THE CLASSIFICATION OF 1 DIMENSIONAL MANIFOLDS

This is a proof of the classification of connected, second countable[1], Hausdorff 1-manifolds in excruciating detail following the basic plan of the appendix of [Mi]. This solves Exercises 1.2.6 and 1.4.9 in Hirsch.

In all that follows M denotes a connected, second countable, Hausdorff manifold of class C^r possibly with boundary with $0 \leq r \leq \infty$.

We will fix the following notation: $\varphi\colon U \to \mathbb{R}$ and $\psi\colon V \to \mathbb{R}$ are charts of M satisfying

- $U \cap V \neq \emptyset$,$U \not\subset V$ and $V \not\subset U$;
- $I = \varphi(U)$ and $J = \psi(V)$ are bounded intervals (possibly containing endpoints);
- $\alpha = \psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$.

Note that $\varphi(U \cap V)$ is a disjoint union of relatively open separated intervals of I.

Lemma 0.1. *Let K be a connected component of $\varphi(U \cap V)$ and a an endpoint of K. Then*

(i) $a \notin K$.

(ii) If $a \in I$ then

$$c = \lim_{\substack{t \to a \\ t \in K}} \alpha(t) \notin J.$$

In particular, c is an endpoint of J where J is open.

Proof. The limit exists because J is bounded and $\alpha_{|K}$ is a homeomorphism of an interval and therefore either increasing or decreasing on K.

If a is not an endpoint of I then $a \notin K$ because K is relatively open in I. Suppose a is an endpoint of I and $a \in K$. Then $\alpha(a)$ must be a boundary point of J (as $\alpha(K)$ is relatively open in J). Let b be the other boundary point of K. Since $U \not\subset V$, $b \in I$ and since $V \not\subset U$, $d = \lim_{t \to b, t \in K} \alpha(t) \in J$, but then $\varphi^{-1}(b)$ and $\psi^{-1}(d)$ are distinct points of M which can't be separated by open sets.

To prove (ii), suppose $c \in J$. Then, as the connected components of $\varphi(U \cap V)$ are separated, $\psi^{-1}(c) \notin U$ and $\varphi^{-1}(a) \notin V$. However, any two neighborhoods of these points in M intersect (they contain points of $\varphi^{-1}(K)$) and this contradicts the assumption that M is Hausdorff. $\square$

Corollary 0.2. *Each interval in $\varphi(U \cap V)$ is open and has at least one endpoint in common with I. In particular, $\varphi(U \cap V)$ consists of either one or two open intervals.*

[1]A connected paracompact Hausdorff manifold is automatically second countable so this is a classification of paracompact Hausdorff 1-manifolds. Sketch proof: Paracompactness allows us to define a Riemannian metric on M. Need to find a countable dense set on M. Pick a point and take a maximal geodesic ball around it. Take the points in this ball with rational coordinates (for some choice of coordinates). Consider maximal geodesic balls with centers in these points and take all points with rational coordinates in these new balls. Continue in this way to infinity to get a countable dense set.

Proof. Lemma 0.1 says that $\varphi(U \cap V)$ is a union of open intervals since the endpoints of its components K do not belong to K.

If a component $K \subset \varphi(U \cap V)$ has both endpoints not on the boundary of I. Lemma 0.1 implies that $\alpha(K) = J$ (the limits at the boundary of K can't be equal as α is increasing or decreasing) contradicting the assumption that $V \not\subset U$. $\square$

Proposition 0.3. *(i) If $\varphi(U \cap V)$ consists of two intervals then I and J are open and $U \cup V$ is homeomorphic to S^1.*

(ii) If $\varphi(U \cap V)$ consists of one interval with the same endpoints as I, then I contains exactly one boundary point and $U \cup V$ is homeomorphic to $[0, 1]$ or $[0, 1[$ according to whether J contains one boundary point or not.

(iii) Otherwise $U \cup V$ is homeomorphic to $]0, 1[$, $[0, 1[$ or $[0, 1]$. I and J each contain at most one boundary point and the total number of these is the number of boundary points of the interval to which $U \cup V$ is homeomorphic.

Furthermore, if $U \cup V$ is homeomorphic to $[0, 1]$ or S^1 then $M = U \cup V$.

Proof. (i) Suppose $\varphi(U \cap V)$ contains two intervals K_1 and K_2. Let a_i be the endpoint of K_i which is not an endpoint of I and b_i be the other endpoint of K_i. Corollary 0.2 says that I and J are open. Define

$$c_i = \lim_{t \to a_i, t \in K_i} \alpha(t); \qquad d_i = \lim_{t \to b_i, t \in K_i} \alpha(t).$$

Then $c_1 \neq c_2$ (even if $a_1 = a_2$) because equality would imply that α is not injective. Hence c_1 and c_2 are the endpoints of J and d_1 and d_2 are interior to J (they could be equal).

Suppose without any loss that $b_1 < a_1 \leq a_2 < b_2$. Then either $c_1 < c_2$ and α is decreasing on $K_1 =]b_1, a_1[$ and $K_2 =]a_2, b_2[$ or $c_1 > c_2$ and α is increasing on K_1 and K_2. In either case it is easy to define a homeomorphism $S^1 \to U \cup V$. Suppose for instance that α is increasing. Then, either $d_1 = d_2$ and $U \cup V$ is the one point compactification of U and hence homeomorphic to S^1 or $d_2 < d_1$. In the latter case pick increasing homeomorphisms $\lambda\colon]0, \pi[\to]b_1, b_2[$ and $\mu\colon [\pi, 2\pi] \to [d_2, d_1]$ and define $f\colon S^1 \to U \cup V$ by

$$f(e^{i\theta}) = \begin{cases} \varphi^{-1}(\lambda(\theta)) & \text{if } 0 < \theta < \pi, \\ \psi^{-1}(\mu(\theta)) & \text{if } \pi \leq \theta \leq 2\pi. \end{cases}$$

(ii) Since $U \not\subset V$, I must contain one of its boundary points. It can not contain both or U would be open and closed in M. Suppose for instance that $I = [a, b[$ (so $K = \phi(U \cap V) =]a, b[$) and let $\psi(U \cap V) =]c, d[$. Then $c = \lim_{t \to a, t \in K} \alpha(t)$ is an endpoint of J where J is open by Lemma 0.1. Suppose for instance that α is increasing and let $e = \lim_{t \to b, t \in K} \alpha(t)$. Since $V \not\subset U$, it must then be that either $J =]c, d[$ with $e < d$ (if V does not contain a boundary point) or $J =]c, d]$ with $e \leq d$ (if V does contain a point in the boundary of M). Assume for example that V has no boundary point. Pick a homeomorphism $\mu\colon [b, b + d - e[\to [e, d[$ and define

$$f\colon [a, b + d - e[\to U \cup V$$

by

$$f(x) = \begin{cases} \varphi^{-1}(x) & \text{if } x < b \\ \psi^{-1}(\mu(x)) & \text{if } b \leq x < b + d - e. \end{cases}$$

Composing f with a homeomorphism $[0,1[\to [a, b+d-e[$ we obtain the desired homeomorphism. The remaining (seven) cases are handled in exactly the same way.

(iii) This is exactly as (ii) and is left as an exercise.

The last statement follows from connectedness of M. □

We need to prove a version of the previous Proposition when $r \geq 1$. The difficulty is that it is harder to extend diffeomorphisms than homeomorphisms: the transition functions α might not extend as the limits of the derivatives may not exist when we approach the boundary. However, if we have charts U and V which intersect, we can shrink them to smaller charts $U' \subset \overline{U'} \subset U$ and $V' \subset \overline{V'} \subset V$ and then it is not hard to glue the charts U' and V' as the following lemma explains.

Proposition 0.4. *Let $r \geq 1$, $A = \varphi(U \cap V)$, $B = \psi(U \cap V)$ and suppose α can be extended to a diffeomorphism between open neighborhoods of $\overline{A}$ and $\overline{B}$. Then Proposition 0.3 holds with "homeomorphic" replaced* [2] *by "C^r diffeomorphic".*

Proof. We will just do the case of Proposition 0.3(iii) when I and J are both open. We will pick particular intervals I,J in order to make the argument easier to follow.

Let $\varphi\colon U \to]0,1[$ and $\psi\colon V \to]1,2[$ and suppose $\varphi(U\cap V) =]\frac{1}{2}, 1[$, $\psi(U\cap V) =]1, \frac{3}{2}[$ (so α is increasing). Suppose α can be extended to an increasing C^r map on $]\frac{1}{2}, 1+\epsilon]$. Let $\lambda\colon]0, \frac{3}{2}[\to [0,1]$ be smooth and such that

$$\lambda(x) = \begin{cases} 1 & \text{if } x \leq 1, \\ 0 & \text{if } x \geq 1+\epsilon \end{cases}$$

and λ is decreasing. Let $g\ :]1, \frac{3}{2}[\to]1,2[$ be an increasing affine function so that $g(\frac{3}{2}) = 2$ and $g(x) \geq \alpha(x)$ for $1 \leq x \leq 1+\epsilon$. Define a map

$$f\colon]0, \tfrac{3}{2}[\to U \cup V$$

by the expression

$$f(x) = \begin{cases} \varphi^{-1}(x) & \text{if} x < 1 \\ \psi^{-1}(\lambda(x)\alpha(x) + (1-\lambda(x))g(x)) & \text{if } x \geq 1. \end{cases}$$

Then f is smooth, has image $U \cup V$, and for $x \geq 1$, we have

$$(\psi \circ f)'(x) = \lambda'(x)(\alpha(x) - g(x)) + \lambda(x)\alpha'(x) + (1-\lambda(x))g'(x) > 0$$

so we can precompose f with a diffeomorphis $]0,1[\to]0, \frac{3}{2}[$ to obtain the required diffeomorphism. □

Theorem 0.5. *Let $0 \leq r \leq \infty$. Every connected, Hausdorff, second countable C^r 1-manifold is C^r diffeomorphic to one of the following:*

$$S^1, \quad]0,1[, \quad [0,1], \quad [0,1[.$$

Proof. Let $\{U_i\}$ be a locally finite countable cover of M by coordinate patches with $\varphi_i(U_i)$ bounded intervals. If $r \geq 1$, replace $\{U_i\}$ by a shrinking (i.e. an open cover $\{V_i\}$ with $V_i \subset \overline{V_i} \subset U_i$). This will guarantee that the transition functions between intersecting charts can be extended to a neighborhood of the closures of their domains. We will abuse notation and still denote the shrinking by $\{U_i\}$.

[2] But note that some cases of Proposition 0.3 are excluded by the assumption that α extends to a neighborhood of the closure of its domain.

Using Zorn's lemma we can pick a subcover (still denoted $\{U_i\}$) with the property that $U_i \not\subset U_j$ when $i \neq j$. Identify the indexing set of the cover with an initial segment of the natural numbers in such a way that

- If M contains a boundary point, then U_1 contains a boundary point,
- $U_n \cap (U_1 \cup \ldots \cup U_{n-1}) \neq \emptyset$.

The second condition can be enforced because M is connected. Note that if M is compact the indexing set must be finite.

If M is compact, a finite number of applications of Propositions 0.3 or 0.4 gives a C^r diffeomorphism from $[0,1]$ or S^1 to M (we disregard U_k if it happens to be contained in $U_1 \cup \ldots \cup U_{k-1}$, otherwise we apply the Propositions to these two intervals).

Suppose M is not compact and suppose for instance that it contains a boundary point. We define inductively a (possibly finite) sequence $a_n \in \mathbb{R}$ with $0 < a_1$, $a_n \leq a_{n+1}$ and $f_n \colon |0, a_n[\to M$ so that f_n^{-1} is a chart for $U_1 \cup \ldots \cup U_n$ and $f_{n|[0,a_{n-1}[} = f_{n-1}$. Set f_1 to be the inverse of a chart for U_1 to start the induction. If $U_n \not\subset U_1 \cup \ldots \cup U_{n-1}$ use Propositions 0.3 or 0.4 to obtain f_n. Otherwise just set $a_n = a_{n-1}$ and $f_n = f_{n-1}$. Note that U_n can not contain a boundary point if $n > 1$ for then M would be either compact or disconnected. Having defined this sequence, let $a = \lim a_n \in]0, \infty]$, and define $f \colon |0, a[\to M$ by

$$f(x) = \lim_{n \to \infty} f(x).$$

Compose this with a diffeomorphism $[0,1[\to [0,a[$ to get the required C^r diffeomorphism.

The argument for the case when M does not contain a boundary point is very similar and is left as an exercise. $\square$

References

[Mi] J. Milnor, *Topology from the differentiable viewpoint*, University of Virginia Press, 1966.