^*a$ for all $a \in Q$ but such that the operation ς defined by (II.3.11) does not preserve joins.

Enveloping quantales. Recall (cf. Definition I.2.16) that by an abstract complete pseudogroup is meant a complete and infinitely distributive inverse monoid. We have seen in the previous section how to obtain an inverse quantale from an arbitrary inverse semigroup. In the case of an abstract complete pseudogroup there is another useful inverse quantale $\mathcal{L}^{\vee}(S)$, here referred to as the *enveloping quantale* of S, which takes into account the joins that exist in S. From here until the end of this section S will be a fixed but arbitrary abstract complete pseudogroup.

Definition II.3.23 By a *compatibly closed ideal* of S is meant a downwards closed set (possibly empty) which is closed under the formation of joins of compatible sets. The set of compatibly closed ideals of S is denoted by $\mathcal{L}^{\vee}(S)$.

Lemma II.3.24 *$\mathcal{L}^{\vee}(S)$ is a quotient of $\mathcal{L}(S)$ both as a locale and as a unital involutive quantale, and it is an inverse quantale.*

Proof. Let $j : \mathcal{L}(S) \to \mathcal{L}(S)$ be the closure operator that to each downwards closed set $U \subseteq S$ assigns the least compatibly closed ideal that contains U. First we remark that j is explicitly defined by

$$j(U) = \left\{ \bigvee X \mid X \subseteq U,\ X \text{ is compatible} \right\} .$$

In order to see this, let $x \le y \in j(U)$. Then y is of the form $\bigvee Y$ for some compatible set $Y \subseteq U$, and thus $x = x \wedge \bigvee Y = \bigvee (x \wedge Y)$, where the set $x \wedge Y = \{x \wedge y \mid y \in Y\}$ is of course compatible. Hence, we have $x \in j(U)$, showing that $j(U)$ is downwards closed. Now let $Z \subseteq j(U)$ be a compatible set. Each element $z \in Z$ is of the form $\bigvee U_z$ for some compatible set $U_z \subseteq U$, and the fact that Z is compatible implies that the set $Z' = \bigcup_{z \in Z} U_z$ is compatible. But we also have $\bigvee Z = \bigvee Z'$, and thus $z \in j(U)$, showing that $j(U)$ is closed under the formation of joins of compatible sets. It is thus a compatible ideal, clearly the smallest one containing U.

Now we shall show that j is both a locale nucleus and a nucleus of involutive quantales, thus proving that $\mathcal{L}^{\vee}(S)$, which coincides with the quotient of $\mathcal{L}(S)$ obtained as the set of fixed-points of S, is a unital involutive quantale, a locale, and a quotient of $\mathcal{L}(S)$.

Let $I, J \in \mathcal{L}(S)$. Let $X \subseteq I$ and $Y \subseteq J$ be compatible sets such that $\bigvee X = \bigvee Y$. Let us denote this join by z. We have $z \in j(I) \cap j(J)$, and all the elements of $j(I) \cap j(J)$ can be obtained in the same way. Now define the set

$$Z = \{x \wedge y \mid x \in X,\ y \in Y\} .$$

We have $Z \subseteq I \cap J$, and

$$\bigvee Z = \bigvee_{x \in X} \bigvee_{y \in Y} (x \wedge y) = \bigvee_{x \in X} \left(x \wedge \bigvee Y\right) = \bigvee_{x \in X} (x \wedge z) = \left(\bigvee X\right) \wedge z = z \ .$$

Hence, $z \in j(I \cap J)$, and we conclude that $j(I) \cap j(J) \subseteq j(I \cap J)$, i.e., j is a nucleus of locales.

Let again $I, J \in \mathcal{L}(S)$. Consider an arbitrary element of $j(I)j(J)$, which is necessarily of the form xy with $x = \bigvee X$ and $y = \bigvee Y$, where $X \subseteq I$ and $Y \subseteq J$ are compatible sets. We shall show that $xy \in j(XY)$, hence proving that j is a nucleus of quantales. First, we remark that XY is a compatible set, since it is bounded above by xy. Its join $\bigvee(XY)$ coincides with xy, due to infinite distributivity, and thus $xy \in j(XY)$. Clearly, j preserves the involution of the quantales, and thus j is a nucleus of involutive quantales. The involution of $\mathcal{L}^\vee(S)$ is, similarly to that of $\mathcal{L}(S)$, given by pointwise inversion.

Finally, j is also a nucleus with respect to the support of $\mathcal{L}(S)$ because the support is, since $\mathcal{L}(S)$ is stably supported, expressed in terms of locale and unital involutive quantale operations:

$$\varsigma U = UU^* \cap E(S) \ .$$

Hence, the conclusion that $\mathcal{L}^\vee(S)$ is stably supported follows, and it is obviously an inverse quantale because it has a basis consisting of the principal ideals $\downarrow s$ with $s \in S$, which are partial units of $\mathcal{L}^\vee(S)$. ∎

Example II.3.25 From the results about groupoids and quantales in Chapter III it will follow (but it can also be verified directly) that if G is a discrete groupoid then we have an isomorphism $\mathcal{L}^\vee(\mathcal{I}(G)) \cong \wp(G)$. In particular, the enveloping quantale $\mathcal{L}^\vee(\mathcal{I}(X))$ of the symmetric inverse monoid $\mathcal{I}(X)$ of a set X is isomorphic to the quantale of binary relations $\wp(X \times X)$ on X.

Theorem II.3.26 *$\mathcal{L}^\vee(S)$ is the quotient of $\mathcal{L}(S)$ (in the category of stably supported quantales StabQu) determined by the condition that joins of $E(S)$ should be preserved by the injection of generators $S \to \mathcal{L}^\vee(S)$.*

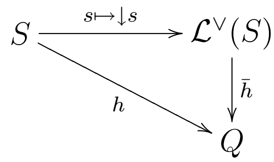
Proof. This is a consequence of exercise I.2.17, from which it follows that the homomorphisms of abstract complete pseudogroups (i.e., those monoid homomorphisms that preserve joins of compatible subsets)

$$h : S \to \mathcal{I}(Q) \ ,$$

where Q is any stably supported quantale, are exactly the monoid homomorphisms that preserve just the joins of sets of idempotents. Hence, the universal properties corresponding to preservation of joins of idempotents, on one hand, and to preservation of arbitrary joins, on the other, are the same. ∎

The universal properties possessed by enveloping quantales are by now essentially obvious, once one takes into account the analogous properties for $\mathcal{L}(S)$. We shall provide an explicit description of them.

Let S be an abstract complete pseudogroup, and let Q be a stably supported quantale. The monoid of partial units $\mathcal{I}(Q)$ is an abstract complete pseudogroup, and if $h : S \to \mathcal{I}(Q)$ is a homomorphism of abstract complete pseudogroups there is a unique homomorphism of unital involutive quantales $\bar{h} : \mathcal{L}^\vee(S) \to Q$ such that the following diagram commutes,



where $\bar{h}$ is explicitly defined by

$$\bar{h}(U) = \bigvee \{h(s) \in Q \mid s \in U\} .$$

In other words, we have:

Corollary II.3.27 *$\mathcal{L}^\vee$ defines a functor from ACPGrp to StabQu, which is left adjoint to the functor $\mathcal{I}$: StabQu $\to$ ACPGrp.*

A consequence of this is also, for S an inverse semigroup, that $\mathcal{L}(S) \cong \mathcal{L}^\vee(C(S))$, where $C(S)$ is the completion of S in the sense of [14, Section 1.4] (but including the join of the empty set), and in fact we have $C(S) \cong \mathcal{I}(\mathcal{L}(S))$.

Chapter III

Localic groupoids

In this chapter we describe the main results of these notes, which involve a three-fold correspondence between localic étale groupoids, abstract complete pseudogroups, and the so-called inverse quantal frames. The reason for working with localic groupoids rather than topological groupoids is essentially of a pragmatic nature: the question of whether or not one can reconstruct a topological étale groupoid from the inverse quantale structure of its topology cannot be expected to have a positive answer unless the groupoid satisfies some topological separation axioms (it should at least be T_0). In practice we shall often need the groupoid to be sober, and this means that we can think of it instead as being a spatial locale. But it may be observed that spatiality really plays no role in the theory, and thus the natural thing to do is to work with arbitrary locales (and luckily so because, roughly, spatiality is the topos dependent part of the theory). The net effect is that topological groupoids become algebraicized in a rather natural way. Indeed we may view the kind of algebra involved as playing, with respect to general topology, a similar role to that of commutative rings in algebraic geometry (from this viewpoint localic groupoids become a topological counterpart of algebraic groupoids).

1 Locales

We shall begin by studying general locale theory. Since locales are examples of quantales, most of the general algebraic properties of locales have already been studied in Chapter II, but now we shall deal with specific aspects of locale theory, namely those which carry analogies with topological spaces.

Frames. We have already provided a definition of the notion of locale (cf. Example II.2.2), as being a particular kind of unital quantale. Let us recall the definition here:

Definition III.1.1 By a *locale* is meant a sup-lattice L satisfying the following distributivity law, for all $x \in L$ and $Y \subseteq L$:

$$x \wedge \bigvee Y = \bigvee_{y \in Y} x \wedge y .$$

Locales are also known as *frames*. The distinction in terminology will be addressed below. The relation to quantales can be described as follows:

Proposition III.1.2 ([10]) *Any frame L is an idempotent unital quantale whose unit is the top. Conversely, any unital quantale with these properties is a frame.*

Proof. If L is a frame then it is obviously a unital quantale with $e = 1$ and the idempotent multiplication given by binary meet. Conversely, let L be an idempotent unital quantale with $e = 1$, and let $a, b \in L$. We shall prove that $ab = a \wedge b$, showing that L is a frame. First, we have $ab \leq a1 = ae = a$ and $ab \leq 1b = b$, showing that ab is a lower bound of a and b. Let c be another lower bound of a and b. Then $c = cc \leq ab$, and thus ab is the greatest lower bound: $ab = a \wedge b$. ∎

Definition III.1.3 A *homomorphism* of frames is a homomorphism of unital quantales between frames. The resulting category, denoted by *Frm*, is the *category of frames*.

Frm is a full subcategory of the category of commutative unital quantales, and algebraically we handle frames in much the same way as we handle commutative unital quantales. For instance, any homomorphic image of a frame is still a frame, and thus frame quotients are described by quantale nuclei. Also, the coproduct of two frames as unital quantales is again a frame, and thus the coproduct of two frames L and M coincides with their tensor product $L \otimes M$ as sup-lattices, with meet defined by

$$(x \otimes y) \wedge (x' \otimes y') = (x \wedge x') \otimes (y \wedge y') ,$$

where the coprojections

$$L \xrightarrow{i_1} L \otimes M \xleftarrow{i_2} M$$

are given by

$$\begin{aligned} i_1(x) &= x \otimes 1_M \\ i_2(y) &= 1_L \otimes y . \end{aligned}$$

The motivation behind locales is topological: the topology $\Omega(X)$ of a topological space X is a frame, and if $f : X \to Y$ is a continuous map then the inverse image $f^{-1} : \wp(Y) \to \wp(X)$ restricts to a frame homomorphism $f^{-1} : \Omega(Y) \to \Omega(X)$. Hence, the category of frames can be regarded as an approximation of (the dual of) category of topological spaces and continuous maps, and locales are often referred to as *point-free* spaces — or, more tongue-in-cheek, *pointless* spaces [6, 7].

Many definitions and properties readily carry over from topological spaces. We provide some simple examples:

Definition III.1.4 Let L be a frame, and let $a \in L$. By a *cover* of a is meant a subset $C \subseteq L$ such that $a \leq \bigvee C$. A *cover of* L is by definition a cover of 1. We say a frame L is *compact* if every cover of L has a finite subcover: that is, for all $C \subseteq L$ such that $\bigvee C = 1$ there is a finite subset $F \subseteq C$ such that $\bigvee F = 1$.

Of course, a space X is compact if and only if its frame $\Omega(X)$ is compact. Another similar situation occurs with the notion of regularity:

Definition III.1.5 Let L be a frame, and let $a, b \in L$. We say that a is *well inside* b, and write $a \lll b$, if there exists $c \in L$ such that $b \vee c = 1$ and $a \wedge c = 0$. Then L is *regular* if every $x \in L$ is the join of the elements well inside itself:

$$x = \bigvee_{a \lll x} a \ .$$

Of course, a space X is regular if and only if the frame $\Omega(X)$ is regular.

Definition III.1.6 Let L be a frame. A *basis* of L is a cover B of L such that for all $a \in L$ we have

$$a = \bigvee \{b \in B \mid b \leq a\} \ .$$

The following simple property of any basis will play an important role later on.

Proposition III.1.7 *Let $h : L \to M$ be a frame homomorphism, and let $B \subseteq L$ be a basis of L which furthermore is a downwards closed subset. Then h is injective if and only if its restriction $h|_B : B \to M$ is.*

Proof. Assume that $h|_B$ is injective. We shall prove that h is injective (the converse is trivial). Let $b \in B$ and $x \in L$ be arbitrary elements. Then,

$$\begin{aligned} h(b) \le h(x) &\iff h(b) \wedge h(x) = h(b) \\ &\iff h(b \wedge x) = h(b) \\ &\iff b \wedge x = b \qquad \text{(Because } b \wedge x \in B.\text{)} \\ &\iff b \le x \,. \end{aligned}$$

Now let x and y be arbitrary elements of L. Then $y = \bigvee Y$ for some $Y \subseteq B$, and we have

$$\begin{aligned} h(y) \le h(x) &\iff h(\bigvee Y) \le h(x) \\ &\iff \bigvee h(Y) \le h(x) \\ &\iff \forall_{b \in Y}\ h(b) \le h(x) \\ &\iff \forall_{b \in Y}\ b \le x \\ &\iff y \le x \,. \end{aligned}$$

Hence, h is an order embedding. ∎

Exercise III.1.8 1. Let L be a frame, and let j be a closure operator on L. Show that j is a nucleus if and only if $j(x \wedge y) = j(x) \wedge j(y)$ for all $x, y \in L$.

2. Show that the sup-lattice tensor product of two frames is a frame.

3. Show that *Frm* is a reflective subcategory of the category of commutative unital quantales.

4. Show that the topological notion of regularity is equivalent to the definition given in Definition III.1.5.

5. Show that if L is a regular frame then so is any of its quotients.

6. Show that the notion of regularity of III.1.5 can be defined equivalently in terms of residuations, by defining the well-inside relation as follows,

$$a \lessdot b \iff \neg a \vee b = 1 \,,$$

where $\neg a = 0/a$.

7. Let $p : E \to X$ be a local homeomorphism. Consider the set $\Gamma(p)$ of all the continuous local sections of p, ordered by restriction. The sets $S \subseteq \Gamma(p)$ for which $\bigvee S$ exists in $\Gamma(P)$ are precisely the sets of sections which are pairwise compatible. Defining $\mathcal{L}^\vee(\Gamma(p))$ to be the set of downwards closed sets of $\Gamma(p)$ which are closed under the formation of joins (as in Definition II.3.23), show that $\mathcal{L}^\vee(\Gamma(p))$ is a locale isomorphic to the topology $\Omega(E)$ of E. (This technique will be used in Theorem III.2.13.)

Frames versus locales. The idea that *Frm* is an approximation of $\mathit{Top}^{\mathrm{op}}$ gives rise to the following notion:

Definition III.1.9 Let A and B be locales. By a *(continuous) map* $f : A \to B$ of locales is meant a frame homomorphism $f^* : B \to A$, referred to as the *inverse image* of f. The resulting category is the dual of *Frm*, and we denote it by *Loc*. It is called the *category of locales and continuous maps*, or simply the *category of locales*. We shall denote by

$$\Omega : \mathit{Top} \to \mathit{Loc}$$

the (obvious) functor that to each topological space X assigns the locale $\Omega(X)$ and to each continuous map $f : X \to Y$ assigns the map

$$\Omega(f) = \Omega(X) \to \Omega(Y)$$

which is defined by $\Omega(f)^* = f^{-1}$.

Before we proceed we shall introduce some more or less traditional notation: the identity functor on *Frm* is a contravariant functor $\mathcal{O} : \mathit{Loc}^{\mathrm{op}} \to \mathit{Frm}$; accordingly, if A is a locale we shall write $\mathcal{O}(A)$ for the locale itself seen as an object of *Frm*. Metaphorically, $\mathcal{O}(A)$ can be thought of as the frame of "open sets of A", in imitation of the notation $\Omega(X)$ for topological spaces. For example, using this notation we could have said, in the above definition, that a locale map $f : A \to B$ is defined to be a frame homomorphism

$$f^* : \mathcal{O}(B) \to \mathcal{O}(A) .$$

This may seem unreasonably redundant, but this notation is often useful. For instance, if A and B are two locales we shall write $A \times B$ for the product of A and B in *Loc*, with the following projections:

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

Of course, this is the coproduct in *Frm* of the frames $\mathcal{O}(A)$ and $\mathcal{O}(B)$, where the coprojections are π_1^* and π_2^*; indeed, we could have defined the locale product $A \times B$ by the equation

$$\mathcal{O}(A \times B) = \mathcal{O}(A) \otimes \mathcal{O}(B) .$$

Accordingly, we shall usually write $\mathcal{O}(A) \times \mathcal{O}(B)$ (rather than again $A \times B$) for the product in *Frm*, without any need to specify the category in question. (But cumbersome expressions like $\mathcal{O}(\Omega(X))$ should be avoided!)

Another aspect in which we distinguish frames and locales is when we refer to quotients and subobjects: a *subframe* is just a subset of a frame which is a unital involutive subquantale; but by a *sublocale* of a locale A will always be meant a frame quotient of the frame $\mathcal{O}(A)$, mimicking the fact that a subspace inclusion $X \to Y$ determines a surjective frame homomorphism $\Omega(Y) \to \Omega(X)$; that is, we should think of a sublocale as the localic analogue of a subspace. We shall be loose about the precise definition of the notions of sublocale, which can often be taken to be a nucleus, or equivalently a subset of a frame closed under arbitrary meets and residuations (cf.II.2.23-3), or an isomorphism class of a frame surjection.

Exercise III.1.10

1. Let $p : X \to B$ and $q : Y \to B$ be maps of locales. Show that the pullback $X \times_B Y$ coincides, as a frame, with the tensor product $\mathcal{O}(X) \otimes_{\mathcal{O}(B)} \mathcal{O}(Y)$ of $\mathcal{O}(X)$ and $\mathcal{O}(Y)$, which are regarded as right and left $\mathcal{O}(B)$-modules, respectively, with the action given by multiplication via change of "base ring":

$$\begin{array}{rcl} \mathcal{O}(X) \otimes \mathcal{O}(B) & \to & \mathcal{O}(X) \\ V \otimes U & \mapsto & V \wedge p^*(U) \end{array}$$

$$\begin{array}{rcl} \mathcal{O}(B) \otimes \mathcal{O}(Y) & \to & \mathcal{O}(Y) \\ U \otimes W & \mapsto & W \wedge q^*(U)\ . \end{array}$$

2. Show that the topology $\Omega(X \times Y)$ of a product space is a frame quotient of $\Omega(X) \otimes \Omega(Y)$ (hint: each $U \otimes V$ is mapped to the open rectangle $U \times V$).

3. Show that if X is a locally compact space then $\Omega(X \times Y) \cong \Omega(X) \otimes \Omega(Y)$ (cf. [8]).

4. Let $\mathbb{Q}$ be the set of rational numbers with the subspace topology of the reals. Show that $\Omega(\mathbb{Q} \times \mathbb{Q})$ and $\Omega(\mathbb{Q}) \otimes \Omega(\mathbb{Q})$ are not isomorphic (cf. [8]).

5. Recall the negation operator $\neg$ of frames (cf. Exercise III.1.8), which is given by $\neg a = 0/a$. Show that the double negation $a \mapsto \neg\neg a$ is a nucleus on any frame.

6. Let $j : L \to L$ be a nucleus on the frame L. We say that j is *dense* if $j(0) = 0$.

(a) Show that if $i : X \to Y$ is the inclusion of X into Y as a subspace then X is dense in Y if and only if the nucleus induced by i (i.e., $i_*^{-1} \circ i^{-1}$) is dense.

(b) Show that $\neg\neg$ is dense, on any frame.

(c) Show that $\neg\neg$ is the least dense nucleus, on any frame.

(d) Let $\mathbb{R}$ be the space of real numbers. Show that the locale $\Omega(\mathbb{R})_{\neg\neg}$ has no points (hint: both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$).

Open sublocales and closed sublocales. We have already given a definition of the notion of sublocale, and now we shall give two important examples, namely the open and the closed sublocales, which are the localic counterpart of open and closed subspaces of topological spaces.

Definition III.1.11 Let L be a locale, and let j be a nucleus on $\mathcal{O}(L)$. The sublocale defined by j is *open* (and j itself is said to be *open*) if j is the nucleus induced by the quotient

$$(-) \wedge a : \mathcal{O}(L) \to {\downarrow}a$$

for some $a \in \mathcal{O}(L)$.

Of course, if L is the topology of a space X, then an open sublocale ${\downarrow}U$ with $U \in \Omega(X)$ is the topology of the open subspace U.

Definition III.1.12 Let L be a locale, and let $a \in \mathcal{O}(L)$. The mapping

$$(-) \vee a : \mathcal{O}(L) \to \mathcal{O}(L)$$

is a nucleus on $\mathcal{O}(L)$, and it is called a *closed* nucleus. The image of this nucleus is just ${\uparrow}a$, which may be regarded as a concrete definition of the *closed sublocale* induced by a.

Again the motivation behind this terminology is topological. If X is a topological space and $C \subseteq X$ is a closed subspace then $\Omega(C)$ is isomorphic to the frame ${\uparrow}(X \setminus C)$. In other words, the sublocale induced by a in the previous definition should be regarded as the analogue of a closed subspace consisting of the "complement of a".

Open and closed sublocales have many interesting properties. For instance, the set $N(L)$ of nuclei on a frame L is itself a frame, called the *assembly* of L. This is the localic generalization of the boolean algebra of all the subspaces of a space, but unlike the topological case it is not complemented. There is a frame homomorphism, which is both a monomorphism and an epimorphism (but not an isomorphism), from L into $N(L)$, which to each $a \in L$ assigns the closed nucleus corresponding to a. We shall not go more into the properties of the assembly of a frame (except for a few exercises below) because we shall not need them in these notes. For more details see [8].

Exercise III.1.13 Show that the open nucleus corresponding to a is given in terms of residuations by $x \mapsto x/a$.

Open maps and local homeomorphisms. A locale map $f : L \to M$ is said to be *semiopen* if the frame homomorphism $f^* : \mathcal{O}(M) \to \mathcal{O}(L)$ preserves all the meets of $\mathcal{O}(M)$; equivalently, if f^* has a left adjoint $f_!$ — the *direct image* of f. (For adjoints between partial orders see [17, Ch. IV] or [8, Ch. I].)

This terminology is motivated by open maps of topological spaces: if $f : X \to Y$ is a continuous open map then the direct image map

$$U \mapsto f_!(U) = \{f(x) \mid x \in U\}$$

is a sup-lattice homomorphism $\Omega(X) \to \Omega(Y)$ and it is left adjoint to $f^{-1} : \Omega(Y) \to \Omega(X)$. However, semiopen maps are insufficient as a localic generalization of open maps, in particular because they are not stable under pullback (cf. exercise III.1.16).

The right localic definition of open map is based, as with topological spaces, on the idea that open subspaces are mapped to open subspaces; in this case open sublocales must be mapped to open sublocales. In order to make sense of this idea first we need to define what we mean by the image of a sublocale. For that we remark that in *Frm* every homomorphism

$$h : \mathcal{O}(M) \to \mathcal{O}(L)$$

can obviously be factorized as $m \circ e$ where $e : \mathcal{O}(M) \to h(\mathcal{O}(M))$ is surjective (equivalently, a regular epimorphism in *Frm*) and $m : h(\mathcal{O}(M)) \overset{\subseteq}{\to} \mathcal{O}(L)$ is a subframe inclusion (hence, a monomorphism in *Frm*). Consequently, *Loc* has a "dual" factorization system: every map $f : L \to M$ factors as an epimorphism e followed by a regular monomorphism m (i.e., a sublocale inclusion)

$$L \xrightarrow{e} L' \xrightarrow{m} M \ .$$

Hence, L' is a sublocale of L, and we define it (up to isomorphism) to be the *image* of f.

Definition III.1.14 Let $f : L \to M$ be a map of locales, and let $i : S \to L$ a sublocale inclusion (i.e., $i^* : \mathcal{O}(L) \to \mathcal{O}(S)$ is a surjective frame homomorphism). The *image* of S by f is the image of the map $f \circ i$. The map f is *open* if the image of every open sublocale of L is an open sublocale of M.

The open locale maps $f : L \to M$ are exactly the semiopen maps that satisfy the following condition for all $a \in \mathcal{O}(L)$ and $b \in \mathcal{O}(M)$,

$$f_!(a \wedge f^*(b)) = f_!(a) \wedge b \ ,$$

known as the *Frobenius reciprocity condition* (see [9, p. 521] or [10, Ch. V]). If $a \in \mathcal{O}(L)$, the open sublocale of M to which $\downarrow a$ is mapped is $\downarrow f_!(a)$.

Definition III.1.15 An open locale map $f : L \to M$ is a *local homeomorphism* if there is a cover C of L such that for each $a \in C$ the frame homomorphism

$$((-) \wedge a) \circ f^* : \mathcal{O}(M) \to \downarrow a$$

is surjective (this is the analogue for locales of a continuous open map of spaces whose restriction to each open set in a given cover is a subspace inclusion).

Exercise III.1.16

1. Let L be a locale and consider the following sublocale inclusion $i : M \to L$. Show that M is an open sublocale if and only if i is an open map.

2. Show that a continuous map of topological spaces $f : X \to Y$ is a local homeomorphism if and only if the map of locales $\Omega(f) : \Omega(X) \to \Omega(Y)$ is a local homeomorphism.

3. Let $f : X \to Y$ be a continuous map of topological spaces. Show that if f is open then so is $\Omega(f)$. Show that the converse fails; that is, f is not necessarily open of $\Omega(f)$ is.

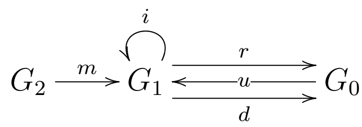
4. Show that both open maps and local homeomorphisms are stable under pullbacks.

5. Show that semiopen maps are in general not stable under pullbacks.

6. Give a suitable definition of local section for maps of locales and repeat Exercise III.1.8-7 for local homeomorphisms of locales.

Groupoids. A *localic groupoid* is an internal groupoid in *Loc*. The definitions of *open* groupoid and of *étale* groupoid are analogous to the corresponding notions for topological groupoids, now with open maps and local homeomorphisms being defined for locales:

Definition III.1.17 Let G be a localic groupoid



We say that G is *open* if d is open, and that G is *étale* if d is a local homeomorphism.

Spatial locales and sober spaces. Just as spaces give rise to locales via the functor Ω, so locales can be approximated by spaces via a *spectrum* functor Σ. Such spatial aspects will be largely irrelevant in these notes, but we include them in this section for the sake of motivation.

We shall write 1, as usual, for some fixed final object in *Loc*. This is an initial frame, which in turn is a free sup-lattice on one generator, so we may use again the $\mathcal{O}$ notation and write

$$\mathcal{O}(1) = \wp(1) \ .$$

The locale 1 is the localic analogue of a "singleton" space, and we may use it in order to define points of locales:

Definition III.1.18 Let A be a locale. By a *point* of A is meant a map $p : 1 \to A$ (equivalently, a frame homomorphism $p^* : \mathcal{O}(A) \to \mathcal{O}(1)$).

The set of points of A is therefore given by the covariant hom functor

$$Loc(1, -) : Loc \to Set \ .$$

There is an obvious pairing

$$Loc(1, A) \times \mathcal{O}(A) \to \mathcal{O}(1)$$

that maps each (p, a) to $p^*(a)$, and thus each $a \in \mathcal{O}(A)$ defines a function

$$\hat{a} : Loc(1, A) \to \mathcal{O}(1) \ ,$$

in turn defining a subset $U_a \subseteq Loc(1, A)$ by

$$U_a = \{p \mid \hat{a}(p) = 1_{\mathcal{O}(1)}\} = \{p \mid p^*(a) = 1_{\mathcal{O}(1)}\} \ .$$

The family (U_a) is a topology on $Loc(1, A)$:

Proposition III.1.19 *For all $a, b \in \mathcal{O}(A)$ and $S \subseteq \mathcal{O}(A)$ we have*

$$\begin{aligned} U_{1_A} &= \Sigma(A) \\ U_{a \wedge b} &= U_a \cap U_b \\ U_{\bigvee S} &= \bigcup_{a \in S} U_a \ . \end{aligned}$$

Definition III.1.20 The *spectrum* of a locale A is the set of points $\mathit{Loc}(1, A)$ equipped with the above topology, and we shall denote the resulting space by $\Sigma(A)$.

For each locale map $f : A \to B$ the function

$$f_* = f \circ (-) : \mathit{Loc}(1, A) \to \mathit{Loc}(1, B)$$

is easily seen to be continuous with respect to the topologies of $\Sigma(A)$ and $\Sigma(B)$, and thus we obtain a functor

$$\Sigma : \mathit{Loc} \to \mathit{Top} \ .$$

Theorem III.1.21 *Ω is left adjoint to Σ.*

Proof. Let A be a locale. Proposition III.1.19 shows that the assigment $a \mapsto U_a$ is a homomorphism of frames $\mathcal{O}(A) \to \Omega(\Sigma(A))$, hence giving us a map of locales $\epsilon : \Omega(\Sigma(A)) \to A$, which will provide the co-unit of the adjunction. Now let X be a topological space, and let $f : \Omega(X) \to A$ be a map of locales. We define a function $\bar{f} : X \to \Sigma(A)$ by

$$\bar{f}(x)(a) = 1_{\mathcal{O}(1)} \iff x \in f^*(a) \ .$$

Then if U_a is an open set of $\Sigma(A)$ we have

$$\bar{f}^{-1}(U_a) = \{x \in X \mid \bar{f}(x)(a) = 1\} = f^*(a) \in \Omega(X) \ ,$$

showing that $\bar{f}$ is continuous. It is equally easy to prove that the following diagram is commutative

$$\begin{array}{ccc} \Omega(\Sigma(A)) & \xrightarrow{\ \epsilon\ } & A \\ {\scriptstyle \Omega(\bar{f})} \uparrow & \nearrow_{f} & \\ \Omega(X) & & \end{array}$$

and that $\hat{f}$ is the only continuous map $X \to \Sigma(A)$ with this property. This concludes the proof. ∎

Definition III.1.22 A locale A is *spatial* if and only if the A component of the co-unit of the adjunction is an isomorphism of locales (equivalently, the assignment $a \mapsto U_a$ is an isomorphism of frames). A topological space X is *sober* if the X component of the unit of the adjunction is a homeomorphism.

Theorem III.1.23 *Let X be a topological space, and A a locale.*

1. *$\Omega(X)$ is a spatial locale.*
2. *$\Sigma(A)$ is a sober space.*
3. *The adjunction $\Omega \dashv \Sigma$ restricts to an equivalence of categories between sober spaces and spatial locales.*

Locales are often important as replacements for the notion of topological space in a *constructive* setting, by which is meant the ability to interpret definitions and theorems in an arbitrary topos. See [6, 7]. For instance, the locale $\mathrm{RIdl}(R)$ of radical ideals of a commutative ring R can be regarded as the "constructive Zariski spectrum" of R because $\Sigma(\mathrm{RIdl}(R))$ is (classically) homeomorphic to the usual space of prime ideals with the Zariski topology (cf. exercises below).

Exercise III.1.24

1. Show that the following two conditions are equivalent:
   - The locale A is spatial;
   - There is a topological space X such that $\Omega(X) \cong A$ in *Loc*.
2. Describe the unit of the adjunction $\Omega \dashv \Sigma$ explicitly.
3. Show that the points of a locale A can be identified with either of the following concepts:
   - *Completely prime filters* of $\mathcal{O}(A)$, by which are meant filters $F \subseteq \mathcal{O}(A)$ such that $\bigvee S \in F$ implies $a \in F$ for some $a \in S$ (these correspond to 1-kernels of the points $p^* : \mathcal{O}(A) \to \mathcal{O}(1)$).
   - *Prime elements* of $\mathcal{O}(A)$, by which are meant elements $p \in \mathcal{O}(A)$ such that $p \neq 1$ and such that the condition $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$, for all $a, b \in \mathcal{O}(A)$ (the prime elements are the joins of the 0-kernels of the points).
4. Describe the topology of the spectrum of a locale directly in terms of prime elements, and of completely prime filters.

5. Let X be a topological space. Show that the points of the locale $\Omega(X)$ can be identified with the (nonempty) irreducible closed sets of X (hint: these are the complements in X of the primes of $\Omega(X)$).

6. Describe the topology of the spectrum of a locale $\Omega(X)$ directly in terms of irreducible closed sets.

7. Show that every Hausdorff space is sober.

8. Give an example of a space which is not sober.

9. Let X and B be topological spaces, and let $f : X \to B$ be a local homeomorphism. Show that if B is sober then so is X.

10. Let X and B be locales, and let $f : X \to B$ be a local homeomorphism. Show that if B is spatial then so is X. (Hint: recall Exercise III.1.16-6.)

11. Let R be a commutative ring. Show that the prime ideals of R are precisely the prime elements of the locale of radical ideals $\mathrm{RIdl}(R)$. Show that the topology of $\Sigma\,\mathrm{RIdl}(R)$ is precisely the Zariski topology.

12. Let X be a compact Hausdorff space. Show that X is homeomorphic to the spectrum $\Sigma(\mathrm{I}(C(X)))$ of the locale $\mathrm{I}(C(X))$ of norm-closed ideals of the commutative C*-algebra $C(X)$ of continuous functions $X \to \mathbb{C}$.

13. Let $f : A \to B$ be a map of locales. Show that if f is a local homeomorphism then so is $\Sigma(f) : \Sigma(A) \to \Sigma(B)$. Show that the analogous assertion for open maps instead of local homeomorphisms is false.

14. Let G be a localic groupoid

$$G_2 \xrightarrow{m} G_1 \overset{i}{\circlearrowleft} \; \underset{d}{\overset{r}{\rightrightarrows}} \; \xleftarrow{u} G_0 .$$

Defining ΣG to be the structure defined by

$$\Sigma(G_2) \xrightarrow{\Sigma(m)} \Sigma(G_1) \overset{\Sigma(i)}{\circlearrowleft} \; \underset{\Sigma(d)}{\overset{\Sigma(r)}{\rightrightarrows}} \; \xleftarrow{\Sigma(u)} \Sigma(G_0)$$

show that $\Sigma(G)$ is a topological groupoid, and that in this way we define a functor Σ from the category of localic groupoids to the category of topological groupoids.

15. Show that if G is a localic étale groupoid then $\Sigma(G)$ is a topological étale groupoid.

16. Let X and Y be topological spaces. Show that if X is locally compact then the locale $\Omega(X) \otimes \Omega(Y)$ is spatial. Give an example showing that $\Omega(X) \otimes \Omega(Y)$ may fail to be spatial in general.

17. Show, based on the previous exercise, that a topological groupoid G does not necessarily give rise to a localic groupoid "$\Omega(G)$".

2 Quantal frames

We have already remarked that the topologies of certain topological groupoids are quantales. Besides this they are also frames, of course, hence suggesting the following definition:

Definition III.2.1 By a *quantal frame* is meant a quantale Q such that for all $a, b_i \in Q$ the following distributivity property holds:

$$a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i \ .$$

In this section we shall see how (localic) groupoids or at least categories can be obtained from suitable quantal frames. The main result (theorem III.2.19) states simply that the localic étale groupoids correspond bijectively, up to isomorphisms, to the inverse quantales that are also quantal frames. To a large extent this is a consequence of a bijective correspondence between these quantales and the abstract complete pseudogroups that furthermore is part of an equivalence of categories (theorem III.2.15). At the end, in section 2, we provide examples whose purpose is to separate all the classes of quantales considered so far.

Stable quantal frames. We begin by considering quantal frames of which nothing is required except that their underlying quantales should be stably supported. This condition can be expressed equivalently as follows:

Definition III.2.2 By a *stable quantal frame* will be meant a unital involutive quantal frame satisfying the following additional conditions:

$$\begin{aligned} a1 \wedge e &\leq aa^* \\ a &\leq (a1 \wedge e)a \ . \end{aligned}$$

[Equivalently, satisfying the equations $a1 \wedge e = aa^* \wedge e$ and $a = (a1 \wedge e)a$.]

Let Q be a stable quantal frame. The sup-lattice inclusion

$$\upsilon : \varsigma Q \to Q \ .$$

has a right adjoint given by

$$\upsilon_*(a) = a \wedge e \ ,$$

which preserves arbitrary joins and is the inverse image frame homomorphism of an (obviously) open locale map

$$u : G_0 \to G_1$$

whose direct image is $u_! = \upsilon$, where G_0 and G_1 are defined by the conditions

$$\mathcal{O}(G_0) = \varsigma Q \text{ and } \mathcal{O}(G_1) = Q \ .$$

Now consider the sup-lattice homomorphism $\delta : Q \to \varsigma Q$ defined by

$$\delta(a) = a1 \wedge e \ .$$

(This is just the support ς with codomain restricted to ςQ.)

Lemma III.2.3 *δ is the direct image $d_!$ of an open map $d : G_1 \to G_0$.*

Proof. Consider the map $\varsigma Q \to Q$ given by $a \mapsto a1$. By the properties of stably supported quantales this is an isomorphism $\varsigma Q \to \mathrm{R}(Q)$ followed by the inclusion $\mathrm{R}(Q) \to Q$. Hence, it is a frame homomorphism and it defines a map of locales $d : G_1 \to G_0$, which furthermore is semiopen with $\delta = d_!$ because d^* is the right adjoint δ_* of δ:

$$\begin{array}{l} \delta(d^*(a)) = \varsigma(a1) = \varsigma a = a \text{ for all } a \leq e \\ d^*(\delta(a)) = \varsigma a1 = a1 \geq a \text{ for all } a \in Q \ . \end{array}$$

In order to see that d is open we check the Frobenius reciprocity condition. Let $a, b \in Q$, with $b \leq e$. Then

$$\begin{array}{rcll} d_!(d^*(b) \wedge a) & = & \varsigma(b1 \wedge a) & \\ & = & \varsigma(ba) & \text{(By II.3.5-5 and } \varsigma b = b.) \\ & = & b\varsigma a & \text{(By II.3.8-2.)} \\ & = & b \wedge \varsigma a & (\varsigma Q \text{ is a locale.}) \\ & = & b \wedge d_!(a) \ . & \blacksquare \end{array}$$

The involution of Q is a frame isomorphism that defines a locale map $i : G_1 \to G_1$ by the condition $i^*(a) = a^*$, and thus we have $i \circ i = \mathrm{id}$ and $i_! = i^*$. Our aim is that ultimately d should be the domain map of a groupoid, and we obtain a candidate for an open range map

$$r : G_1 \to G_0$$

just by defining $r = d \circ i$. These maps satisfy the appropriate relations:

Lemma III.2.4 *Consider the locale maps*

$$i \circlearrowleft G_1 \begin{array}{c} \xrightarrow{\;d\;} \\ \xleftarrow{\;u\;} \\ \xrightarrow[\;r\;]{} \end{array} G_0$$

as defined above. We have

$$\begin{aligned} d \circ u &= \mathrm{id} \\ r \circ u &= \mathrm{id} \\ d \circ i &= r \\ r \circ i &= d\ . \end{aligned}$$

Proof. The first condition is equivalent to $u^* \circ d^* = \mathrm{id}$, which holds because for all $a \le e$ we have $\varsigma a = a$, and thus

$$u^*(d^*(a)) = a1 \wedge e = \varsigma a = a\ .$$

Similarly, the second condition is true: for all $a \le e$ we have $a = a^*$, and thus

$$u^*(r^*(a)) = \varsigma(a^*) = \varsigma a = a\ .$$

The third condition is the definition of r, and the fourth follows from this because $i \circ i = \mathrm{id}$. ∎

So far we have obtained from the stable quantal frame Q a localic graph G that is equipped with an involution i, and whose maps d, r, and u, are all open. There is additional structure on G, consisting of a certain kind of multiplication defined on the "locale of composable pairs of edges" $G_1 \times_{G_0} G_1$, although not (yet) necessarily the multiplication of a groupoid or even of a category. In order to see this let us first notice that the frame $\mathcal{O}(G_0) = \varsigma Q$ is, as a quantale, a unital subquantale of Q because meet in ςQ coincides with multiplication in Q (see II.3.3). Hence, there are two immediate ways in which Q is a module over ςQ: multiplication on the left defines an action of ςQ on Q, and multiplication on the right defines another. We shall regard Q as an ςQ-ςQ-bimodule with respect to these two actions, namely letting the left (resp. right) action be left (resp. right) multiplication (in fact each action makes Q both a right and a left ςQ-module because ςQ is a commutative quantale, but this is irrelevant). We shall denote the corresponding tensor product over ςQ by $Q \otimes_{\varsigma Q} Q$. This coincides (fortunately for our notation) with $\mathcal{O}(G_1 \times_{G_0} G_1)$:

Lemma III.2.5 *The tensor product $Q \otimes_{\varsigma Q} Q$ coincides with the pushout of the homomorphisms d^* and r^*.*

Proof. This is equivalent to showing, for all $a, b, c \in Q$, with $a \leq e$, that the equality

$$(b \wedge r^*(a)) \otimes c = b \otimes (d^*(a) \wedge c)$$

is equivalent to

$$ba \otimes c = b \otimes ac \;,$$

which is immediate from II.3.5-5:

$$d^*(a) \wedge c = a1 \wedge c = \varsigma ac = ac \;.$$

[For r^* it is analogous, using the obvious dual of II.3.5-5.] ∎

Now we notice that the quantale multiplication $Q \otimes Q \to Q$ factors through the above pushout because it is associative and thus in particular it respects the relations $ba \otimes c = b \otimes ac$ for $a \leq e$ that determine the quotient $Q \otimes Q \to Q \otimes_{\varsigma Q} Q$. This enables us to make the following definition:

Definition III.2.6 Let Q be a stable quantal frame, and let

$$G = \; i \circlearrowleft G_1 \underset{r}{\overset{d}{\underset{\longleftarrow u \longrightarrow}{\rightrightarrows}}} G_0$$

be the corresponding involutive localic graph. The *quantal multiplication induced by Q on G* is the sup-lattice homomorphism

$$\mu : \mathcal{O}(G_1 \underset{G_0}{\times} G_1) \to \mathcal{O}(G_1)$$

which is defined by, for all $a, b \in Q$,

$$\mu(a \otimes b) = ab \;.$$

Multiplicative quantal frames. Let us now examine a condition under which the localic graph associated to a stable quantal frame has the additional structure of a localic category.

Definition III.2.7 By a *multiplicative quantal frame* will be meant a stable quantal frame for which the right adjoint μ_* of the quantal multiplication $\mu : Q \otimes_{\varsigma Q} Q \to Q$ preserves arbitrary joins.

The multiplicativity condition is that under which μ is the direct image $m_!$ of a (semiopen) locale map m, which then gives us a category, as the following theorem shows.

Theorem III.2.8 *Let Q be a multiplicative quantal frame. Then the locale map*

$$m : G_1 \underset{G_0}{\times} G_1 \to G_1$$

which is defined by $m^* = \mu_*$ (equivalently, $m_! = \mu$), together with the maps d, r, and u, defines a localic category.

Proof. In III.2.4 we have obtained many of the needed conditions. The only ones missing are the unit laws, the associativity of m, and those that specify the domain and range of a product of two arrows:

$$d \circ m = d \circ \pi_1 \tag{III.2.9}$$

$$r \circ m = r \circ \pi_2 \; . \tag{III.2.10}$$

We shall begin by proving these. In fact we shall prove (III.2.9) only, as (III.2.10) is analogous. We have to show, for frame homomorphisms, that $m^* \circ d^* = \pi_1^* \circ d^*$. In order to do this we shall prove that $d_! \circ m_!$ is left adjoint to $\pi_1^* \circ d^*$ (this identifies $m^* \circ d^*$ and $\pi_1^* \circ d^*$ because adjoints between partial orders are uniquely determined), in order to take advantage of the following simple formulas:

$$\begin{aligned} d_!(m_!(a \otimes b)) &= \varsigma(ab) \\ \pi_1^*(d^*(a)) &= a1 \otimes 1 \qquad (a \le e) \end{aligned}$$

We prove that the adjunction exists by proving the following two inequalities (resp. the co-unit and the unit of the adjunction):

$$\begin{aligned} d_!(m_!(\pi_1^*(d^*(a)))) &\le a \text{ for all } a \le e \\ \pi_1^*(d^*(d_!(m_!(a \otimes b)))) &\ge a \otimes b \text{ for all } a, b \in Q \; . \end{aligned}$$

Let us prove the first inequality. Consider $a \leq e$. Then

$$d_!(m_!(\pi_1^*(d^*(a)))) = d_!(m_!(a1 \otimes 1)) = \varsigma(a11) = \varsigma(a1) = \varsigma a = a\ .$$

Now let us prove the second inequality. Let $a, b \in Q$. Then

$$\begin{aligned} \pi_1^*(d^*(d_!(m_!(a \otimes b)))) &= \pi_1^*(d^*(\varsigma(ab))) = \varsigma(ab)1 \otimes 1 = \\ &= ab1 \otimes 1 = a\varsigma b1 \otimes 1 \geq a\varsigma b \otimes 1 = \\ &= a \otimes \varsigma b1 = a \otimes b1 \geq a \otimes b\ . \end{aligned}$$

Hence, (III.2.9) holds, and for analogous reasons so does (III.2.10).

Let us now prove the unit laws, which state that the following diagram is commutative:

(III.2.9)
$$\begin{array}{ccccc} G_0 \underset{G_0}{\times} G_1 & \xrightarrow{u\times \mathrm{id}} & G_1 \underset{G_0}{\times} G_1 & \xleftarrow{\mathrm{id}\times u} & G_1 \underset{G_0}{\times} G_0 \\ {\scriptstyle\langle d,\mathrm{id}\rangle}\uparrow & & \downarrow{\scriptstyle m} & & \uparrow{\scriptstyle\langle \mathrm{id},r\rangle} \\ G_1 & = & G_1 & = & G_1 \end{array}$$

First we remark that, since the frame homomorphism u^* has a left adjoint $u_!$, the maps $u \times \mathrm{id}$ and $\mathrm{id} \times u$ are semiopen, with

$$\begin{aligned} (u \times \mathrm{id})_! &= u_! \otimes \mathrm{id} \\ (\mathrm{id} \times u)_! &= \mathrm{id} \otimes u_!\ , \end{aligned}$$

because the operations $\mathrm{id} \otimes -$ and $- \otimes \mathrm{id}$ are functorial and thus preserve the conditions $u_! \circ u^* \leq \mathrm{id}$ and $u^* \circ u_! \geq \mathrm{id}$ that define the adjunction $u_! \dashv u^*$. Secondly, the maps $\langle d, \mathrm{id}\rangle$ and $\langle \mathrm{id}, r\rangle$ are isomorphisms whose inverses are, respectively, the projections $\pi_2 : G_0 \times_{G_0} G_1 \to G_1$ and $\pi_1 : G_1 \times_{G_0} G_0 \to G_1$. Hence, in particular, these maps are semiopen, and we have

$$\begin{aligned} \langle d, \mathrm{id}\rangle_! &= \pi_2^* \\ \langle \mathrm{id}, r\rangle_! &= \pi_1^*\ . \end{aligned}$$

Hence, since m^*, too, has a left adjoint $m_!$, we conclude that the commutativity of (III.2.9) is equivalent to that of the following diagram:

(III.2.9)
$$\begin{array}{ccccc} \mathcal{O}(G_0 \underset{G_0}{\times} G_1) & \xrightarrow{u_!\otimes \mathrm{id}} & \mathcal{O}(G_1 \underset{G_0}{\times} G_1) & \xleftarrow{\mathrm{id}\otimes u_!} & \mathcal{O}(G_1 \underset{G_0}{\times} G_0) \\ {\scriptstyle\pi_2^*}\uparrow & & \downarrow{\scriptstyle m_!} & & \uparrow{\scriptstyle\pi_1^*} \\ \mathcal{O}(G_1) & = & \mathcal{O}(G_1) & = & \mathcal{O}(G_1) \end{array}$$

Now the commutativity of the left square of (III.2.9), i.e., the condition

$$m_! \circ (u_! \otimes \mathrm{id}) \circ \pi_2^* = \mathrm{id} ,$$

follows from the fact that Q is a unital quantale with $e = u_!(1_{G_0})$, since for all $a \in Q$ we obtain

$$(m_! \circ (u_! \otimes \mathrm{id}) \circ \pi_2^*)(a) = m_! \circ (u_! \otimes \mathrm{id})(1_{G_0} \otimes a) = m_!(u_!(1_{G_0}) \otimes a) = ea = a .$$

Similarly, the right square of (III.2.9) follows from $ae = a$.

Finally, we shall prove that the multiplication map m is associative. Let $m' : \mathcal{O}(G_1) \otimes \mathcal{O}(G_1) \to \mathcal{O}(G_1)$ be the quantale multiplication,

$$m' = m_! \circ q ,$$

where q is is the quotient homomorphism

$$q : \mathcal{O}(G_1) \otimes \mathcal{O}(G_1) \to \mathcal{O}(G_1) \underset{\mathcal{O}(G_0)}{\otimes} \mathcal{O}(G_1) .$$

It is clear that the associativity of m', which is equivalent to the commutativity of

$$\begin{array}{ccc}
(\mathcal{O}(G_1) \otimes \mathcal{O}(G_1)) \otimes \mathcal{O}(G_1) & \xrightarrow{m' \otimes \mathrm{id}} & \mathcal{O}(G_1) \otimes \mathcal{O}(G_1) \\
\cong \downarrow & & \\
\mathcal{O}(G_1) \otimes (\mathcal{O}(G_1) \otimes \mathcal{O}(G_1)) & & \downarrow m' \\
\mathrm{id} \otimes m' \downarrow & & \\
\mathcal{O}(G_1) \otimes \mathcal{O}(G_1) & \xrightarrow{m'} & \mathcal{O}(G_1) ,
\end{array}$$

implies (in fact it is equivalent to) the commutativity of

$$\begin{array}{ccc}
(\mathcal{O}(G_1) \underset{\mathcal{O}(G_0)}{\otimes} \mathcal{O}(G_1)) \underset{\mathcal{O}(G_0)}{\otimes} \mathcal{O}(G_1) & \xrightarrow{m_! \otimes \mathrm{id}} & \mathcal{O}(G_1) \underset{\mathcal{O}(G_0)}{\otimes} \mathcal{O}(G_1) \\
\cong \downarrow & & \\
\mathcal{O}(G_1) \underset{\mathcal{O}(G_0)}{\otimes} (\mathcal{O}(G_1) \underset{\mathcal{O}(G_0)}{\otimes} \mathcal{O}(G_1)) & & \downarrow m_! \\
\mathrm{id} \otimes m_! \downarrow & & \\
\mathcal{O}(G_1) \underset{\mathcal{O}(G_0)}{\otimes} \mathcal{O}(G_1) & \xrightarrow{m_!} & \mathcal{O}(G_1) ,
\end{array}$$

as the following diagram chase shows:

$$\begin{array}{ccccc}
(a\otimes b)\otimes c & \xmapsto{m_!\otimes \mathrm{id}} & & & (ab)\otimes c \\
\cong \downarrow & & & & \\
a\otimes(b\otimes c) & & & & \downarrow m_! \\
\mathrm{id}\otimes m_! \downarrow & & & & \\
a\otimes(bc) & \xmapsto{m_!} & a(bc) & = & (ab)c\ .
\end{array}$$

Taking the right adjoints of all the above morphisms gives us the frame version of the associativity of m,

$$\begin{array}{ccc}
(G_1 \times_{G_0} G_1) \times_{G_0} G_1 & \xrightarrow{m\times \mathrm{id}} & G_1 \times_{G_0} G_1 \\
\cong \downarrow & & \\
G_1 \times_{G_0} (G_1 \times_{G_0} G_1) & & \downarrow m \\
\mathrm{id}\times m \downarrow & & \\
G_1 \times_{G_0} G_1 & \xrightarrow{m} & G_1\ ,
\end{array}$$

because, similarly to what we have argued for u, we have

$$\begin{aligned}
(m\times \mathrm{id})_! &= m_!\otimes \mathrm{id} \\
(\mathrm{id}\times m)_! &= \mathrm{id}\otimes m_!\ . \qquad \blacksquare
\end{aligned}$$

Now we see a fundamental example of multiplicative quantal frame.

Theorem III.2.9 *Let S be an abstract complete pseudogroup. Then $\mathcal{L}^\vee(S)$ is a multiplicative quantal frame.*

Proof. The right adjoint of the quantal multiplication,

$$\mu_* : \mathcal{L}^\vee(S) \to \mathcal{L}^\vee(S) \otimes_{\varsigma\mathcal{L}^\vee(S)} \mathcal{L}^\vee(S)\ ,$$

is given by the formula

$$\mu_*(U) = \bigvee\{V\otimes W \mid VW\subseteq U\}\ .$$

Due to the universal property of $\mathcal{L}^\vee(S)$ as a sup-lattice, the question of whether $\mathcal{L}^\vee(S)$ is multiplicative, that is of whether μ_* preserves joins, is equivalent to asking whether the map

$$f : S \to \mathcal{L}^\vee(S) \otimes_{\varsigma\mathcal{L}^\vee(S)} \mathcal{L}^\vee(S)$$

defined by

$$f(x) = \bigvee\{\downarrow y \otimes \downarrow z \mid yz \le x\}$$

preserves all the joins that exist, in which case μ_* is the unique homomorphic extension of f; that is, for each compatible set $X \subseteq S$ we need to see that

$$f(\bigvee X) \subseteq \bigvee f(X) .$$

Equivalently, we need to see that $yz \le \bigvee X$ implies $\downarrow y \otimes \downarrow z \subseteq \bigvee f(X)$ for all $y, z \in S$.

Consider then $y, z \in S$ such that $yz \le \bigvee X$. For each $x \in X$ we have $yzx^{-1}x \le x$ because

$$yzx^{-1}x \le (\bigvee X)x^{-1}x = \bigvee_{w\in X} wx^{-1}x \le x ,$$

where the equality is a consequence of distributivity, and the last inequality follows from the fact that X is compatible and therefore $wx^{-1} \in E(S)$ for all $w \in X$. Hence,

$$y(y^{-1}yzx^{-1}x) = (yy^{-1}y)zx^{-1}x = yzx^{-1}x \le x$$

and thus, by definition of f, we obtain

$$\downarrow y \otimes \downarrow(y^{-1}yzx^{-1}x) \subseteq f(x) .$$

From here, using distributivity, it follows that

$$\begin{aligned}
\downarrow y \otimes \downarrow \left(y^{-1}yz \bigvee_{x\in X} x^{-1}x\right) &= \downarrow y \otimes \downarrow \left(\bigvee_{x\in X} y^{-1}yzx^{-1}x\right) \\
&= \downarrow y \otimes \left(\bigvee_{x\in X} \downarrow(y^{-1}yzx^{-1}x)\right) \\
&= \bigvee_{x\in X} \downarrow y \otimes \downarrow(y^{-1}yzx^{-1}x) \\
&\subseteq \bigvee f(X) .
\end{aligned}$$

Since $\bigvee_x x^{-1}x = (\bigvee X)^{-1}(\bigvee X)$ [cf. I.2.7], we further conclude that

$$yz \bigvee_{x\in X} x^{-1}x = yz\left(\bigvee X\right)^{-1}\left(\bigvee X\right) = yz$$

(the last equality follows from the fact that for any elements a and b of an inverse semigroup the condition $a \leq b$ implies $a = ab^{-1}b$), and thus

$$\begin{aligned} \downarrow y \otimes \downarrow z &= \downarrow(yy^{-1}y) \otimes \downarrow z = \downarrow y \downarrow(y^{-1}y) \otimes \downarrow z = \downarrow y \otimes \downarrow(y^{-1}y) \downarrow z \\ &= \downarrow y \otimes \downarrow(y^{-1}yz) = \downarrow y \otimes \downarrow \left(y^{-1}yz \bigvee_{x \in X} x^{-1}x \right) \subseteq \bigvee f(X) \,. \quad \blacksquare \end{aligned}$$

Inverse quantal frames. Now we shall prove some facts about those quantal frames that are also inverse quantales. In particular we shall see that such quantal frames are necessarily multiplicative and of the form $\mathcal{L}^\vee(S)$, up to isomorphism, and that this gives us a category which is equivalent to the category of abstract complete pseudogroups *ACPGrp*.

Definition III.2.10 By an *inverse quantal frame* Q will be meant a supported quantal frame whose maximum is a join of partial units:

$$1 = \bigvee \mathcal{I}(Q) \,.$$

We remark that any inverse quantal frame Q is an inverse quantale in the sense of our original definition, due to distributivity: if $a \in Q$ then

$$a = a \wedge 1 = a \wedge \bigvee \mathcal{I}(Q) = \bigvee \{a \wedge s \mid s \in \mathcal{I}(Q)\} \,,$$

where each $a \wedge s$ is of course a partial unit. Hence, in particular, any inverse quantal frame is a stable quantal frame.

Recall the adjunction $\mathcal{L}^\vee \dashv \mathcal{I}$ of II.3.27, between the category *ACPGrp* of abstract complete pseudogroups and the category *StabQu* of stably supported quantales. For each stably supported quantale Q we shall denote by

$$\varepsilon_Q : \mathcal{L}^\vee(\mathcal{I}(Q)) \to Q$$

the corresponding component of the co-unit of the adjunction. This is a homomorphism of unital involutive quantales that is explicitly defined by

$$\varepsilon_Q(U) = \bigvee U \,.$$

In the following results we shall apply again the technique of Exercise III.1.8-7 for the sheaf topology:

Lemma III.2.11 *Let Q be an inverse quantal frame. Then ε_Q is a surjective frame homomorphism whose restriction to the set of principal ideals of $\mathcal{L}^\vee(\mathcal{I}(Q))$ is injective.*

Proof. ε_Q is surjective because Q is an inverse quantale. Hence, it remains to show that ε_Q preserves binary meets. First, $\mathcal{I}(Q)$ is an abstract complete pseudogroup and thus in particular it is a meet semilattice. The fact that ε_Q preserves binary meets is now an essentially immediate consequence of the coverage theorem for frames [8] (with minor adaptations due to the possible absence of a maximum in $\mathcal{I}(Q)$), but a direct proof using the explicit formula for the co-unit is also immediate and we give it here: for any $U, V \in \mathcal{L}^{\vee}(\mathcal{I}(Q))$ we have

$$U \cap V = \{s \wedge t \mid s \in U,\ t \in V\}\ ,$$

and thus using the frame distributivity of Q we obtain

$$\begin{aligned} \varepsilon_Q(U \cap V) &= \varepsilon_Q(\{s \wedge t \mid s \in U,\ t \in V\}) = \bigvee_{s \in U,\ t \in V} s \wedge t \\ &= \bigvee U \wedge \bigvee V = \varepsilon_Q(U) \wedge \varepsilon_Q(V)\ . \end{aligned}$$

Finally, the restriction of ε_Q to principal ideals is the assignment

$$\downarrow s \mapsto \bigvee \downarrow s = s\ ,$$

and, of course, this is an order embedding. ∎

Lemma III.2.12 *Let S be an abstract complete pseudogroup. The principal ideals of $\mathcal{L}^{\vee}(S)$ form a downwards closed set.*

Proof. Let $s \in S$, and let $U \in \mathcal{L}^{\vee}(S)$ be such that $U \subseteq \downarrow s$. For all $t, u \in U$ we have $tu^{-1} \leq ss^{-1} \leq e$, and, similarly, $t^{-1}u \leq e$; that is, t and u are compatible and we conclude that U is a compatible subset of S. Since S is complete the join $\bigvee U$ exists in S, and since U is closed under joins it must contain $\bigvee U$. Hence, U is the principal ideal $\downarrow \bigvee U$. ∎

Theorem III.2.13 *Let Q be an inverse quantal frame. Then there is an isomorphism*

$$\mathcal{L}^{\vee}(\mathcal{I}(Q)) \cong Q$$

of unital involutive quantales.

Proof. ε_Q is a homomorphism of unital involutive quantales, and the fact that it is an isomorphism follows immediately from III.1.7 and the previous two lemmas. ∎

Corollary III.2.14 *Any inverse quantal frame is multiplicative.*

Denoting by *InvQuF* the full subcategory of Qu_e^* whose objects are the inverse quantal frames, we have:

Theorem III.2.15 *The categories ACPGrp and InvQuF are equivalent.*

Proof. From III.2.13 it follows that the adjunction $\mathcal{L}^\vee \dashv \mathcal{I}$ between *ACPGrp* and *InvQu* restricts to a reflection between *ACPGrp* and *InvQuF*. Now let S be an abstract complete pseudogroup. The unit of the adjunction gives us the injective homomorphism of abstract complete pseudogroups

$$\eta_S : S \to \mathcal{I}(\mathcal{L}^\vee(S))$$

defined by $s \mapsto \downarrow s$, and in order to prove that the reflection is in fact an equivalence it remains to see that η_S is surjective. Let then $U \in \mathcal{I}(\mathcal{L}^\vee(S))$. By definition of partial unit this is an element of $\mathcal{L}^\vee(S)$ such that $UU^* \subseteq E(S)$ and $U^*U \subseteq E(S)$. Hence, $st^{-1} \in E(S)$ and $s^{-1}t \in E(S)$ for all $s, t \in U$, which means that U is a compatible subset of S. Hence, again as in III.2.12, U must coincide with the principal ideal $\downarrow \bigvee U$; that is, $\eta_S(\bigvee U) = U$, showing that η_S is surjective. ∎

Étale groupoids from quantales. Now we determine the conditions under which the category associated to a multiplicative quantal frame Q is a groupoid. As we shall see, this happens if and only if Q is an inverse quantal frame. A first step is given by the following result, which basically produces a straightforward translation of the inversion law of groupoids into the language of quantales.

Lemma III.2.16 *Let Q be a multiplicative quantal frame. The localic category $(G_1, G_0, d, r, u, m, i)$ associated to Q is a groupoid (with inversion i) if and only if Q satisfies the following two conditions, for all $a \in Q$:*

$$(a \wedge e)1 = \bigvee_{xy^* \leq a} x \wedge y \,, \tag{III.2.17}$$

$$1(a \wedge e) = \bigvee_{x^*y \leq a} x \wedge y \,. \tag{III.2.18}$$

Proof. Recall the two groupoid inverse laws, namely the commutativity of the following diagram:

$$\begin{array}{ccccc}
G_1 & \xrightarrow{\langle \mathrm{id}, i\rangle} & G_1 \times_{G_0} G_1 & \xleftarrow{\langle i, \mathrm{id}\rangle} & G_1 \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle r} \\
G_0 & \xrightarrow[u]{} & G_1 & \xleftarrow[u]{} & G_0 .
\end{array}$$

Consider its dual frame version:

(III.2.17)
$$\begin{array}{ccccc}
\mathcal{O}(G_1) & \xleftarrow{[\mathrm{id},i^*]} & \mathcal{O}(G_1 \times_{G_0} G_1) & \xrightarrow{[i^*,\mathrm{id}]} & \mathcal{O}(G_1) \\
\uparrow d^* & & \uparrow m^* & & \uparrow r^* \\
\mathcal{O}(G_0) & \xleftarrow[u^*]{} & \mathcal{O}(G_1) & \xrightarrow[u^*]{} & \mathcal{O}(G_0)
\end{array}$$

The commutativity of the left square of (III.2.17) is equivalent, for each $a \in Q = \mathcal{O}(G_1)$, to the equation

(III.2.17)
$$d^*(u^*(a)) = [\mathrm{id}, i^*](m^*(a)) \ .$$

Taking into account the following formulas, for all $b, x, y \in Q$,

$$\begin{array}{rcl}
i^*(b) & = & b^* \\
d^*(u^*(b)) & = & (b \wedge e)1 \\
m^*(b) & = & \bigvee\{x \otimes y \mid xy \leq b\} \\
[f,g](x \otimes y) & = & f(x) \wedge g(y) \ ,
\end{array}$$

we see that (III.2.17) is equivalent to

$$(a \wedge e)1 = \bigvee_{xy \leq a} x \wedge y^* \ ,$$

which is equivalent to (III.2.17). Similarly, the right square of the diagram (III.2.17) is equivalent to (III.2.18). ∎

The following two lemmas are motivated by the equations (III.2.17) and (III.2.18). We remark that, even though the equations have been introduced in the context of multiplicative quantal frames, the lemmas hold for more general quantal frames.

Lemma III.2.17 *Let Q be a stable quantal frame. Then for all $a \in Q$ the following inequalities hold:*

(III.2.18)
$$(a \wedge e)1 \geq \bigvee_{xy^* \leq a} x \wedge y$$

(III.2.19)
$$1(a \wedge e) \geq \bigvee_{x^*y \leq a} x \wedge y \ .$$

Proof. Recall the property (II.3.18) of supported quantales:

$$\varsigma(x \wedge y) \leq xy^* .$$

The support ς coincides with the sup-lattice homomorphism

$$u_! \circ d_! : Q \to Q ,$$

and thus from (II.3.18) we obtain, by adjointness, $x \wedge y \leq d^*(u^*(xy^*))$. This is equivalent to the statement that $x \wedge y \leq d^*(u^*(a))$ for all $a \in Q$ such that $xy^* \leq a$, and thus we obtain

$$d^*(u^*(a)) \geq \bigvee_{xy^* \leq a} x \wedge y .$$

Then (III.2.18) is a consequence of this and of the equality $(a \wedge e)1 = d^*(u^*(a))$, and (III.2.19) is proved analogously taking into account that $1(a \wedge e) = r^*(u^*(a))$ and, again using (II.3.18), $u_!(r_!(x \wedge y)) = u_!(d_!((x \wedge y)^*)) = \varsigma(x^* \wedge y^*) \leq x^* y$. ∎

This has as a consequence that the conditions (III.2.17) and (III.2.18) are equivalent, for any stable quantal frame, to the following lax version of them,

$$(a \wedge e)1 \leq \bigvee_{xy^* \leq a} x \wedge y \qquad \text{(III.2.18)}$$

$$1(a \wedge e) \leq \bigvee_{x^* y \leq a} x \wedge y , \qquad \text{(III.2.19)}$$

leading us to the second lemma:

Lemma III.2.18 *Let Q be a unital involutive quantal frame. Then Q satisfies the two conditions (III.2.18) and (III.2.19) if and only if $\bigvee \mathcal{I}(Q) = 1$.*

Proof. Let us assume that $\bigvee \mathcal{I}(Q) = 1$ and prove from there that Q satisfies (III.2.18):

$$\begin{aligned}
(a \wedge e)1 &= (a \wedge e) \bigvee \{x \mid xx^* \leq e \text{ and } x^* x \leq e\} \\
&\leq (a \wedge e) \bigvee \{x \mid xx^* \leq e\} \\
&= \bigvee \{(a \wedge e)x \mid xx^* \leq e\} \\
&\leq \bigvee \{(a \wedge e)x \mid xx^* a \leq a\} \\
&= \bigvee \{(a \wedge e)^* x \mid xx^* a \leq a\} \\
&\leq \bigvee \{a^* x \wedge x \mid xx^* a \leq a\} \\
&= \bigvee \{a^* x \wedge x \mid x(a^* x)^* \leq a\} \\
&\leq \bigvee \{x \wedge y \mid xy^* \leq a\} .
\end{aligned}$$

Proving (III.2.19) is done in an analogous way, but in the beginning retaining the inequality $x^*x \le e$ instead of $xx^* \le e$.

For the converse let us assume that both (III.2.18) and (III.2.19) hold, and from there let us prove that $\bigvee \mathcal{I}(Q) = 1$. From (III.2.18) we obtain

$$\begin{aligned} 1 = (e \wedge e)1 &\le \bigvee \{x \wedge y \mid xy^* \le e\} \\ &\le \bigvee \{x \wedge y \mid (x \wedge y)(x \wedge y)^* \le e\} = \bigvee \{x \mid xx^* \le e\} , \end{aligned}$$

and, similarly, from (III.2.19) we obtain

$$1 \le \bigvee \{y \mid y^*y \le e\} .$$

Hence,

$$\begin{aligned} 1 &\le \bigvee \{x \mid xx^* \le e\} \wedge \bigvee \{y \mid y^*y \le e\} \\ &= \bigvee \{x \wedge y \mid xx^* \le e \text{ and } y^*y \le e\} \\ &\le \bigvee \{x \wedge y \mid (x \wedge y)(x \wedge y)^* \le e \text{ and } (x \wedge y)^*(x \wedge y) \le e\} \\ &= \bigvee \{x \mid xx^* \le e \text{ and } x^*x \le e\} = \bigvee \mathcal{I}(Q) . \quad \blacksquare \end{aligned}$$

We finally arrive at the main result of this section.

Theorem III.2.19 *The following conditions are equivalent.*

1. *Q is an inverse quantal frame.*
2. *Q is a multiplicative quantal frame and the category associated to Q is a groupoid.*
3. *Q is a multiplicative quantal frame and the category associated to Q is an étale groupoid.*

Proof. We already know that inverse quantal frames are multiplicative, and it is clear from the previous three lemmas that the category associated to a multiplicative quantal frame Q is a groupoid if and only if Q is an inverse quantal frame. What remains to be proved is therefore that this groupoid is necessarily étale. Let then Q be an inverse quantal frame. Then $\mathcal{I}(Q)$ is a cover, and, since we already know that d is open, in order to show that d is a local homeomorphism it suffices to prove, for each $a \in \mathcal{I}(Q)$, that the frame homomorphism

$$f = ((-) \wedge a) \circ d^* : \varsigma Q \to \downarrow a$$

is surjective. Let $x \in \downarrow a$. Then $xx^* \in \varsigma Q$ because x is a partial unit. The conclusion that f is surjective follows from the fact that $f(xx^*) = x$:

- By (II.3.13) we have $xx^*1 = x1$. Then $x \le xx^*1$, and thus $x \le xx^*1 \wedge a = f(xx^*)$;
- By II.3.5-5 we have $f(xx^*) = xx^*1 \wedge a = xx^*a \le xa^*a \le xe = x$. ∎

Definition III.2.20 Let Q be an inverse quantal frame. We denote its associated localic étale groupoid by $\mathcal{G}(Q)$.

Separating examples. We have studied various kinds of quantales, and in particular we have obtained the following inclusions:

$$\begin{array}{ccccc} \left\{\begin{array}{c}\text{inverse}\\ \text{quantales}\end{array}\right\} & & \subset & & \left\{\begin{array}{c}\text{stably}\\ \text{supported}\\ \text{quantales}\end{array}\right\} \\ \cup & & & & \cup \\ \left\{\begin{array}{c}\text{inverse}\\ \text{quantal}\\ \text{frames}\end{array}\right\} & \subset & \left\{\begin{array}{c}\text{multiplicative}\\ \text{quantal}\\ \text{frames}\end{array}\right\} & \subset & \left\{\begin{array}{c}\text{stable}\\ \text{quantal}\\ \text{frames}\end{array}\right\}. \end{array} \tag{III.2.21}$$

The examples that follow show that all the inclusions are strict.

Example III.2.21 A stable quantal frame which is not multiplicative is the commutative quantale $Q = \wp(X)$ with

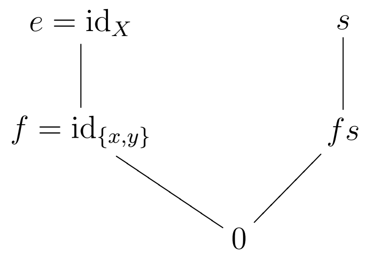
- $X = \{1, x\}$ (with $1 \neq x$),
- trivial involution,
- $\varsigma U = \{1\}$ for all $U \neq \emptyset$,
- $e = \{1\}$,
- multiplication defined on the atom $\{x\}$ by the condition $\{x\}\{x\} = \{1, x\}$ (and freely extended to unions in each variable).

This also shows that not every stably supported quantale is an inverse quantale.

Example III.2.22 Let M be the idempotent ordered monoid which, besides the unit 1, contains only one additional element x such that $1 \le x$. The set $Q = \mathcal{L}(M)$ of downwards closed subsets of M is an idempotent unital quantale under pointwise multiplication, with $e = \{1\}$, and it is commutative. With trivial involution, and with a support defined by $\varsigma(U) = \{1\}$ for all $U \neq \emptyset$, we obtain a multiplicative quantal frame that is not an inverse quantale because M is not a union of partial units.

We remark that this example differs from the previous one because the quantale is not a powerset. There is a good reason for this: if the powerset of (the set of arrows of) a small discrete involutive category is a supported quantale at all, then it is necessarily an inverse quantale. This is because the axiom $\varsigma a \leq aa^*$ alone forces the category to be a groupoid, since on singletons the axiom gives us $d(x) = \varsigma(\{x\}) \subseteq \{xx^*\}$, i.e., $d(x) = xx^*$, and also $r(x) = d(x^*) = x^*x$, and thus the involution of the category coincides with inversion: $x^* = x^{-1}$.

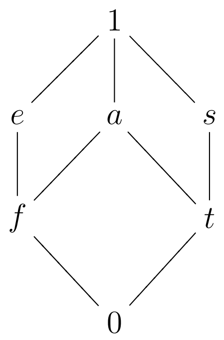
Example III.2.23 Now we present an example of an inverse quantale that is not a frame, hence showing that the two vertical inclusions of the diagram (III.2.21) are strict. Consider a non-T_0 topological space $X = \{x, y, z\}$ with three open sets $\emptyset$, $\{x, y\}$, and X. This space has exactly one non-idempotent automorphism, namely the bijection s that permutes x and y. Hence, the order structure of its pseudogroup $S = \mathcal{I}(X)$ is as follows:



The multiplication of S is commutative, it is defined by the conditions $s^2 = e$ and $E(S) = \{0, f, e\}$, and each element is its own inverse. The inverse quantal frame $Q = \mathcal{L}^{\vee}(S)$, which in this example coincides with $\mathcal{L}(S)$, has nine elements (we write s instead of $\downarrow s$, $f \vee s$ instead of $\downarrow f \cup \downarrow s$, etc.), namely 0, f, $t = fs$, e, $a = f \vee fs$, s, $b = e \vee fs$, $c = f \vee s$, and $1 = e \vee s$, where of course we have $a = f1$. It is now straightforward to obtain the multiplication table of Q; we present only the upper triangle because Q is commutative:

	0	f	t	e	a	s	b	c	1
0	0	0	0	0	0	0	0	0	0
f		f	t	f	a	t	a	a	a
t			f	t	a	f	a	a	a
e				e	a	s	b	c	1
a					a	a	a	a	a
s						e	c	b	1
b							b	c	1
c								b	1
1									1

Now consider the equivalence relation θ on Q whose only non-singular equivalence class is $\{b, c, 1\}$. The rightmost three entries of each line of the table are always equivalent, which means that θ is a congruence for the multiplication. Similarly, any join of an element of Q with either b, c or 1 necessarily produces an element in $\{b, c, 1\}$ (because b and c are maximal elements of Q), and thus θ is also a congruence for binary joins (and hence for all joins because Q is finite). Since θ is trivially also a congruence for the involution, the quotient Q/θ is a unital involutive quantale with seven elements ordered as follows:



This lattice is not distributive (for instance we have $s \wedge (e \vee a) = s \wedge 1 = s$ and $(s \wedge e) \vee (s \wedge a) = 0 \vee t = t \neq s$), but it is a supported quantale because θ is a congruence also with respect to the support, since in Q we have $\varsigma b = \varsigma c = \varsigma 1 = e$. Hence, Q/θ is an inverse quantale but not an inverse quantal frame.

3 Groupoid quantales

In section 2 we have described the class of inverse quantal frames Q, showing that they have associated localic étale groupoids $\mathcal{G}(Q)$. Now we shall do the converse, namely showing that any localic étale groupoid G has an associated inverse quantal frame $\mathcal{O}(G)$, such that the two constructions $\mathcal{G}$ and $\mathcal{O}$ are inverse to each other up to isomorphism.

Quantal groupoids. As we have seen, if the topology $\Omega(G)$ of a topological groupoid G is closed under pointwise multiplication of open sets then $\Omega(G)$ is a unital involutive quantale. The localic analogue of this is of course a localic groupoid G whose multiplication map m is open, but even just by assuming that m is semiopen relevant conclusions are obtained. In particular, as we shall see below, in that case $\mathcal{O}(G_1)$ is a quantal frame, which motivates the following definition.

Definition III.3.1 By a *quantal groupoid* is meant a localic groupoid whose multiplication map is semiopen. If G is a quantal groupoid, the *groupoid quantale* of G, denoted by $\mathcal{O}(G)$, is defined to be the involutive quantale of the following theorem.

Theorem III.3.2 *Let G be a localic groupoid:*

$$G_1 \underset{G_0}{\times} G_1 \xrightarrow{m} G_1 \underset{d}{\overset{r}{\rightrightarrows}} G_0 . \quad (i : G_1 \to G_1,\ u : G_0 \to G_1)$$

If G is quantal then $\mathcal{O}(G_1)$ is a quantale whose multiplication

$$m' : \mathcal{O}(G_1) \otimes \mathcal{O}(G_1) \to \mathcal{O}(G_1)$$

is the sup-lattice homomorphism $m' = m_! \circ q$, where

$$q : \mathcal{O}(G_1 \times G_1) \to \mathcal{O}(G_1 \underset{G_0}{\times} G_1)$$

is the frame quotient that defines $G_1 \times_{G_0} G_1$ as a sublocale of $G_1 \times G_1$. Furthermore, this quantale has an involution given by

$$a^* = i_!(a) = i^*(a) .$$

Proof. The proof of associativity of m' is, with direction reversed, entirely analogous to the proof of associativity in III.2.8.

Let us prove that $i_!$ is an involution on $\mathcal{O}(G)$. The first condition, namely $a^{**} = a$, follows from $i \circ i = \mathrm{id}$, as does the fact that $i_! = i^*$. For the second condition, $(ab)^* = b^*a^*$, we begin by recalling the equation

$$i \circ m = m \circ \chi ,$$

where χ is the isomorphism $\langle i \circ \pi_2, i \circ \pi_1 \rangle$, which satisfies $\chi_! = \chi^*$ because $\chi \circ \chi = \mathrm{id}$. In particular, we have

$$\begin{aligned} \chi_!(a \otimes b) &= \chi^*(a \otimes b) = [\pi_2^* \circ i^*, \pi_2^* \circ i^*](a \otimes b) = \pi_2^*(i^*(a)) \wedge \pi_1^*(i^*(b)) \\ &= 1 \otimes a^* \wedge b^* \otimes 1 = b^* \otimes a^* . \end{aligned}$$

Hence, noting that $i_! \circ m_! = m_! \circ \chi_!$, we obtain

$$(ab)^* = i_!(m_!(a \otimes b)) = m_!(\chi_!(a \otimes b)) = m_!(b^* \otimes a^*) = b^*a^* . \quad \blacksquare$$

Lemma III.3.3 *Let G be a quantal groupoid. Then $d^*(a)$ is right-sided and $r^*(a)$ is left-sided, for all $a \in \mathcal{O}(G_0)$.*

Proof. Let $a \in \mathcal{O}(G_0)$. We have, in $\mathcal{O}(G)$,

$$d^*(a)1 = m_!(d^*(a) \otimes 1) = m_!(\pi_1^*(d^*(a))) \ , \tag{III.3.4}$$

where $\pi_1 : G_1 \times_{G_0} G_1 \to G_1$ is the first projection. One of the defining conditions of G as a localic groupoid is $d \circ \pi_1 = d \circ m$, and thus we can replace π_1^* by m^* in (III.3.4), which leads to

$$d^*(a)1 = m_!(m^*(d^*(a))) \leq d^*(a) \ .$$

In a similar way one proves that $r^*(a)$ is left-sided. ∎

Lemma III.3.4 *Let G be a quantal groupoid. Then, for all $a \in \mathcal{O}(G)$, we have*

$$d^*(u^*(a)) = \bigvee_{bc^* \leq a} b \wedge c \tag{III.3.5}$$

$$r^*(u^*(a)) = \bigvee_{b^* c \leq a} b \wedge c \ . \tag{III.3.6}$$

Proof. This is entirely similar to the proof of III.2.16, where the groupoid inversion law was seen to be equivalent to the two conditions (III.2.17) and (III.2.18), except that now we cannot assume equations like $d^*(u^*(a)) = (a \wedge e)1$ or $r^*(u^*(a)) = 1(a \wedge e)$, which make no sense because we do not even have a unit e. ∎

Lemma III.3.5 *Any quantal groupoid has semiopen domain and range maps, with $d_!(a) = u^*(a1)$ and $r_!(a) = u^*(1a)$.*

Proof. Let us verify the conditions $d_! d^* \leq \mathrm{id}$ and $d^* d_! \geq \mathrm{id}$ with respect to the proposed definition of $d_!$ (for r it is analogous). Let $a \in \mathcal{O}(G_0)$. Taking into account that $d^*(a)$ is right-sided we obtain

$$d_!(d^*(a)) = u^*(d^*(a)1) \leq u^*(d^*(a)) = a \ .$$

Now let $a \in \mathcal{O}(G_1)$. Then $d^*(d_!(a))$ equals $d^*(u^*(a1))$ which, by (III.3.5), equals

$$\bigvee \{b \wedge c \mid bc^* \leq a1\} \ ,$$

and this is greater or equal to a (for instance, let $b = c = a$). ∎

Étale groupoids. In section 2 we have shown how to obtain localic étale groupoids from inverse quantal frames, and now we shall do the converse, showing not only any localic étale groupoid yields an inverse quantal frame, but that indeed this is the case for any quantal groupoid G whose sublocale of units is open; as a consequence it follows that such a groupoid G is necessarily étale.

Definition III.3.6 A localic groupoid G is said to be *unital* if the map

$$u : G_0 \to G_1$$

is open (and thus G_0 is an open sublocale of G_1).

Lemma III.3.7 *Let G be a unital quantal groupoid. The involutive quantale $\mathcal{O}(G)$ is unital, and the multiplicative unit is $e = u_!(1_{G_0})$.*

Proof. The proof is the same, with direction reversed, as the proof of the unit laws in III.2.8. ∎

Lemma III.3.8 *Let G be a unital quantal groupoid. The following conditions hold in $\mathcal{O}(G)$, for all $a \in \mathcal{O}(G_0)$ and $b \in \mathcal{O}(G_1)$:*

$$b \wedge d^*(a) = u_!(a)b \qquad \text{(III.3.7)}$$

$$b \wedge r^*(a) = bu_!(a) \qquad \text{(III.3.8)}$$

Proof. Let $a \in \mathcal{O}(G_0)$. We have

$$e \wedge r^*(a) = u_!(1_{G_0}) \wedge r^*(a) = u_!(1_{G_0} \wedge u^*(r^*(a))) = u_!(u^*(r^*(a))) = u_!(a) \ ,$$

where the second equality follows from the Frobenius reciprocity condition for u, and the last equality is a consequence of the condition $r \circ u = \mathrm{id}$. Using this, and noticing that the pushout of d^* and r^*

$$\mathcal{O}(G_1) \underset{\mathcal{O}(G_0)}{\otimes} \mathcal{O}(G_1)$$

satisfies the condition (cf. Exercise III.1.10-1)

$$b \otimes (d^*(a) \wedge c) = (b \wedge r^*(a)) \otimes c \ , \qquad \text{(III.3.9)}$$

we prove (III.3.7):

$$b \wedge d^*(a) = m_!(e \otimes (b \wedge d^*(a))) = m_!((e \wedge r^*(a)) \otimes b) = m_!(u_!(a) \otimes b) = u_!(a)b \ .$$

Equation (III.3.8) is proved in a similar way, this time starting from the condition $e \wedge d^*(a) = u_!(a)$, which is an instance of (III.3.7). ∎

Lemma III.3.9 *Let G be a unital quantal groupoid. Then, for all $a \in \mathcal{O}(G_1)$,*

$$\begin{aligned} d^*(u^*(a)) &= (a \wedge e)1 \\ r^*(u^*(a)) &= 1(a \wedge e) . \end{aligned}$$

Proof. Let $a \in \mathcal{O}(G_1)$. From (III.3.7) we have

$$d^*(u^*(a)) = 1 \wedge d^*(u^*(a)) = u_!(u^*(a))1 .$$

And we have $u_!(u^*(a)) = a \wedge e$ because u is open:

$$u_!(u^*(a)) = u_!(u^*(a) \wedge 1_{G_0}) = a \wedge u_!(1_{G_0}) = a \wedge e .$$

Hence, $d^*(u^*(a)) = (a \wedge e)1$. For $r^*(u^*(a))$ we use (III.3.8) and everything is analogous. ∎

Theorem III.3.10 *Let G be a unital quantal groupoid. Then $\mathcal{O}(G)$ is an inverse quantal frame, and its groupoid $\mathcal{G}(\mathcal{O}(G))$ is isomorphic to G.*

Proof. First, we show that the sup-lattice homomorphism

$$\varsigma = u_! \circ d_!$$

defines a support:

- $\varsigma a = u_!(d_!(a)) \leq u_!(1_{G_0}) = e$, which proves (II.3.2).
- An instance of (III.3.5) gives us

$$d^*(u^*(aa^*)) = \bigvee_{xy^* \leq aa^*} x \wedge y ,$$

  and thus $a \leq d^*(u^*(aa^*))$ (make $a = x = y$). Hence, by adjointness we obtain $u_!(d_!(a)) \leq aa^*$, i.e., we have proved (II.3.3).
- Now we prove (II.3.4):

$$\begin{aligned} \varsigma aa &= m_!(\varsigma a \otimes a) \\ &= m_!(u_!(d_!(a)) \otimes a) \\ &= m_!((e \wedge r^*(d_!(a))) \otimes a) && \text{[By (III.3.8)]} \\ &= m_!(e \otimes (d^*(d_!(a)) \wedge a)) && \text{[By (III.3.9)]} \\ &= d^*(d_!(a)) \wedge a \\ &= a && (d^* \circ d_! \geq \mathrm{id}) . \end{aligned}$$

Hence, $\mathcal{O}(G)$ is a supported quantal frame. Furthermore, from III.3.4 and III.3.9 we obtain the equations

$$\begin{aligned} (a \wedge e)1 &= \bigvee_{xy^* \leq a} x \wedge y \\ 1(a \wedge e) &= \bigvee_{x^* y \leq a} x \wedge y \ , \end{aligned}$$

which show, by III.2.18, that $\mathcal{O}(G)$ is an inverse quantal frame. Hence, $\mathcal{O}(G)$ has an associated (étale) groupoid $\mathcal{G}(\mathcal{O}(G))$. Let us denote this by $\widehat{G}$, with structure maps $\hat{d}$, $\hat{r}$, $\hat{u}$, $\hat{m}$, and $\hat{\imath}$. We shall prove that G and $\widehat{G}$ are isomorphic. Since obviously we have $G_1 = \widehat{G}_1$, it is natural to look for an isomorphism $(f_1, f_0) : G \to \widehat{G}$ with $f_1 = \mathrm{id}_{G_1}$. Then f_0 must be given by $f_0 = \hat{d} \circ f_1 \circ u = \hat{d} \circ u$, and, similarly, its inverse must be given by $d \circ \hat{u}$. Let us verify that the pair (f_1, f_0) commutes with d and $\hat{d}$, i.e., that the following diagram commutes:

(III.3.10)
$$\begin{array}{ccc} G_1 & \xrightarrow{f_1 = \mathrm{id}} & \widehat{G}_1 = G_1 \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \hat{d}} \\ G_0 & \xrightarrow[f_0 = \hat{d} \circ u]{} & \widehat{G}_0 \end{array}$$

In order to do this, first we remark that the results of section 2 on stable quantal frames give for the homomorphisms $\hat{d}^* \circ \hat{u}^*$ and $\hat{r}^* \circ \hat{u}^*$ the formulas

$$\begin{aligned} (\hat{d}^* \circ \hat{u}^*)(a) &= (a \wedge e)1 \\ (\hat{r}^* \circ \hat{u}^*)(a) &= 1(a \wedge e) \ , \end{aligned}$$

which are identical to those of III.3.9 for $d^* \circ u^*$ and $r^* \circ u^*$, thus yielding the following identities of locale maps:

(III.3.11) $$\hat{u} \circ \hat{d} = u \circ d$$

(III.3.12) $$\hat{u} \circ \hat{r} = u \circ r \ .$$

Hence, using (III.3.11) we have

$$f_0 \circ d = \hat{d} \circ u \circ d = \hat{d} \circ \hat{u} \circ \hat{d} = \mathrm{id} \circ \hat{d} = \hat{d} \circ \mathrm{id} = \hat{d} \circ f_1 \ ;$$

that is, the diagram (III.3.10) commutes. Using again (III.3.11) we show that (f_1, f_0) commutes with u and $\hat{u}$,

$$\hat{u} \circ f_0 = \hat{u} \circ \hat{d} \circ u = u \circ d \circ u = u \circ \mathrm{id} = \mathrm{id} \circ u = f_1 \circ u \ ,$$

and using (III.3.12) we conclude that (f_1, f_0) commutes with r and $\hat{r}$:

$$f_0 \circ r = \hat{d} \circ u \circ r = \hat{d} \circ \hat{u} \circ \hat{r} = \mathrm{id} \circ \hat{r} = \hat{r} \circ \mathrm{id} = \hat{r} \circ f_1 \ .$$

Hence, (f_1, f_0) is a morphism of reflexive graphs. The fact that it is an isomorphism with $f_0^{-1} = d \circ \hat{u}$ follows again from (III.3.11):

$$\begin{aligned}(\hat{d} \circ u) \circ (d \circ \hat{u}) &= \hat{d} \circ \hat{u} \circ \hat{d} \circ \hat{u} = \mathrm{id} \circ \mathrm{id} = \mathrm{id} \ ,\\ (d \circ \hat{u}) \circ (\hat{d} \circ u) &= d \circ u \circ d \circ u = \mathrm{id} \circ \mathrm{id} = \mathrm{id} \ .\end{aligned}$$

From these results it follows that the pullback of d and r coincides with the pullback of $\hat{d}$ and $\hat{r}$ (both pullbacks are, as frame pushouts, given by the same quotient of $\mathcal{O}(G_1) \otimes \mathcal{O}(G_1)$), and thus it is obvious that $m = \hat{m}$, since both m^* and $\hat{m}^*$ are right adjoint to the same quantal multiplication $\mathcal{O}(G_1 \times_{G_0} G_1) \to \mathcal{O}(G_1)$. Similarly, $i = \hat{i}$ because both i^* and $\hat{i}^*$ coincide with the quantale involution, and we conclude that (f_1, f_0) is an isomorphism of groupoids. ∎

Our results provide equivalent but new alternative definitions for the notion of étale groupoid:

Corollary III.3.11 *For any localic groupoid G, the following are equivalent:*

1. *G is étale.*
2. *G is quantal and unital.*
3. *G is open and unital.*

Proof. $2 \Rightarrow 1$: Immediate consequence of III.2.19 and III.3.10.
$3 \Rightarrow 2$: Immediate because being open implies being quantal.
$1 \Rightarrow 3$: For an étale groupoid all the structure maps are local homeomorphisms, and thus, in particular, both m and u are open. ∎

Groupoid maps and quantale homomorphisms. We remark that as consequences of III.2.19 and III.3.10 we have obtained a duality between étale groupoids and inverse quantal frames, which is given by isomorphisms

$$\begin{aligned} G &\cong \mathcal{G}(\mathcal{O}(G)) \\ Q &\cong \mathcal{O}(\mathcal{G}(Q)) \ . \end{aligned}$$

[Indeed there is an obvious equality $Q = \mathcal{O}(\mathcal{G}(Q))$.] It is natural to ask how well this duality behaves with respect to morphisms, a question that we shall briefly address now.

Lemma III.3.12 *Let Q_1 and Q_2 be inverse quantal frames, and $f : Q_1 \to Q_2$ a sup-latttice homomorphism. Let also m_1 and m_2 be the multiplication maps of $\mathcal{G}(Q_1)$ and $\mathcal{G}(Q_2)$, respectively. The following conditions are equivalent:*

1. *$f(a)f(b) \leq f(ab)$ for all $a, b \in Q_1$.*
2. *$(f \otimes f) \circ m_1^* \leq m_2^* \circ f$.*

Proof. $1 \Rightarrow 2$: Condition 1 is equivalent to $(m_2)_! \circ (f \otimes f) \leq f \circ (m_1)_!$, which by adjointness is equivalent to $f \otimes f \leq m_2^* \circ f \circ (m_1)_!$. Composing with m_1^* on the right we obtain $(f \otimes f) \circ m_1^* \leq m_2^* \circ f \circ (m_1)_! \circ m_1^*$, and thus we obtain condition 2 because $(m_1)_! \circ m_1^* \leq \mathrm{id}$.
$2 \Rightarrow 1$: From condition 2 we obtain, by adjointness, $(m_2)_! \circ (f \otimes f) \circ m_1^* \leq f$. Now composing, on both sides of this inequality, with $(m_1)_!$ on the right we obtain condition 1 because $m_1^* \circ (m_1)_! \geq \mathrm{id}$. ∎

Theorem III.3.13 *Let G and G' be étale groupoids, and let*

$$h = (h_1, h_0) : G' \to G$$

be a morphism of groupoids. Then for all $a, b \in \mathcal{O}(G)$ we have

$$h_1^*(a)h_1^*(b) \leq h_1^*(ab) .$$

Proof. This a corollary of the previous lemma, since a morphism (h_1, h_0) of groupoids preserves multiplication and that is equivalent to the equality $(h_1^* \otimes h_1^*) \circ m^* = (m')^* \circ h_1^*$. ∎

The following example shows that the inequality in the above theorem is in general not an equality.

Example III.3.14 As an example of an étale groupoid consider a nontrivial discrete group G (written multiplicatively), and let $h : G \to G$ be an endomorphism. In general $h^{-1} : \wp(G) \to \wp(G)$ is not a homomorphism of quantales: for instance, if $h(g) = 1$ for all $g \in G$, and $U = V^{-1} = \{g\}$ with $g \neq 1$, then $h^{-1}(U) = h^{-1}(V) = \emptyset$, whence $h^{-1}(U)h^{-1}(V) = \emptyset$ but $h^{-1}(UV) = h^{-1}(\{1\}) = G$.

This shows that there is no immediate contravariant functor from étale groupoids to quantales, and that in order to find a duality between étale groupoids and inverse quantal frames in the categorical sense one must be willing to change the morphisms under consideration, for instance allowing more general homomorphisms of quantales, in particular homomorphisms that are lax on multiplication as in III.3.13, or restricting consideration of maps of groupoids to those whose inverse images preserve quantale multiplication, etc. (Similar problems apply to multiplicative units — for instance, in III.3.14 we have $e = \{1\}$ and $h^*(e) = \ker h$, and thus $h^*(e) = e$ if and only if h is injective.)

Open groupoids. Recall that an open localic groupoid G is a localic groupoid

$$G_2 \xrightarrow{m} G_1 \overset{i}{\circlearrowleft} \; \underset{d}{\overset{r}{\rightrightarrows}} \; G_0 \qquad (u : G_0 \to G_1)$$

whose domain map d is open. Similarly to topological groupoids, we have (cf. Exercise I.1.8):

Proposition III.3.15 *Let G be a localic groupoid as above. The following properties are equivalent:*

1. *G is open;*
2. *r is open;*
3. *m is open.*

Proof. The first two conditions are clearly equivalent because $d = r \circ i$ and $r = d \circ i$ and i is an isomorphism. Now assume that G is open. Then m is open because it is a pullback of d (cf. Exercise III.1.16). Conversely, assume that m is open. By III.3.5 we know that d is semiopen because m is, and furthermore it satisfies the Frobenius reciprocity condition because m does, which can be proved as follows:

$$\begin{aligned}
d_!(a \wedge d^*(b)) &= u^*((a \wedge d^*(b))1) && \text{(By III.3.5.)}\\
&= u^*(m_!((a \wedge d^*(b)) \otimes 1)) &&\\
&= u^*(m_!(\pi_1^*(a \wedge d^*(b)))) && (\pi_1^*(-) = - \otimes 1\,.)\\
&= u^*(m_!(\pi_1^*(a) \wedge \pi_1^* d^*(b))) &&\\
&= u^*(m_!(\pi_1^*(a) \wedge m^* d^*(b))) && (d \circ m = d \circ \pi_1\,.)\\
&= u^*(m_!(\pi_1^*(a)) \wedge d^*(b)) && \text{(Frobenius cond. for } m\,.)\\
&= u^*(a1 \wedge d^*(b)) && (\pi_1^*(a) = a \otimes 1\,.)\\
&= u^*(a1) \wedge u^* d^*(b) &&\\
&= d_!(a) \wedge b && \text{(By III.3.5 and } d \circ u = \mathrm{id}\,.)
\end{aligned}$$

∎

The quantales $\mathcal{O}(G)$ of open groupoids G have interesting properties, but we shall not address them in these notes.

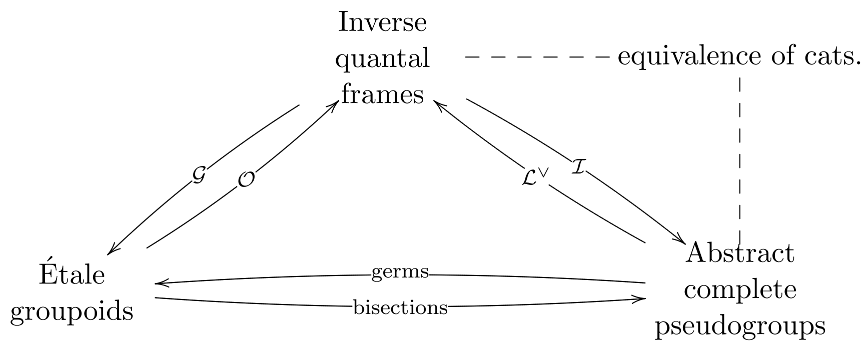
Germ groupoids revisited. In these notes we have provided a detailed account of a threefold interplay between inverse quantal frames, abstract complete pseudogroups, and étale groupoids, beginning, in Chapter I, with a description of direct relations between topological étale groupoids and abstract complete pseudogroups over a space. Chapters II and III, on the other hand, have developed a point free version of this where quantales act as mediating objects between groupoids and inverse semigroups.

Although we have not said so explicitly in Chapters II and III, the relation between sober étale groupoids and spatial inverse semigroups that results from this is a generalization of that of Chapter I: this is obvious when constructing inverse semigroups from groupoids; and in the reverse direction the construction of groupoids from abstract complete pseudogroups is easily seen to coincide, in the case of spatial locales, with the germ groupoid construction of Chapter I, once we take into account the following two facts:

- local homeomorphisms reflect sobriety of the base spaces (cf. Exercise III.1.24-9);
- the locale $\mathcal{L}^{\vee}(S)$ of an abstract complete pseudogroup S is necessarily the topology of the germ groupoid of S, due to an argument similar to that of Exercise III.1.8-7.

(See also the Exercise III.3.16 at the end of this section.)

In other words, the groupoid $\mathcal{G}(\mathcal{L}^{\vee}(S))$ of an abstract complete pseudogroup S is the localic version of the germ groupoid of S. In particular, if $E(S)$ is a spatial locale then the spectrum of the groupoid is exactly the germ groupoid of S, and we can summarize the results described in these notes in the following diagram:



Exercise III.3.16 1. Let S be an inverse semigroup. Let $s \in S$, and let $F \subseteq E(S)$ be a filter of $E(S)$ such that $ss^{-1} \in F$. The *germ of s at F* is defined to be the set

$$\mathrm{germ}_F\, s = \{t \in S \mid tt^{-1} \in F \text{ and } ft = fs \text{ for some } f \in F\} .$$

Prove the following assertions:

(a) $fs \in \text{germ}_F\, s$, for all $f \in F$;

(b) $tt^{-1} \in F$, for all $t \in \text{germ}_F\, s$;

(c) F coincides with the set $\uparrow\{tt^{-1} \mid t \in \text{germ}_F\, s\}$;

(d) $\text{germ}_F\, s$ is a filter of S.

2. Let S be an inverse semigroup, and Φ a filter of S. Prove that the set $d(\Phi)$ defined by
$$d(\Phi) = \uparrow\{tt^{-1} \mid t \in \Phi\}$$
is a filter of $E(S)$, and that for all $s \in \Phi$ we have
$$\Phi = \text{germ}_{d(\Phi)}\, s\ .$$

3. Let S be an abstract complete pseudogroup. Show that a point of $\mathcal{L}^\vee(S)$ can be identified with any of the following two things:

(a) A map $p : S \to \wp(1)$ such that
- $\bigvee \varphi(S) = 1$,
- φ preserves binary meets,
- φ preserves joins of compatible sets;

(b) A filter $\Phi \subseteq S$ such that for every compatible set $X \subseteq S$ such that $\bigvee X \in \Phi$ we have $x \in \Phi$ for some $x \in X$.

4. Calling the filters of the previous exercise *compatibly prime*, show that the compatibly prime filters of S are exactly the same as the germs of S defined on completely prime filters of $E(S)$. (This shows by explicit calculation that the points of the germ groupoid of S are the locale points of $\mathcal{L}^\vee(S)$.)

5. The classical groupoid $\widetilde{S}$ of an inverse semigroup S (see [14]) is defined as follows:
$$\begin{aligned}
\widetilde{S}_1 &= S \\
\widetilde{S}_0 &= E(S) \\
d(s) &= ss^{-1} \\
r(s) &= s^{-1}s \\
i(s) &= s^{-1} \\
m(s,t) &= st \quad \text{if } d(t) = r(s)\ .
\end{aligned}$$

(a) Show that S is a topological étale groupoid whose topology is the (co-)Alexandroff topology obtained from the natural order of S (the open sets are the downwards closed sets).

(b) Show that the quantale of this groupoid, $\Omega(\widetilde{S})$, coincides with $\mathcal{L}(S)$.

(c) Show that the principal filters of S are exactly the germs $\mathrm{germ}_F\, s$ with F a principal filter of $E(S)$.

(d) Show that $\widetilde{S}$ is isomorphic to the full subgroupoid of $\Sigma(\mathcal{G}(\mathcal{L}(S)))$ whose points are the principal filters of S. (Hence, $\Sigma(\mathcal{G}(\mathcal{L}(S))$ is in an obvious sense the soberification of $\widetilde{S}$, and therefore $\mathcal{G}(\mathcal{L}(S))$ is the closest possible localic counterpart of $\widetilde{S}$.)

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