Displacing Lagrangian Toric Fibers by Extended Probes

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Introduction

Question: given a Lagrangian submanifold $L \subset (M, \omega)$, does there exist $\phi \in \text{Ham}(M, \omega)$ such that $\phi(L) \cap L = \emptyset$?

If YES then L is displaceable, otherwise L is non-displaceable.

- This question, and its several variations, play a fundamental role in symplectic topology.
- Examples: Arnold Conjectures; Fukaya Category and Mirror Symmetry.
- Main tool: Floer homology and its several variations.

Introduction

Toric orbifolds provide examples that are both interesting and tractable.

Detecting non-displaceability of Lagrangian torus orbits: several papers in recent years by Biran-Entov-Polterovich, Cho, Entov-Polterovich, Fukaya-Oh-Ohta-Ono, Woodward et al.

Displacing Lagrangian torus orbits: McDuff (2009).

Simplest toric example: $(S^2, \sigma_{\text{area}})$ where the equator is a non-displaceable Lagrangian submanifold and all other S^1-orbits are displaceable.

Toric Symplectic Orbifolds

- A toric symplectic orbifold is a connected symplectic $2n$-orbifold (M, ω), equipped with an effective Hamiltonian action of the n-torus: $\mathbb{T}^n \hookrightarrow \mathrm{Ham}(M^{2n}, \omega)$.

- The corresponding moment map, unique up to an additive constant, will be denoted by $\mu : M \to \mathbb{R}^n$.

- Atiyah-Guillemin-Sternberg'82, Lerman-Tolman'95: the image of the moment map is the convex polytope given by the convex hull of the images of the fixed points of the action. This will be called the moment polytope and denoted by $\Delta := \mu(M) \subset \mathbb{R}^n$.

- Delzant'82, Lerman-Toman'95: the moment polytope together with a positive integer label attached to each of its facets give a complete invariant of a compact toric symplectic orbifold.

Labeled Moment Polytopes

The labeled moment polytope $\Delta \subset \mathbb{R}^n$ of a toric symplectic orbifold (M^{2n}, ω) can be defined by

$$x \in \Delta \Leftrightarrow \ell_i(x) := \langle x, \nu_i \rangle + a_i \geq 0\,, \; i = 1, \ldots, d,$$

where

- d is the number of facets of Δ,
- each vector $\nu_i \in \mathbb{Z}^n$ is an integral interior normal to the facet F_i of Δ,
- the a_i's are real numbers that determine $[\omega] \in H^2(M; \mathbb{R})$.

(M, ω) is monotone, i.e. $[\omega] = \lambda(2\pi c_1(\omega)) \in H^2(M; \mathbb{R})$ with $\lambda \in \mathbb{R}^+$, iff $\Delta \subset \mathbb{R}^n$ can be defined as above with

$$a_1 = \cdots = a_d = \lambda\,.$$

Such a Δ will be called a monotone labeled polytope and note that, in this case, $0 \in \Delta$.

Lagrangian Toric Fibers

- Toric Fibers: $x \in \breve{\Delta} :=$ interior$(\Delta) \Rightarrow T_x := \mu^{-1}(x) \cong \mathbb{T}^n \equiv$ Lagrangian submanifold of (M, ω). When Δ is monotone T_0 is called the centered, special or monotone torus fiber.
- Question: $\exists\ \phi \in \mathrm{Ham}(M, \omega)$ such that $\phi(T_x) \cap T_x = \emptyset$?
- Example: $\mathbb{C}\mathrm{P}(1, m_1, \ldots, m_n) =$ weighted projective space

$$\Delta = \{(x_1, \ldots, x_n) : x_j + 1 \geq 0, j = 1, \ldots, n; -\sum_{j=1}^{n} m_j x_j + 1 \geq 0\}.$$

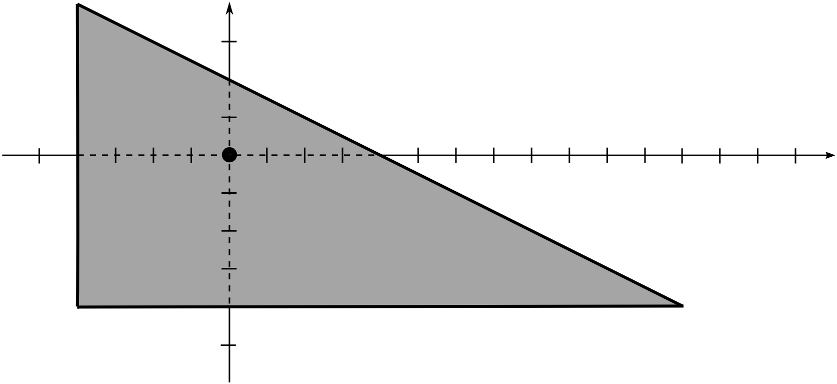
$T_0 =$ Clifford torus.

- Biran-Entov-Polterovich, Woodward, Cho-Poddar:

$$\psi(T_0) \cap T_0 \neq \emptyset\,,\ \forall\, \psi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}(1, m_1, \ldots, m_n)).$$

Example: $\mathbb{C}\mathrm{P}(1,1,2)$

$$\Delta = \{(x_1, x_2) : x_1 + 1 \geq 0,\ x_2 + 1 \geq 0;\ -x_1 - 2x_2 + 1 \geq 0\}$$

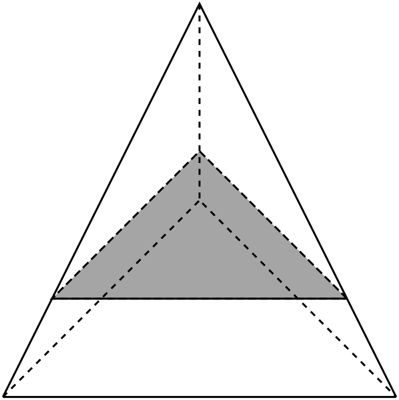


Symplectic Reduction

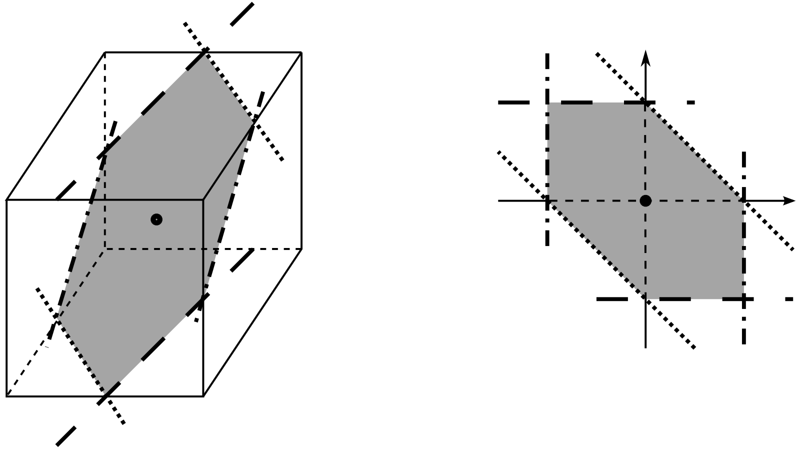
- $(\tilde{M}, \tilde{\omega})$: $\tilde{\mathbb{T}}$-manifold of dimension $2N$, with moment map $\tilde{\mu} : \tilde{M} \to (\mathbb{R}^N)^*$.
- $K \subset \tilde{\mathbb{T}}$: subtorus of dimension N-n, determined by inclusion of Lie algebras $\iota : \mathbb{R}^{N-n} \to \mathbb{R}^N$.
- Induced action of K on $\tilde{M}$ has moment map $\tilde{\mu}_K = \iota^* \circ \tilde{\mu} : \tilde{M} \to (\mathbb{R}^{N-n})^*$.
- $c \in \tilde{\mu}_K(\tilde{M}) \subset (\mathbb{R}^{N-n})^*$ regular value and consider level set $Z := \tilde{\mu}_K^{-1}(c) \subset \tilde{M}$.
- $(M := Z/K, \omega)$ is $\mathbb{T} := \tilde{\mathbb{T}}/K$-manifold of dimension $2n$, with moment map $\mu : M \to \Delta \subset (\mathbb{R}^n)^* \cong \ker(\iota^*)$ characterized by

$$
\begin{array}{ccc}
\tilde{M} \supset Z & \xrightarrow{\tilde{\mu}} & \tilde{\Delta} \subset (\mathbb{R}^N)^* \\
\pi \downarrow & & \uparrow \\
M & \xrightarrow{\mu} & \Delta \subset (\mathbb{R}^n)^*
\end{array}
$$

Example: $\mathbb{C}P^2$ as reduction of $\mathbb{C}P^3$



Example: $\mathbb{C}P^2 \sharp 3\overline{\mathbb{C}P}^2$ as reduction of $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$



Reduction and (Non-)Displaceability of Torus Fibers (A.-Macarini)

Note

$T_x := \mu^{-1}(x)\,,\; x \in \mathrm{int}(\Delta) \subset \mathrm{int}(\tilde{\Delta}) \;\Rightarrow\; \pi^{-1}(T_x) = \tilde{T}_x$.

Proposition

$$T_x \subset (M, \omega) \textit{ displ.} \quad \Rightarrow \quad \tilde{T}_x \subset (\tilde{M}, \tilde{\omega}) \textit{ displ.}$$

Equivalently

$$\tilde{T}_x \subset (\tilde{M}, \tilde{\omega}) \textit{ non-displ.} \quad \Rightarrow \quad T_x \subset (M, \omega) \textit{ non-displ.}$$

Remark

Tamarkin uses similar idea in his work on microlocal analysis of sheaves and Lagrangian intersections. Also present in the work of Borman on reduction properties of quasi-morphisms and quasi-states.

Application (A.-Macarini)

Theorem (Entov-Polterovich, Cho, FOOO)

The centered Lagrangian torus fiber T_0 of any monotone compact toric (M^{2n}, ω) is non-displaceable.

Proof.

Any compact toric (M^{2n}, ω) can be obtained as reduction of a weighted projective space. If (M^{2n}, ω) is monotone then that reduction is centered, i.e. its level set contains the Clifford torus. □

Remark (A. – Borman – McDuff)

The following should be true: any compact toric (M^{2n}, ω) can be obtained as centered reduction of a product of weighted projective spaces. Hence, any compact toric (M^{2n}, ω) has at least one non-displaceable torus fiber (Entov-Polterovich, FOOO, Woodward).

Example: Hirzebruch surfaces

FOOO, Woodward: any Hirzebruch surface has at least one non-displaceable torus fiber.

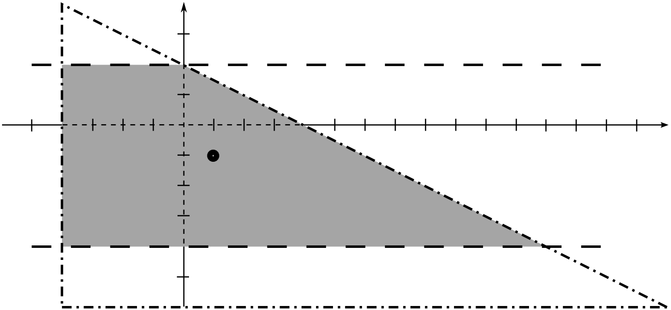


Figure: Hirzebruch surface H_2 as reduction of $\mathbb{C}P(1,1,2) \times \mathbb{C}P^1$.

Example: $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P}^2$

FOOO, Woodward: with blow-ups of different sizes, one small and the other big, get closed segment of non-displaceable fibers.

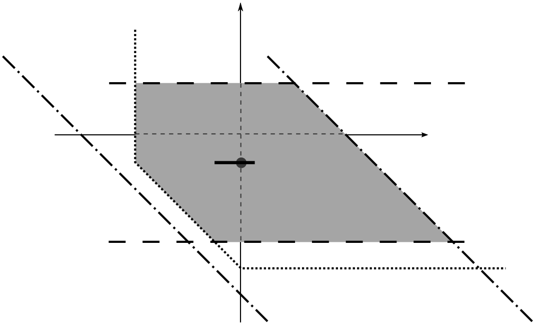
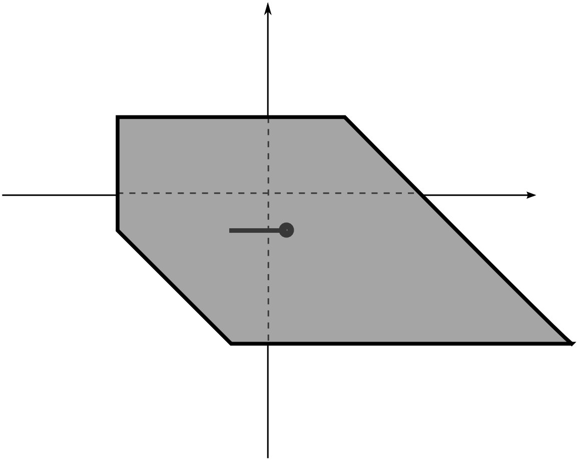


Figure: reduction of $\mathcal{O}(-1) \times \mathbb{C}P^1 \times \mathbb{C}P^1$.

Example: $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P}^2$ via TMMP (Gonzalez-Woodward)



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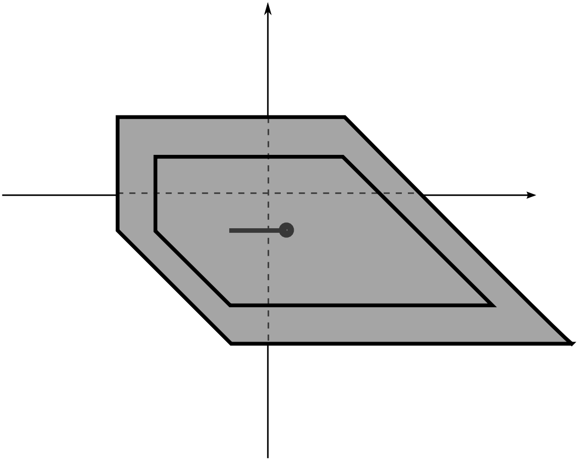

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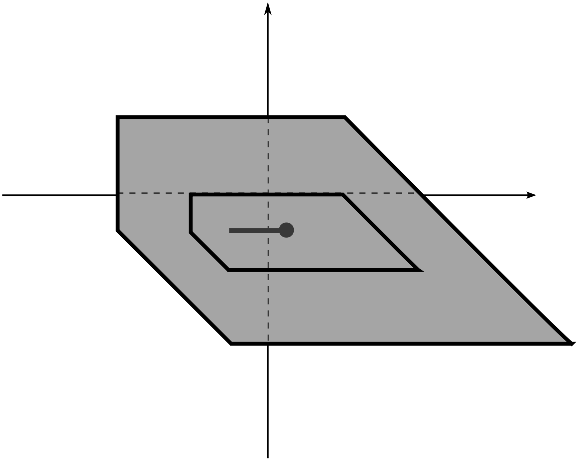

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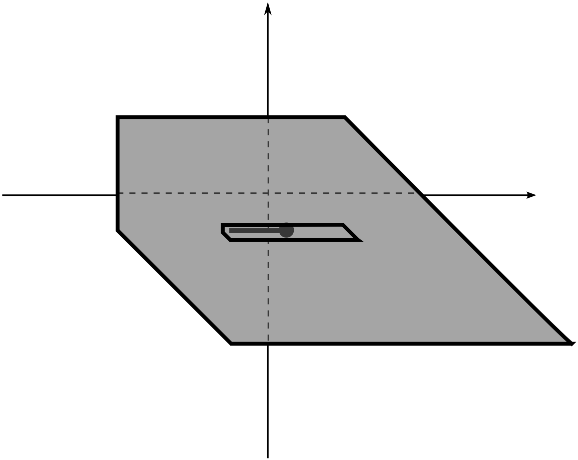


Probes (McDuff)

Definition

A probe P in a moment polytope $\Delta = \mu(M) \subset \mathbb{R}^n$ is a line segment in Δ based at $b_P \in \breve{F}_P =$ interior of a facet $F_P \subset \Delta$, with primitive direction vector $v_P \in \mathbb{Z}^n$ such that $\langle \nu_{F_P}, v_P \rangle = 1$. The length $\ell(P)$ of P is defined as

$$\ell(P) := \max \left\{ t \in \mathbb{R}^+ \ : \ b_P + t\, v_p \in P \subset \Delta \right\} \ .$$

Theorem

If $u = b_P + t\, v_p$ with $t < \ell(P)/2$, then the Lagrangian torus fiber $T_u = \mu^{-1}(u)$ is displaceable.

Remark

Such Lagrangian torus fibers are said to be displaceable by probes.

Beyond Probes (A. – Borman – McDuff)

Observation 1: odd Hirzebruch surfaces have segment of fibers that cannot be displaced by probes.

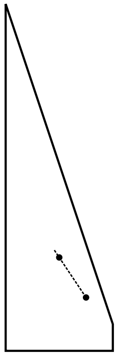


Figure: Hirzebruch surface H_3.

Beyond Probes (A. – Borman – McDuff)

Observation 2: use Karshon's S^1-equiv. symplectomorphisms $H_{\mathrm{odd}} \to H_1 = \mathbb{C}\mathrm{P}^2 \sharp \overline{\mathbb{C}\mathrm{P}}^2$ to prove that fibers with $HF = 0$ are, in fact, displaceable.

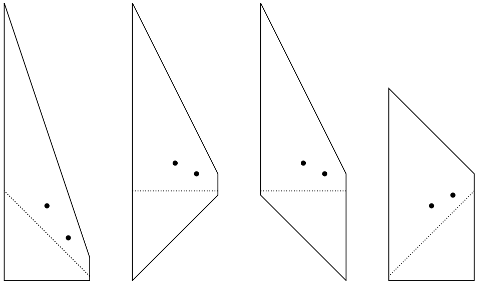


Figure: H_3 - left dot is unique non-displaceable fiber.

Question: how to understand and generalize this example to "probe" further the set of displaceable fibers in toric manifolds?

Symmetric Extended Probes (A. – Borman – McDuff)

Definition

A probe $Q \subset \Delta = \mu(M) \subset \mathbb{R}^n$ *is symmetric if it exits* Δ *at an interior point of a facet* F'_Q *with* $\langle \nu_{F'_Q}, v_Q \rangle = -1$.

Definition

A symmetric extended probe $\mathcal{P}$ *is formed by deflecting a probe* P *with a symmetric probe* Q*:*

$$\mathcal{P} = P \cup Q \cup P' \subset \Delta\,.$$

The length $\ell(\mathcal{P})$ *of* $\mathcal{P}$ *is defined as the sum*

$$\ell(\mathcal{P}) := \ell(P) + \ell(P')\,.$$

Symmetric Extended Probes (A. – Borman – McDuff)

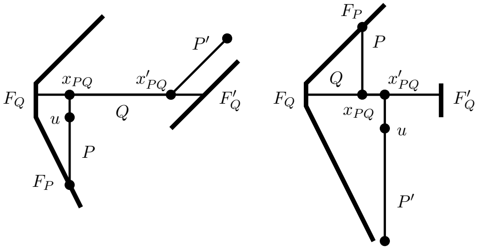


Figure: Two ways of using symmetric extended probes.

Theorem

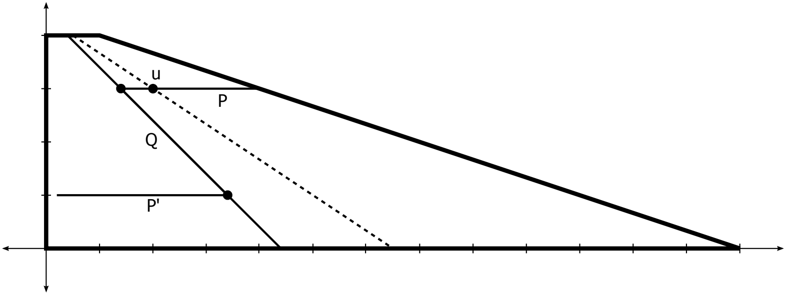
If

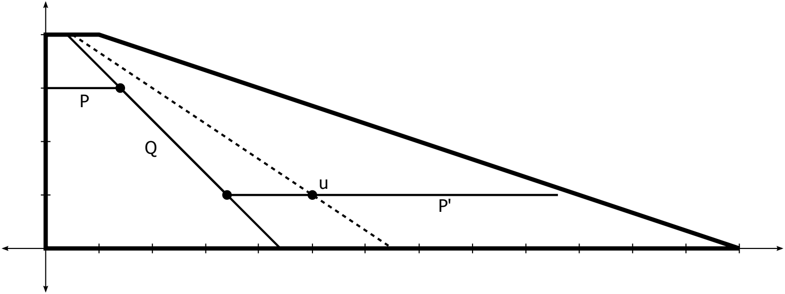
(i) $u = b_p + t v_P \in \check{P}$ *with* $t < \ell(\mathcal{P})/2$, *or*

(ii) $u = x' + t v_{P'} \in \check{P}'$ *with* $\ell(P) + t < \ell(\mathcal{P})$,

then the Lagrangian torus fiber $T_u = \mu^{-1}(u)$ *is displaceable.*

Application to Hirzebruch surfaces





Proof

- Let $\ell := \ell(\mathcal{P})$,

  $$\mathbb{D}(a) := \{z \in \mathbb{C}\,,:\ |z|^2 \leq a\} \subset (\mathbb{C}, \omega_0) \quad \text{with} \quad \int_{\mathbb{D}(a)} \omega_0 = a\,,$$

  $$S^1(a) := \partial\mathbb{D}(a) \quad \text{and} \quad \mathbb{A}(b,c) := \mathbb{D}(c) \setminus \operatorname{int}\mathbb{D}(b)\,,\ 0 \leq b < c\,.$$

- It suffices to construct an embedding

  $$\psi_{\mathcal{P}} : \mathbb{D}(\ell) \times \mathbb{T}^{n-1} \to (M^{2n}, \omega)$$

  such that, for some $a < \ell/2$,

  $$\psi_{\mathcal{P}}^* \omega = \pi^* \omega_0 \quad \text{and} \quad \psi_{\mathcal{P}}(S^1(a) \times \mathbb{T}^{n-1}) = T_u\,,$$

  where $\pi : \mathbb{D}(\ell) \times \mathbb{T}^{n-1} \to \mathbb{D}(\ell)$ is the projection.

Proof

- Step 1: for $\ell_P := \ell(P)$, construct embedding $\psi_P : \mathbb{D}(\ell_P) \times \mathbb{T}^{n-1} \to (M^{2n}, \omega)$ such that $\psi_P^* \omega = \pi^* \omega_0$ and $\mathrm{im}(\psi_P) \subset \mu^{-1}(P)$. In case (i), need also that $\psi_P(S^1(a) \times \mathbb{T}^{n-1}) = T_u$ with $a = t$.

- Step 2: construct embedding $\psi_{P'} : \mathbb{A}(\ell_P, \ell) \times \mathbb{T}^{n-1} \to (M^{2n}, \omega)$ such that $\psi_{P'}^* \omega = \pi^* \omega_0$ and $\mathrm{im}(\psi_{P'}) \subset \mu^{-1}(P')$. In case (ii), need also that $\psi_{P'}(S^1(a) \times \mathbb{T}^{n-1}) = T_u$ with $a = \ell(P) + t$.

- Step 3: use deflecting symmetric probe Q to construct symplectomorphism $\Psi : (M^{2n}, \omega) \to (M^{2n}, \omega)$, with support in neighborhood of $\mu^{-1}(Q)$, such that $\Psi \circ \psi_P$ and $\psi_{P'}$ glue to give embedding $\psi_{\mathcal{P}} : \mathbb{D}(\ell) \times \mathbb{T}^{n-1} \to (M^{2n}, \omega)$ with required properties.

$\mathbb{C}P(1,3,5)$ (McDuff, Wilson-Woodward)

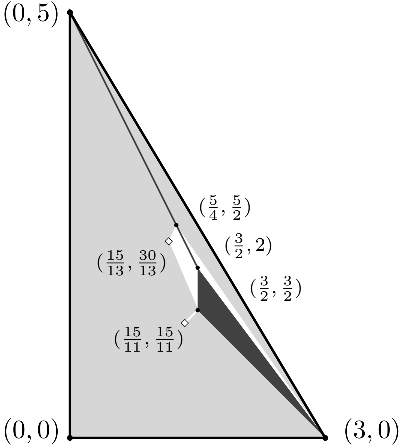


Figure: The fibers in gray are displaceable by standard probes.
The points in black and the two diamonds are non-displaceable.
Displaceability is still unknown for the points in white.
One cannot do better with extended probes.

Partial resolution of $\mathbb{C}P(1,3,5)$

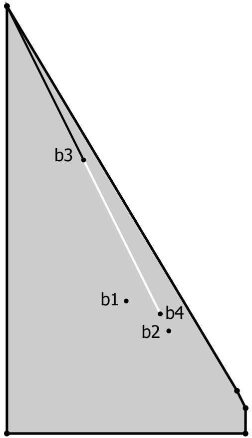


Figure: Fibers in gray are displaceable by (symmetric extended) probes. b_1 and points on the black line are non-displaceable. Displaceability is still unknown for b_2, b_4 and points on the white line.

Minimal resolutions of A_n-singularities

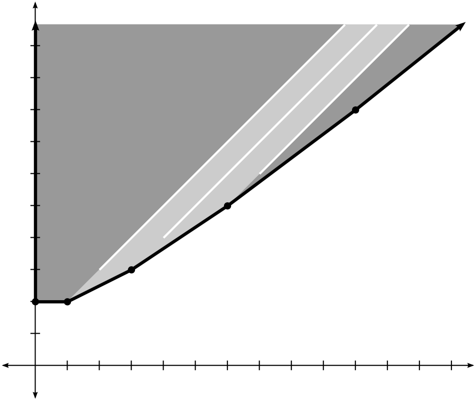


Figure: minimal resolution of A_4-singularity, with normals $\nu_1 = (1, 0)$, $\nu_2 = (0, 1)$, $\nu_3 = (-1, 2)$, $\nu_4 = (-2, 3)$, $\nu_5 = (-3, 4)$, $\nu_6 = (-4, 5)$. Fibers in light and medium gray are displaceable by standard and/or extended probes. Displaceability is still unknown for points on the white rays.

Minimal resolutions of (n, m)-singularities

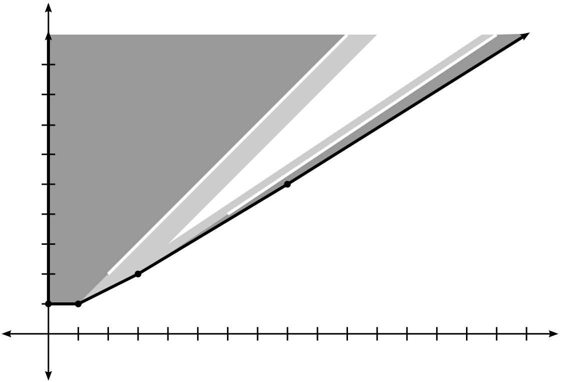


Figure: minimal resolution of $(5, 8)$-singularity, with normals
$\nu_1 = (1, 0)$, $\nu_2 = (0, 1)$, $\nu_3 = (-1, 2)$, $\nu_4 = (-3, 5)$, $\nu_5 = (-5, 8)$.
Fibers in light and medium gray are displaceable by standard and/or extended probes. Displaceability is still unknown for points on the white rays and regions.