CAUSAL RELATIONS ON THE MANIFOLD OF LIGHT RAYS

by

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Hertford College

A Thesis submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy

Mathematical Institute

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Trinity Term, 2000

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ABSTRACT

The aim of this work is to try to understand spacetime geometry (and in particular causal relations) in the manifold of light rays of a given (globally hyperbolic) spacetime.

After defining the fundamental notions of the manifold of light rays and the sky of a spacetime point (chapter 1), some important ideas about static spacetimes are reviewed (chapter 2), and examples of skies in such spacetimes (both (2+1) and (3+1)-dimensional) are examined (chapters 3 and 4). The conjectured relationship between linking and causality is discussed (chapter 5). The contact structure of the manifold of light rays is studied and the idea of Legendrian linking described (chapter 6). This contact structure is then used to analyze (2+1)-dimensional skies (chapter 8). Results are obtained for the (2+1) and (3+1)-dimensional cases (chapters 9 and 10).

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CHAPTER 1

The manifold of light rays

1.1 The set of light rays

Definition 1 *Given a spacetime (M, g) (M being a real 4-manifold and g a smooth Lorentzian metric on M), its* set of light rays *is just the set N of its null geodesics.*

Proposition 2 *This set can be thought of as a quotient set $D^nM/\sim$, where D^nM is a certain fibre bundle over M with fibre S^2 and $\sim$ is a certain equivalence relation on D^nM.*

Proof: The tangent bundle TM possesses a sub-bundle T^nM obtained by selecting the nonvanishing null vectors on each tangent space T_xM ($x \in M$). Notice that T^nM (which is *not* a vector bundle) admits an obvious effective, free action by $\mathbb{R}^*$(the multiplicative group of real numbers). Taking the quotient of T^nM by this action originates a fibre bundle over M with fibre S^2, which we shall call the *bundle of null directions* of (M, g), and shall denote by D^nM. It is obvious that every null geodesic on M has a unique lift to D^nM, and that these lifted null geodesics foliate D^nM by one-dimensional manifolds. Thus taking $\sim$ to be the equivalence relation of belonging to the same lifted null geodesic clearly gives us the bijection $N \approx D^nM/\sim$.□

Of course that since any Lorentzian metric is by definition nondegenerate, we have a natural diffeomorphism $TM \approx T^*M$, and could therefore make this construction using the cotangent bundle instead of the tangent bundle; this is done in [L].

For certain classes of spacetimes, it is possible to endow N with a differentiable structure. The most obvious example will be discussed in the next section.

1.2 Globally hyperbolic spacetimes

Proposition 3 *Let (M, g) be a globally hyperbolic spacetime and Σ any Cauchy surface. Then N has a natural structure of a fibre bundle over Σ with fibre S^2.*

Proof: Since every null geodesic intersects Σ exactly once, it is clear that we have the bijection $D^n M/\sim\, \approx D^n M|_\Sigma$. Because $D^n M|_\Sigma$ is clearly a fibre bundle over Σ with fibre S^2, one can use the bijection to introduce this structure on N.□

Of course that we are not so much interested in the fibre bundle structure mentioned above but just in the differentiable structure it implies. To this end the following result is important:

Proposition 4 *The differentiable structure induced in N by the identification $D^n M/\sim\, \approx D^n M|_\Sigma$ is independent of the choice of Σ.*

Proof: Let Σ_1, Σ_2 be two Cauchy surfaces. It is obvious that because $D^n M/\sim\, \approx D^n M|_{\Sigma_1}$ and $D^n M/\sim\, \approx D^n M|_{\Sigma_2}$, we have the bijection $D^n M|_{\Sigma_1} \approx D^n M|_{\Sigma_2}$. All that remains to prove then is that this bijection is in fact a diffeomorphism. But this is a consequence of the smooth dependence on initial data of the solutions of the geodesic equations (recall that we are assuming g to be a smooth tensor field).□

Thus for any globally hyperbolic spacetime N has a natural differentiable structure.

Definition 5 *We shall define the* manifold of light rays *of a globally hyperbolic spacetime to be its set of light rays N endowed with this differentiable structure.*

It is clear from the identification $N \approx D^n M|_\Sigma$ that

$$\dim(N) = \dim(\, D^n M|_\Sigma) = \dim(\Sigma) + \dim(S^2) = 3 + 2 = 5.$$

Any Cauchy surface Σ of our globally hyperbolic spacetime (M, g) is of course a Riemannian manifold with (minus) the induced metric. We have the following

Definition 6 *Let (Σ, h) be a Riemannian manifold. Then its* tangent sphere bundle *is the sub-bundle $TS(\Sigma)$ of $T\Sigma$ consisting of all unit tangent vectors to Σ.*

By considering at each point $x \in \Sigma$ the natural decomposition $T_x M = T_x\Sigma \oplus (T_x\Sigma)^\perp$ one readily obtains

Proposition 7 *Let (M, g) be a globally hyperbolic spacetime and Σ any Cauchy surface. Then N is diffeomorphic to $TS(\Sigma)$.*

1.3 Other spacetimes

There are of course non-globally hyperbolic spacetimes whose set of light rays possesses a natural differentiable structure. Nevertheless we shall from this point on assume that all spacetimes we're dealing with are globally hyperbolic. For technical reasons, we shall also assume them to be time orientable.

1.4 Skies and light rays

Every point γ of the manifold of light rays N of a spacetime (M, g) represents a null geodesic of this spacetime. One might think of reversing this correspondence and for every point $x \in M$ consider the set $X \subseteq N$ of all null geodesics through x.

Definition 8 *Given $x \in M$, the set $X = \{\gamma \in N : x \in \gamma\}$ shall be called the* sky *of the point x.*

Proposition 9 *The sky X of any point $x \in M$ is an embedded submanifold of N diffeomorphic to S^2.*

Proof: Recall we are assuming (M, g) to be globally hyperbolic. Let Σ be a Cauchy surface through x. Using the identification $N \approx TS(\Sigma)$ it is clear that X is just the fibre over x, and is thus obviously an embedded submanifold diffeomorphic to S^2.□

This leads us to try to reinterpret spacetime geometry in the manifold of light rays, the causal properties being of course our first main concern. In doing this, we must have in mind that some information about the geometry is lost as one goes to the manifold of light rays. For instance, we have the well known (see [W])

Proposition 10 *Let (M, g) and $(M, \widehat{g})$ be conformally related spacetimes. Then they have the same null geodesics.*

Proof: Let $\widehat{g} = \Omega^2 g$ and $\{x^{\mathbf{a}}\}$ be local coordinates on M (we will be using the notation conventions of [PR]). The Christoffel symbols for the Levi-Civita connection associated to $\widehat{g}$ are related to those associated to g through the expression

$$\begin{aligned} \widehat{\Gamma}^{\mathbf{a}}_{\mathbf{bc}} &= \frac{1}{2}\widehat{g}^{\mathbf{ad}}\left(\nabla_{\mathbf{b}}\widehat{g}_{\mathbf{cd}} + \nabla_{\mathbf{c}}\widehat{g}_{\mathbf{bd}} - \nabla_{\mathbf{d}}\widehat{g}_{\mathbf{bc}}\right) \\ &= \frac{1}{2}\Omega^{-2}g^{\mathbf{ad}}\left[\nabla_{\mathbf{b}}\left(\Omega^2 g_{\mathbf{cd}}\right) + \nabla_{\mathbf{c}}\left(\Omega^2 g_{\mathbf{bd}}\right) - \nabla_{\mathbf{d}}\left(\Omega^2 g_{\mathbf{bc}}\right)\right] \\ &= \Gamma^{\mathbf{a}}_{\mathbf{bc}} + \Omega^{-1}\left(\nabla_{\mathbf{b}}\Omega g_{\mathbf{c}}{}^{\mathbf{a}} + \nabla_{\mathbf{c}}\Omega g_{\mathbf{b}}{}^{\mathbf{a}} - \nabla^{\mathbf{a}}\Omega g_{\mathbf{bc}}\right). \end{aligned}$$

Now let $x^{\mathbf{a}} = x^{\mathbf{a}}(s)$ be an affinely parametrized geodesic for the metric g. We have

$$\ddot{x}^{\mathbf{a}} + \Gamma^{\mathbf{a}}_{\mathbf{bc}}\,\dot{x}^{\mathbf{b}}\dot{x}^{\mathbf{c}} = 0$$

and therefore

$$\begin{aligned} \ddot{x}^{\mathbf{a}} + \widehat{\Gamma}^{\mathbf{a}}_{\mathbf{bc}}\,\dot{x}^{\mathbf{b}}\dot{x}^{\mathbf{c}} &= \ddot{x}^{\mathbf{a}} + \Gamma^{\mathbf{a}}_{\mathbf{bc}}\,\dot{x}^{\mathbf{b}}\dot{x}^{\mathbf{c}} + \Omega^{-1}\left(\nabla_{\mathbf{b}}\Omega g_{\mathbf{c}}{}^{\mathbf{a}} + \nabla_{\mathbf{c}}\Omega g_{\mathbf{b}}{}^{\mathbf{a}} - \nabla^{\mathbf{a}}\Omega g_{\mathbf{bc}}\right)\dot{x}^{\mathbf{b}}\dot{x}^{\mathbf{c}} \\ &= \Omega^{-1}\left(2\frac{d\Omega}{ds}\,\dot{x}^{\mathbf{a}} - g_{\mathbf{bc}}\,\dot{x}^{\mathbf{b}}\dot{x}^{\mathbf{c}}\,\nabla^{\mathbf{a}}\Omega\right). \end{aligned}$$

Thus affinely parametrized null geodesics of the metric g (which of course satisfy $g_{\mathbf{ab}}\,\dot{x}^{\mathbf{a}}\dot{x}^{\mathbf{b}} = 0$) are (in general) non-affinely parametrized geodesics of the metric $\widehat{g}$, which are necessarily null geodesics in view of the relationship $\widehat{g} = \Omega^2 g$. The roles of the two metric tensors can obviously be interchanged and it is therefore clear that the null geodesics for both these metrics coincide.□

On the other hand, we also have

Proposition 11 *Let g, $\widehat{g}$ be two Lorentzian metrics on the same 4-manifold M such they have the same null vectors. Then they are conformally related.*

Proof: Let $\{t^a, x^a, y^a, z^a\}$ be an orthonormal frame on some coordinate neighbourhood U with respect to g. Since both metric tensors yield the same null vectors we have

$$g(t^a \pm x^a, t^a \pm x^a) = 0 \Rightarrow \widehat{g}(t^a \pm x^a, t^a \pm x^a) = 0$$

which is equivalent to

$$\widehat{g}(t^a, t^a) + \widehat{g}(x^a, x^a) \pm 2\widehat{g}(t^a, x^a) = 0$$

yielding

$$\widehat{g}(t^a, x^a) = 0$$

and

$$\widehat{g}(x^a, x^a) = -\widehat{g}(t^a, t^a).$$

This is of course also true for y^a and z^a and thus we get

$$\widehat{g} = \widehat{g}(t^a, t^a)g$$

clearly showing that g and $\widehat{g}$ are conformally related.□

We thus see that

Proposition 12 *Let (M, g) be a spacetime; then the knowledge of its (parametrized) null geodesics is equivalent to the knowledge of its conformal structure.*

So merely by considering only null geodesics we are throwing away information about the scale factor of the metric tensor. Things are even worse if we are only given the manifold of light rays N and information about which 2-submanifolds are associated with points of spacetime. For one thing, different points may have the same sky. This happens when all light rays through a certain point refocus on another point, as is the case for instance in the Einstein universe, or in the closed matter-dominated Friedmann-Robertson-Walker universe. As a matter of fact it would not be possible to distinguish between these two spacetimes based solely on the manifold of light rays, although they are clearly distinct (and are not even conformally related).

CHAPTER 2

Static spacetimes

2.1 Space manifold

Null geodesics on static spacetimes are particularly simple. Studying these will allow us to understand the manifold of light rays of a number of spacetimes (namely those conformally equivalent to static spacetimes). We shall review some ideas about static spacetimes, beginning with the well known

Definition 13 *A spacetime* (M, g) *is said to be* stationary *if it possesses a complete nowhere vanishing timelike Killing vector field.*

Definition 14 *A stationary spacetime* (M, g) *is said to be* static *if the orthogonal distribution to its Killing vector field* t^a *is integrable.*

We shall restrict ourselves to (globally hyperbolic) static spacetimes such that the integral surfaces of this distribution are Cauchy surfaces. Thus from this point on "static spacetime" means "globally hyperbolic static spacetime such that the integral surfaces of the orthogonal distribution to the Killing field are Cauchy surfaces".

If Σ is an arbitrary such surface, one can extend any local chart $\left\{x^{\mathbf{i}}\right\}$ on Σ to a local chart $\left\{t, x^{\mathbf{i}}\right\}$ on M by using the integral curves of t^a, t being the parameter along these curves (here we will take the indices $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, ... to run from 1 to 3). On these coordinates, the fact that t^a is a Killing field translates into

$$\frac{\partial g_{\mathbf{ab}}}{\partial t} = 0$$

whereas the fact that Σ is an integral surface of t_a is translated into

$$g_{\mathrm{oi}}|_{\Sigma} = 0$$

These two facts taken together obviously imply that on the coordinate chart mentioned above the metric tensor is written

$$(g_{\mathbf{ab}}) = \begin{pmatrix} g_{00} & 0 \\ 0 & -h_{\mathrm{ij}} \end{pmatrix}$$

where the functions g_{00} and h_{ij} do not depend on t. Notice that the fact that g is a Lorentzian metric obviously implies that $g_{00} > 0$ and that the matrix (h_{ij}) is positive definite, thus defining a Riemannian metric h on Σ. This is enough to prove

Proposition 15 *Let (M, g) be a (globally hyperbolic) static spacetime with a timelike Killing vector field t^a. Then all the integral surfaces of t_a with the Riemannian metric induced by g are isometric, the isometry being provided by the flow of t^a. Moreover, there exists a global time function t such that $t^a = \frac{\partial}{\partial t}$ and the integral surfaces of t_a are the time slices $t = const.$*

Thus one may think of a static spacetime as having a *space manifold* (namely the Riemannian manifold (Σ, h)), which remains unchanged in time, and a *global time* in which events unfold. We shall tend to do so, committing frequent abuses of language: for instance, we will talk of light rays either referring to null geodesics on M or to their (obvious) projections on Σ parametrized by the global time.

It should be noted that Minkowski spacetime is a very atypical case of a static spacetime in the sense that it admits a 3-parameter family of timelike Killing vector fields (corresponding to the set of nonvanishing constant future-pointing timelike vector fields), and thus a 3-parameter family of global time functions (this being the reason for the non-existence of a unique notion of simultaneity for inertial observers in Minkowski spacetime). A more typical example would be the region $r > 2M$ of Schwarzschild spacetime, which possesses just a one-parameter family of timelike Killing vector fields, and consequently just a one-parameter family of global time functions. Since these functions are constant multiples of each other, one may in fact say that Schwarzschild spacetime admits essentially one global time function (which in a sense is the time measured by an observer at infinity). What happens here is of course that by introducing matter in our static spacetime we single out a special class of observers, namely those at rest with respect to that matter, and consequently a preferred notion of simultaneity.

2.2 Fermat's principle

The main reasons why the null geodesics of static spacetimes are easier to understand are of course contained in

Proposition 16 *Let* (M, g) *be a static spacetime. Then its geodesics possess a* time translation invariance, *in the following sense: if* $t = f^0(s)$, $x^i = f^i(s)$ *is a* *solution of the geodesic equations then* $t = f^0(s) + c$, $x^i = f^i(s)$ *is also a solution for any* $c \in \mathbb{R}$.

Proof: The geodesic Lagrangian in our chosen coordinate chart is of course

$$L = \frac{1}{2}\left(g_{00}\,\dot{t}^2 - h_{ij}\,\dot{x}^i \dot{x}^j\right)$$

meaning that the geodesic equations are

$$\frac{d}{ds}\left(g_{00}\,\dot{t}\right) = 0 \tag{2.1}$$

(which can be thought of as an energy conservation law), and

$$\frac{d}{ds}\left(h_{ij}\,\dot{x}^j\right) + \frac{1}{2}\nabla_i g_{00}\,\dot{t}^2 - \frac{1}{2}\nabla_i h_{jk}\,\dot{x}^j \dot{x}^k = 0 \tag{2.2}$$

We thus see that the result stated above is a trivial consequence of the fact that for each value of s both sets of functions have the same values of $\dot{t}$, $\dot{x}^i$ and x^i, the value of t being of course irrelevant as far as the geodesic equations are concerned. It is easy to see that this result in fact holds for general stationary spacetimes.□

Proposition 17 *Let* (M, g) *be a static spacetime. Then its geodesics possess a* time reflection invariance, *in the following sense: if* $t = f^0(s)$, $x^i = f^i(s)$ *is a solution of the geodesic equations then* $t = 2f^0(0) - f^0(-s)$, $x^i = f^i(-s)$ *is also a solution.*

Proof: One has but to check that, assuming that the geodesic equations 2.1 and 2.2 hold for the first set of functions, they also hold for the second set. It can be shown that, unlike the previous result, this proposition does not hold for general stationary spacetimes.□

Light rays single out preferred curves along the space manifold of a static spacetime. Indeed, proposition 16 guarantees that a light ray through a certain point of the space manifold in a certain direction will always transverse the same spatial path irrespective of the instant in time in which it happens to pass through that point. On the other hand, proposition 17 tells us that light rays travelling in opposite spatial directions follow the same path. This is characteristic of geodesic curves, and as a matter of fact we have

Theorem 18 *Let (M, g) be a static spacetime and (Σ, h) its space manifold. Then the light rays on Σ are the geodesics of the Riemannian manifold $(\Sigma, \frac{1}{g_{00}}h)$ (where $g_{00} = t_a t^a$ is well defined), and the global time t is an affine parameter for these geodesics.*

Proof: As is well known any null geodesic will satisfy

$$L = 0 \Leftrightarrow \dot{t}^2 = \frac{1}{g_{00}} h_{ij}\, \dot{x}^i \dot{x}^j .$$

Using this on equation 2.2 one gets

$$\frac{d}{ds}\left(h_{ij}\, \dot{x}^j\right) + \frac{1}{2g_{00}} \nabla_i g_{00} h_{jk}\, \dot{x}^j \dot{x}^k - \frac{1}{2} \nabla_i h_{jk}\, \dot{x}^j \dot{x}^k = 0. \tag{2.3}$$

Now one can use equation 2.1 and the scale freedom in the affine parameter s of the null geodesic to set

$$g_{00}\, \dot{t} = 1 \Leftrightarrow \dot{t} = \frac{1}{g_{00}}$$

allowing us to write 2.3 as

$$\frac{d}{dt}\left(\frac{1}{g_{00}}h_{ij}\frac{dx^j}{dt}\right)+\frac{1}{2g_{00}^2}\nabla_i g_{00}h_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}-\frac{1}{2g_{00}}\nabla_i h_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=0$$

$$\Leftrightarrow \frac{d}{dt}\left(\frac{1}{g_{00}}h_{ij}\frac{dx^j}{dt}\right)-\frac{1}{2}\nabla_i\left(\frac{1}{g_{00}}h_{jk}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}=0$$

which are clearly the Euler-Lagrange equations for the Lagrangian

$$L=\frac{1}{2g_{00}}h_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}$$

or, in other words, the geodesics for the line element

$$dt^2=\frac{1}{g_{00}}h_{ij}dx^i dx^j.\square$$

One can think about this result in a number of ways. The first and most obvious one is to think that light rays are determined by this new Riemannian metric on Σ (which we shall call the *light metric*), which is conformally equivalent to the natural metric h. Stationary observers (i.e., observers whose worldlines are the integral curves of t^a) would measure (a quantity proportional to) the distance determined by this metric in radar experiments. However a much more interesting way to think about this is to use the fact that t is an affine parameter of the geodesics of the light metric to realize that theorem 18 is really saying that we have a *Fermat's principle* (see [A]) for the propagation of light on our space manifold: a light ray will travel between two points using the path that allows it to do so in minimum time, the speed of light at each point being given by $\sqrt{g_{00}}$.

We shall be interested in studying light wavefronts emanating from one point from our space manifold and propagating across it. To this end it is interesting to recall the following

Theorem 19 *Let (Σ, h) be an arbitrary Riemannian manifold and $x \in \Sigma$. Let $B_\varepsilon(x)$ be the set of all points lying at a distance $\varepsilon > 0$ from x (along some geodesic), and $y \in B_\varepsilon(x)$ a point not conjugate to x. Then the geodesic joining x to y is orthogonal to $B_\varepsilon(x)$.*

Proof: Let T^a be the unit tangent vector to the congruence of geodesics emanating from x. Consider an arbitrary nonzero tangent vector to $B_\varepsilon(x)$ at y. Since y is not conjugated to x, there exists a Jacobi field J^a along the geodesic connecting x to y which coincides with the given tangent vector at y. Such Jacobi field can be chosen to satisfy

$$[T, J] = 0 \Leftrightarrow T^a \nabla_a J^b = J^a \nabla_a T^b$$

and since T^a satisfies the geodesic equation,

$$T^a \nabla_a T^b = 0$$

it follows that

$$\begin{aligned} T^a \nabla_a \left(J_b T^b \right) &= T^b T^a \nabla_a J_b + J_b T^a \nabla_a T^b \\ &= T^b J^a \nabla_a T_b = J^a \nabla_a \left(\frac{1}{2} T^b T_b \right) = 0. \end{aligned}$$

Now since we are considering the congruence of geodesics through x, and chose J^a to satisfy $[T, J] = 0$, we have $J^a = 0$ at x, and therefore $J_a T^a = 0$ along the geodesic. Therefore the tangent space to $B_\varepsilon(x)$ at y is orthogonal to T^a.□

It is then clear that a light wavefront emanating from one point x in our space manifold Σ is orthogonal to the light rays that comprise it (with relation to either the light metric or the space metric, as these are conformally related). This information will be useful later on.

There are many other interesting aspects of static spacetimes, which we shall not mention as they are not directly relevant to our problem. We cannot however resist stating

Theorem 20 *Let (M, g) be a static spacetime with Killing vector field t^a, G_{ab} its Einstein tensor and $\widetilde{R}$ the Ricci scalar curvature of its space manifold. Then in any orthonormal frame containing a timelike vector parallel to t^a we have $G_{00} = \frac{1}{2}\widetilde{R}$.*

The proof of this curious connection between the geometry of the space manifold and Einstein's equations is fairly straightforward. Notice that if no cosmological constant term is present we can write $\widetilde{R} = -16\pi\rho$, where ρ is the energy density measured by the static observers. Thus for instance for static vacuum solutions of Einstein's field equations (e.g. the region $r > 2M$ of Schwarzschild's spacetime) the space manifold satisfies $\widetilde{R} = 0$.

CHAPTER 3

Examples in (2+1)-spacetimes

3.1 (2+1)-spacetimes

Definition 21 *A* (2+1)-spacetime *is a pair* (M, g)*, where* M *is a real 3-manifold and* g *is a smooth Lorentzian metric on* M*.*

Again we shall assume all (2+1)-spacetimes to be globally hyperbolic. All that has been said about ordinary (3+1)-dimensional spacetimes (except for theorem 20) carries over in a fairly straightforward way to (2+1)-spacetimes, the main difference being the dimensionality of the various objects that have been introduced. So if (M, g) is a globally hyperbolic (2+1)-spacetime and Σ a Cauchy surface, its set of light rays N is diffeomorphic to $TS(\Sigma)$, thus being a differentiable manifold with

$$\dim(N) = \dim(\Sigma) + \dim(S^1) = 2 + 1 = 3.$$

This makes the manifold of light rays for a (2+1)-spacetime easy to visualize.

Given $x \in M$, we shall again call the set $X = \{\Gamma \in N : x \in \gamma\}$ the *sky* of the point x; it is clearly an embedded submanifold of N diffeomorphic to S^1.

Proposition 22 *Any Cauchy surface* Σ *of an orientable (globally hyperbolic) (2+1)-spacetime* (M, g) *is necessarily orientable.*

Proof: Recall that any globally hyperbolic spacetime possesses a nonvanishing timelike vector field t^a, which can obviously never be tangent to Σ. We can use this and the orientability of M to define an orientation on the tangent space to Σ as follows: fix an orientation on M; if $x \in \Sigma$ we shall say that the basis $\{x^a, y^a\} \subseteq T_x\Sigma$ is right-handed *iff* $\{t^a, x^a, y^a\}$ is right-handed. It is obvious that this gives us indeed a consistent orientation for Σ.□

We can use this to give the sky of any point $x \in M$ an orientation (assuming that M is orientable). In fact, because Σ is orientable, we can clearly give an orientation to the fibres of $TS(\Sigma)$ (say the right-handed one); and since any sky is a fibre of $TS(\Sigma)$ for some Σ, any sky is thus endowed with an orientation. It is easy to see that the orientation attributed to the sky of a point $x \in M$ is independent of the particular Cauchy surface $\Sigma \ni x$ used to do so.

3.2 Minkowski (2+1)-spacetime

Definition 23 *We shall define* Minkowski (2+1)-spacetime *as* $\mathbb{R}^3$ *with Cartesian coordinates* $\{t, x, y\}$ *and the flat Lorentzian metric*

$$ds^2 = dt^2 - dx^2 - dy^2.$$

Proposition 24 *The manifold of light rays of Minkowski (2+1)-spacetime is* $\mathbb{R}^2 \times S^1$.

Proof: This is clear from the fact that the plane $t = 0$ is a Cauchy surface for Minkowski (2+1)-spacetime.□

An easy way to visualize this manifold is to consider $\mathbb{R}^2 \times [0, 2\pi] \subseteq \mathbb{R}^3$ with the planes $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{2\pi\}$ identified (see figure 3.1). In this manner, the sky of a point $(0, x, y)$ on our Cauchy surface $t = 0$ will be represented as the line $\{(x, y)\} \times [0, 2\pi]$, which is a closed curve by virtue of the identification. The orientation of this sky will obviously be upwards, corresponding to the right-handed orientation of S^1 as a fibre of the tangent sphere bundle to $t = 0$, if the S^1 coordinate θ is chosen in the usual way as the angle from $\frac{\partial}{\partial x}$.

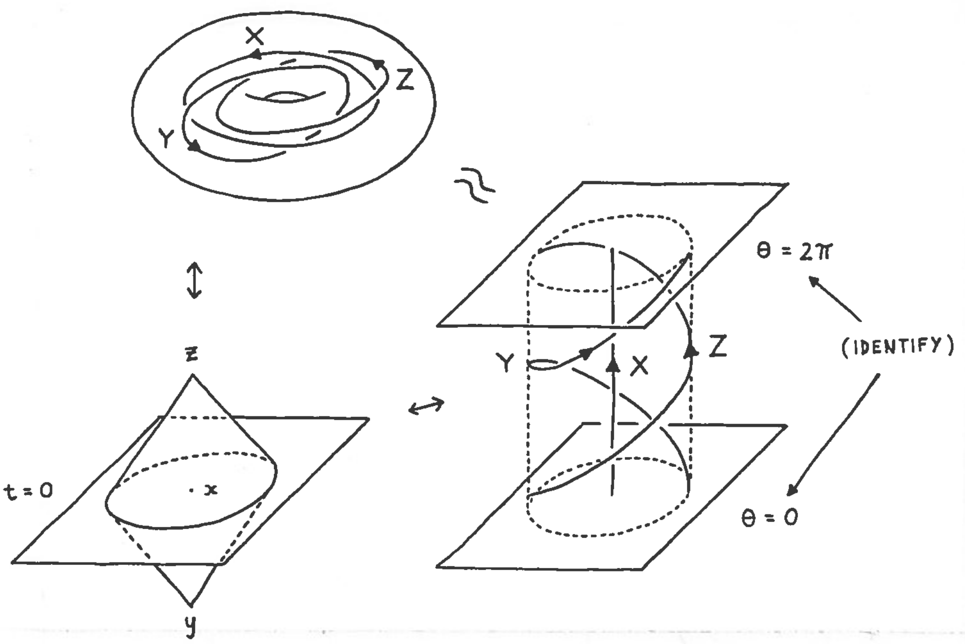

Figure 3.1: The manifold of light rays of Minkowski (2+1)-spacetime

If we now think of the point (t, x, y) with $t < 0$, we see that the null geodesics through this point intersect $t = 0$ along a circle with centre $(0, x, y)$ and radius $|t|$. Also these are *outgoing* null geodesics; it is therefore not hard to see that the sky of this point will be

$$\{(x + |t| \cos\theta, y + |t| \sin\theta, \theta) : 0 \leq \theta \leq 2\pi\}.$$

The orientation of this sky is again upwards.

If $t > 0$ the null geodesics through (t, x, y) will intersect $t = 0$ along a circle with centre $(0, x, y)$ and radius t. Since now these null geodesics are *ingoing*, we get as the sky of this point the curve

$$\{(x + t\cos(\theta + \pi), y + t\sin(\theta + \pi), \theta) : 0 \leq \theta \leq 2\pi\}.$$

The sky orientation is *still* upwards; this can be thought of as a consequence of the fact that the antipodal map on S^2 is orientation-reversing.

We thus see that the skies of points in Minkowski spacetime are represented by certain vertical helices on $\mathbb{R}^2 \times [0, 2\pi]$ going once around. Notice that not every such helix represents the sky of a point: if, for instance, we take an helix that does represent a sky and rotate it slightly about its symmetry axis we end up with an helix that does not.

It is interesting to see that the points timelike-separated from $(0, x, y)$ correspond to helices that wind around $\{(0, x, y)\} \times [0, 2\pi]$, always in the positive sense. Another way to put this is as follows: $\mathbb{R}^2$ is diffeomorphic to the 2-dimensional open ball B^2; therefore the manifold of light rays of Minkowski (2+1)-spacetime is diffeomorphic to $B^2 \times S^1$, which in turn is diffeomorphic to the interior of a 3-dimensional solid torus (regarded as a subset of $\mathbb{R}^3$). We can therefore apply the well known notions of linking and linking number to skies by simply interpreting them as closed oriented curves on $\mathbb{R}^3$. This, of course, depends on the choice of solid torus and diffeomorphism.

Definition 25 *The* standard solid torus *in* $\mathbb{R}^3$ *is the set*

$$T = \{(2\cos\varphi + r\cos\theta, 2\sin\varphi + r\cos\theta, r\sin\theta) : 0 \leq r < 1, 0 \leq \theta, \varphi < 2\pi\}.$$

Definition 26 *The* standard diffeomorphism $f : B^2 \times S^1 \to T$ *is the map defined by*

$$f\left(r\cos\theta, r\cos\theta, \varphi\right) = \left(2\cos\varphi + r\cos\theta, 2\sin\varphi + r\cos\theta, r\sin\theta\right)$$

for all $0 \leq r < 1, 0 \leq \theta, \varphi < 2\pi$.

From this point on we shall identify any subset of $B^2 \times S^1$ with a subset of $\mathbb{R}^3$ via the standard diffeomorphism, thus being able to discuss linking and linking numbers on such subset of $B^2 \times S^1$. It is important that we fix a particular diffeomorphism as there are many ways to embed a torus in $\mathbb{R}^3$. Our definition is natural in that it prevents any knotting or twisting of the torus in the embedding.

We can then say that the skies of points timelike-separated from $(0, x, y)$ are linked with the sky of $(0, x, y)$, the linking number being $+1$. Notice as well that points null-separated from $(0, x, y)$ correspond to skies intersecting the sky of $(0, x, y)$ in one point (the null geodesic going through both points), and that the skies of points spacelike-separated from $(0, x, y)$ clearly do not link the sky of $(0, x, y)$.

For convenience, we make the following

Definition 27 *The* generators *of* $T \approx B^2 \times S^1$ *are the curves of the form* $\{x\} \times S^1$.

We can now summarize what we've been saying in the following

Proposition 28 *The manifold of light rays of Minkowski (2+1)-spacetime is (diffeomorphic to) the standard torus* $T \subseteq \mathbb{R}^3$. *The skies of spacetime points are certain curves homotopic to the generators of the torus. Two spacetime points are* *timelike-separated if and only if their skies are linked (the linking number being necessarily +1).*

3.3 Chronological and causal relations

As one moves on to consider curved spacetimes, the notions of timelike, null and spacelike separation must be reconsidered. It is still possible to say that two points are timelike- or null-separated when they belong to the same timelike or null geodesic; but consideration of such examples as $\mathbb{R}^3 \setminus \{(0, x, y) : x \geq 0\}$ with the Minkowski metric $ds^2 = dt^2 - dx^2 - dy^2$ suggests that more adequate definitions are as follows:

Definition 29 *Let (M, g) be a (2+1)-spacetime and $x, y \in M$. The point x is said to* chronologically precede *the point y if there exists a continuous, piecewise geodesic curve $\gamma : [0,1] \to M$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\dot{\gamma}$ is timelike and future-pointing whenever it exists. Two points are said to be* chronologically related *when one of them chronologically precedes the other.*

Definition 30 *Let (M, g) be a (2+1)-spacetime and $x \in M$. The set $I^+(x)$ of all points $y \in M$ such that x chronologically precedes y is called the* chronological future *of x. Also, the set $I^-(x)$ of all points $y \in M$ which chronologically precede x is called the* chronological past *of x.*

Definition 31 *Let (M, g) be a (2+1)-spacetime and $x, y \in M$. The point x is said to* causally precede *the point y if there exists a continuous, piecewise geodesic curve $\gamma : [0,1] \to M$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\dot{\gamma}$ is non-spacelike and* *future-pointing (or zero) whenever it exists. Two points are said to be* causally related *when one of them causally precedes the other.*

Definition 32 *Let* (M,g) *be a (2+1)-spacetime and* $x \in M$*. The set* $J^+(x)$ *of all points* $y \in M$ *such that* x *causally precedes* y *is called the* causal future *of* x*. Also, the set* $J^-(x)$ *of all points* $y \in M$ *which causally precede* x *is called the* causal past *of* x*.*

These definitions also make sense, and shall be adopted, for (3+1)-dimensional spacetimes. Notice that for Minkowski spacetime "chronologically related" is equivalent to "timelike-separated", and "causally related" is equivalent to "timelike- or null-separated" (here we adopt the convention that the zero vector is a null vector). This may however not be the case for other spacetimes, as the above example clearly shows. In globally hyperbolic spacetimes, the chronological and causal future and past are related as follows:

Theorem 33 *If* (M,g) *is a globally hyperbolic spacetime and* $x \in M$ *then* $I^{\pm}(x) = \mathrm{int} J^{\pm}(x)$ *and* $J^{\pm}(x) = \overline{I^{\pm}(x)}$*.*

Proof: See [W].□

3.4 Wavefronts in static (2+1)-spacetimes

As we have pointed out before, the study of null geodesics in static spacetimes is considerably simplified. We start by stating a trivial proposition:

Proposition 34 *Let (M, g) be a (globally hyperbolic) (2+1)-spacetime and Σ one of its Cauchy surfaces. If Σ is diffeomorphic to a subset of $\mathbb{R}^2$ then the manifold of light rays of (M, g) is diffeomorphic to a subset of the standard torus. The skies of spacetime points are certain (unknotted) closed curves homotopic to the generators of the torus.*

Proof: This arises from the facts that if Σ is diffeomorphic to a subset of $\mathbb{R}^2 \approx B^2$ then $TS(\Sigma)$ is a trivial fibre bundle. Since $B^2 \times S^1$ is diffeomorphic to the standard torus and one can use *any* Cauchy surface to build the manifold of light rays, the rest of the proposition easily follows.□

Thus one can in this case again use the familiar concepts of linking and linking number as defined for closed oriented curves on $\mathbb{R}^3$ applied to skies. Notice that a similar line of reasoning allows us to define the winding number of a closed oriented curve about a point in Σ.

Definition 35 *Let (M, g) be a static (2+1)-spacetime, (Σ, h) its space manifold and $\xi \in \Sigma$. The* wavefront originating at ξ at time $t \geq 0$ *is the geodesic sphere $\Phi(\xi, t)$ of radius t for the light metric $\frac{1}{g_{00}}h$, oriented by considering the right-handed orientation on $T_\xi\Sigma$ of the initial condition of each geodesic.*

Proposition 36 *Let (M, g) be a static (2+1)-spacetime, (Σ, h) its space manifold and t the global time, Σ being diffeomorphic to a subset of B^2. Let $x, y \in M$ be given by $x = (t_1, \xi)$ and $y = (t_2, \eta)$, where $t_1, t_2 \in \mathbb{R}$ and $\xi, \eta \in \Sigma$, and suppose that $t_1 < t_2$ and that x, y do not belong to the same null geodesic. Then the linking* *number of X and Y is given by the winding number of $\Phi(\xi, t_2 - t_1)$ around η on Σ.*

Proof: This is a consequence of the way in which we identify the space of light rays (regarded as the sphere bundle of $t^{-1}(t_2)$) with a subset of the standard torus, and of the way in which we have chosen our orientations. In fact, as shown if figure 3.2, by fixing a surface of constant φ on the standard torus and sliding the sky of x along the generators one can deform it into a curve as close as we wish from $\Phi(\xi, t_2 - t_1)$ (seen as a subset of that slice) plus a number of generators which obviously do not link the sky of y (which is itself a generator), and can therefore be ignored in the calculation of the linking number (in many cases, such as the one depicted in figure 3.2, these generators may be deformed away). One then performs the standard trick of deforming Y into a straight line (by considering a sequence of deformations consisting of a line segment and half a circumference with radii approaching infinity) to conclude that $link(X, Y)$ equals the linking number of $\Phi(\xi, t_2 - t_1)$ (seen, for convenience, as a subset of $\mathbb{R}^2 \times \{0\}$) and the z-axis with positive orientation (which can be thought of as a curve closing at infinity). Let $\Phi(\xi, t_2 - t_1)$ be parametrized by $\mathbf{r}_1(s)$, and the z-axis with positive orientation by $\mathbf{r}_2(s)$; then we have (see [DFN])

$$\begin{aligned} link(X,Y) &= \frac{1}{4\pi}\oint\oint \frac{(\mathbf{r}_1 - \mathbf{r}_2)\cdot(d\mathbf{r}_1 \times d\mathbf{r}_2)}{\|\mathbf{r}_1 - \mathbf{r}_2\|^3} \\ &= \frac{1}{4\pi}\oint\int_{-\infty}^{+\infty} \frac{(\mathbf{r}_1 - s\mathbf{e}_z)\cdot(d\mathbf{r}_1 \times \mathbf{e}_z)}{\|\mathbf{r}_1 - s\mathbf{e}_z\|^3} ds \\ &= \frac{1}{4\pi}\oint\int_{-\infty}^{+\infty} \frac{\mathbf{r}_1 \cdot (d\mathbf{r}_1 \times \mathbf{e}_z)}{\left(\|\mathbf{r}_1\|^2 + s^2\right)^{\frac{3}{2}}} ds \end{aligned}$$

where $\mathbf{e}_z$ is the unit vector along the positive z-direction. Now since

$$\int_{-\infty}^{+\infty} \frac{ds}{(a^2+s^2)^{\frac{3}{2}}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a\sec^2\varphi d\varphi}{a^3\sec^3\varphi} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{a^2}\cos\varphi d\varphi = \frac{2}{a^2}$$

we see that

$$link(X,Y) = \frac{1}{2\pi}\oint \frac{\mathbf{r}_1 \cdot (d\mathbf{r}_1 \times \mathbf{e}_z)}{\|\mathbf{r}_1\|^2}$$

which is just the winding number of $\Phi(\xi, t_2 - t_1)$ about the origin. In fact, if we set $\mathbf{r}_1(s) = x(s)\mathbf{e}_x + y(s)\mathbf{e}_y$, we have

$$\mathbf{r}_1 \cdot (d\mathbf{r}_1 \times \mathbf{e}_z) = \begin{vmatrix} x & y & 0 \\ dx & dy & 0 \\ 0 & 0 & 1 \end{vmatrix} = xdy - ydx$$

and consequently

$$link(X,Y) = \frac{1}{2\pi}\oint \frac{xdy - ydx}{x^2+y^2}$$

which is the more usual formula for the winding number.□

Notice that clearly $link(X,Y) = link(Y,X)$, and therefore one cannot use the linking number of skies to tell which point is to the future of which. We shall come back to this later on.

Theorem 37 *Let (M,g) be a static (2+1)-spacetime, (Σ, h) its space manifold and t the global time. Let $x \in M$ be given by $x = (t_1, \xi)$, where $t_1 \in \mathbb{R}$ and $\xi \in \Sigma$. Suppose that $t_1 < t_2 \in \mathbb{R}$ and let $\Xi = t^{-1}(t_2)$. Then the set of points on Ξ causally related to x is $\{t_2\} \times \bigcup_{t_1 \leq t \leq t_2} \Phi(\xi, t)$.*

Proof: This can be seen to be a consequence of Huygens's principle (see [A]).□

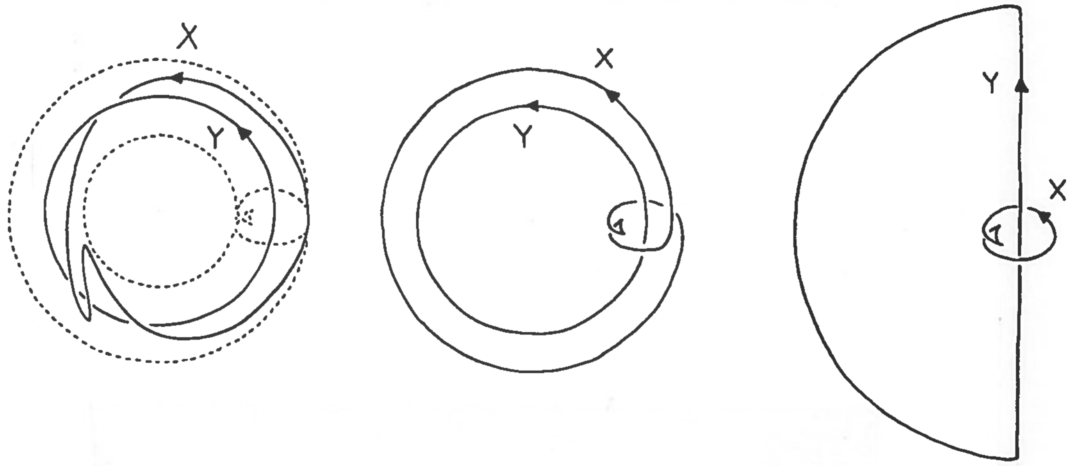

Figure 3.2: Deforming a sky into a wavefront

We have now the tools to compute the linking number of the skies of two points on a static (2+1)-spacetime, as well as to determine their causal relationship. In both cases this amounts to studying wavefronts on the space manifold. It should be noted that because our spacetime is static the particular global time coordinates of the events are irrelevant: only their *difference* is meaningful. This is an additional simplification.

3.5 Schwarzschild static (2+1)-spacetime

Definition 38 *We shall define the* Schwarzschild static (2+1)-spacetime with mass parameter $M > 0$ *as* $\mathbb{R}^3 \setminus \{(t, r, \varphi) : r \leq 2M\}$ *(where* $\{t, r, \varphi\}$ *are cylindrical coordinates) endowed with the Lorentzian metric*

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\varphi^2.$$

The reason for this definition should be obvious. We consider here only positive mass parameters since the Schwarzschild (2+1)-spacetime with $M = 0$ is just Minkowski (2+1)-spacetime and the Schwarzschild (2+1)-spacetime with $M < 0$ is not globally hyperbolic (because of the naked singularity at $r = 0$).

The space manifold is of course diffeomorphic to $\mathbb{R}^2 / \{0\}$; this and proposition 34 make clear that

Proposition 39 *The manifold of light rays of Schwarzschild static (2+1)-spacetime with mass parameter $M > 0$ is (diffeomorphic to) the standard torus $T \subseteq \mathbb{R}^3$ without a generator.*

The geodesic equations for the light metric can be obtained from the Lagrangian

$$L = \frac{1}{2}\left(\left(1 - \frac{2M}{r}\right)^{-2} \dot{r}^2 + r^2 \left(1 - \frac{2M}{r}\right)^{-1} \dot{\varphi}^2\right)$$

and are thus given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Leftrightarrow \ddot{r} - \frac{2M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 - (r - 3M)\, \dot{\varphi}^2 = 0$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0 \Leftrightarrow \frac{d}{dt}\left(r^2 \left(1 - \frac{2M}{r}\right)^{-1} \dot{\varphi}\right) = 0$$

or better yet, integrating once the φ equation,

$$\dot{\varphi} = \frac{l}{r^2}\left(1 - \frac{2M}{r}\right)$$

and

$$\ddot{r} = \frac{2M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + \frac{l^2}{r^3}\left(1 - \frac{3M}{r}\right)\left(1 - \frac{2M}{r}\right)^2$$

To calculate the wavefront originating at a given point (which we can assume without loss of generality to have coordinates $(R, 0)$) we must integrate the system of differential equations above for all initial conditions at point $(R, 0)$. Of course that we have the Hamiltonian integral

$$L = \frac{1}{2} \Leftrightarrow \left(1 - \frac{2M}{r}\right)^{-2} \dot{r}^2 + r^2 \left(1 - \frac{2M}{r}\right)^{-1} \dot{\varphi}^2 = 1.$$

Therefore, the easiest way to obtain all initial conditions at the mentioned point is to choose

$$\left(1 - \frac{2M}{R}\right)^{-1} \dot{r} = \cos\theta \Leftrightarrow \dot{r} = \left(1 - \frac{2M}{R}\right) \cos\theta$$

and

$$R\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} \dot{\varphi} = \sin\theta \Leftrightarrow l = R\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} \sin\theta$$

with θ ranging from 0 to 2π.

Unfortunately, we must rely on numerical calculations to integrate the system of differential equations above. The evolution of a typical wavefront is depicted on figure 3.3. Based on this one can state the following

Proposition 40 *Two points in Schwarzschild static (2+1)-spacetime are causally related if and only if their skies either intersect or are linked; if they are linked, the linking number is exactly the number of non-homotopic timelike curves with the given points as endpoints, and can therefore take any value in* $\mathbb{N}$.

Proof: These conclusions are clear from the numerically computed shape of the evolving wavefront. To make sure we are justified in believing in the numerical calculation we prove that none of the light geodesics contains conjugate points (which

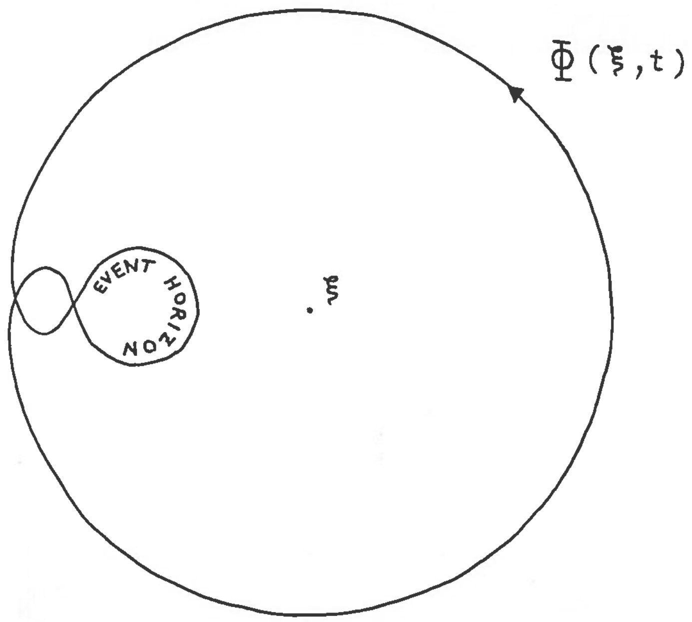

Figure 3.3: Typical wavefront on Schwarzschild (2+1)-spacetime

would lead to singularities in the wavefront, as we shall see in the next section; these singularities could conceivably be unresolved in the numerical calculation, or arise only after a very large amount of time).

To do this it suffices to show that the light metric has positive scalar curvature (i.e., negative Gaussian curvature, if we take our conventions about the Riemann tensor into account). In fact, it is well known (see for instance [W]) that for 2-dimensional Riemannian manifolds

$$R_{abcd} = \frac{1}{2}R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$$

and that consequently Jacobi's equation along a geodesic with tangent unit vector T^a is written

$$D^2 J_a \quad = \quad R_{bcda}T^b J^c T^d$$

$$= \frac{1}{2}R\left(J_a - T_a J_b T^b\right).$$

where $D = T^a \nabla_a$ and we've used $T_a T^a = 1$. Since $DT^a = 0$, we then have

$$D^2\left(J_a T^a\right) = 0$$

and consequently

$$D^2\left(J_a - J_b T^b T_a\right) = \frac{1}{2}R\left(J_a - J_b T^b T_a\right).$$

Let

$$K^a = J^a - J_b T^b T^a$$

be the component of the Jacobi field orthogonal to the geodesic. Then we have

$$D^2 K^a = \frac{1}{2}RK^a$$

and consequently

$$\begin{aligned} D^2\left(K_a K^a\right) &= 2D\left(K_a DK^a\right) \\ &= 2K_a D^2 K^a + 2DK_a DK^a \\ &= RK_a K^a + 2DK_a DK^a. \end{aligned}$$

So we see that if $R > 0$ then $K_a K^a$ is a non-negative function with non-negative second derivative and hence either vanishes identically or has at most one zero. If K^a vanishes identically then J^a has exactly the same number of zeros as the affine function $J_a T^a$, and hence cannot have more than one zero. We conclude that if $R > 0$ then J^a has at most one zero, and hence no conjugate points occur on the geodesic.

The light metric is

$$dt^2 = \left(1 - \frac{2M}{r}\right)^{-2} dr^2 + r^2 \left(1 - \frac{2M}{r}\right)^{-1} d\varphi^2$$

and an orthonormal coframe is therefore

$$\begin{aligned} \theta^r &= \left(1 - \frac{2M}{r}\right)^{-1} dr \\ \theta^\varphi &= r\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} d\varphi \end{aligned}$$

Now

$$\begin{aligned} d\theta^r &= 0 = \theta^\varphi \wedge \omega_\varphi{}^r = -\theta^\varphi \wedge \omega_r{}^\varphi \\ d\theta^\varphi &= \left(\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} - \frac{M}{r}\left(1 - \frac{2M}{r}\right)^{-\frac{3}{2}}\right) dr \wedge d\varphi \\ &= \left(1 - \frac{3M}{r}\right)\left(1 - \frac{2M}{r}\right)^{-\frac{3}{2}} dr \wedge d\varphi \\ &= \left(1 - \frac{3M}{r}\right)\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \theta^r \wedge d\varphi = \theta^r \wedge \omega_r{}^\varphi \end{aligned}$$

and consequently the connection 1-form is

$$\omega_r{}^\varphi = \left(1 - \frac{3M}{r}\right)\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} d\varphi$$

yielding the curvature 2-form

$$\begin{aligned} \Omega_r{}^\varphi &= d\omega_r{}^\varphi \\ &= \left(\frac{3M}{r^2}\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} - \frac{M}{r^2}\left(1 - \frac{3M}{r}\right)\left(1 - \frac{2M}{r}\right)^{-\frac{3}{2}}\right) dr \wedge d\varphi \\ &= \frac{M}{r^2}\left(2 - \frac{3M}{r}\right)\left(1 - \frac{2M}{r}\right)^{-\frac{3}{2}} dr \wedge d\varphi \\ &= \frac{M}{r^3}\left(2 - \frac{3M}{r}\right) \theta^r \wedge \theta^\varphi \end{aligned}$$

Thus the scalar curvature is

$$R = 2\frac{M}{r^3}\left(2 - \frac{3M}{r}\right)$$

and is clearly positive for $r > 2M$.□

3.6 Optical (2+1)-spacetimes

The previous example might lead one to suggest that in a (2+1)-spacetime with Cauchy surface diffeomorphic to a open subset of B^2 two points are causally related if and only if their skies either intersect or are linked with positive linking number. To see that this is not the case we introduce an admittedly artificial class of examples.

Definition 41 *An* optical (2+1)-spacetime *is a static (2+1)-spacetime conformally related to* $\mathbb{R}^3$ *with the Lorentzian metric*

$$ds^2 = dt^2 - A(x, y)\left(dx^2 + dy^2\right)$$

where $A > 0$ *is an otherwise arbitrary* C^∞ *function.*

The reason for the above definition is of course that these spacetimes have a light metric given by

$$dt^2 = A(x, y)\left(dx^2 + dy^2\right)$$

and therefore their light rays mimic the light rays of a dispersive medium with refraction index $n = A^{\frac{1}{2}}$. A simple example of an optical (2+1)-spacetime can be got (symmetries allowing) from the line element of a static classical solution of the linearized Einstein equations (see for instance [S]),

$$\begin{aligned} ds^2 &= (1+2\phi)\,dt^2 - (1-2\phi)\left(dx^2+dy^2+dz^2\right) \\ &= (1+2\phi)\left(dt^2 - \frac{1-2\phi}{1+2\phi}\left(dx^2+dy^2+dz^2\right)\right) \end{aligned}$$

where ϕ is the classical gravitational potential. In this case one therefore has

$$A = \frac{1-2\phi}{1+2\phi} \simeq (1-2\phi)^2 \simeq 1-4\phi$$

The light rays for an optical (2+1)-spacetimes can be obtained from the Lagrangian

$$L = \frac{1}{2}A\left(\dot{x}^2+\dot{y}^2\right) = \frac{1}{2}A\,\dot{\mathbf{x}}^2$$

where $\mathbf{x} = (x, y)$ and we shall use the ordinary vector calculus notation in $\mathbb{R}^2$. The Euler-Lagrange equations are

$$\frac{d}{dt}\left(A\,\dot{\mathbf{x}}\right) - \frac{1}{2}\,\dot{\mathbf{x}}^2\,\nabla A = 0 \Leftrightarrow \ddot{\mathbf{x}} + \frac{1}{A}\left(\nabla A\cdot\dot{\mathbf{x}}\right)\dot{\mathbf{x}} - \frac{1}{2A}\,\dot{\mathbf{x}}^2\,\nabla A = 0$$

and they admit the first integral

$$L = \frac{1}{2} \Leftrightarrow A\,\dot{\mathbf{x}}^2 = 1$$

which allows us to write

$$\ddot{\mathbf{x}} = \frac{1}{2A^2}\nabla A - \frac{1}{A}\left(\nabla A\cdot\dot{\mathbf{x}}\right)\dot{\mathbf{x}}$$

It is interesting to note that this equation requires only that ∇A be piecewise continuous, i.e., that A be continuous and piecewise C^1. The light rays obtained in such a metric would be the limit (in the appropriate sense) of the light rays of a sequence of C^∞ Lorentzian metrics approaching our C^1 metric. This is important as it will simplify our choice of A. One of the simplest choices one may think of is to take

$$A = 1 + BH(R - r)\frac{R - r}{R}$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$, $R > 0$, $B > -1$ and H is the Heaviside function. This corresponds to a sphere of radius R centered at the origin of the coordinate system lying in a vacuum and with refraction index going from 1 at its boundary to $(1 + B)^{\frac{1}{2}}$ at its center. For convenience let us call this a *thickening sphere*. Taking $B > 0$ and computing numerically the shape of a wavefront originating outside the thickening sphere after being scattered by it one typically gets something like as shown on figure 3.4. Although one gets only +1 and +2 linking numbers with this wavefront, one gets an interesting new feature - *cusps*.

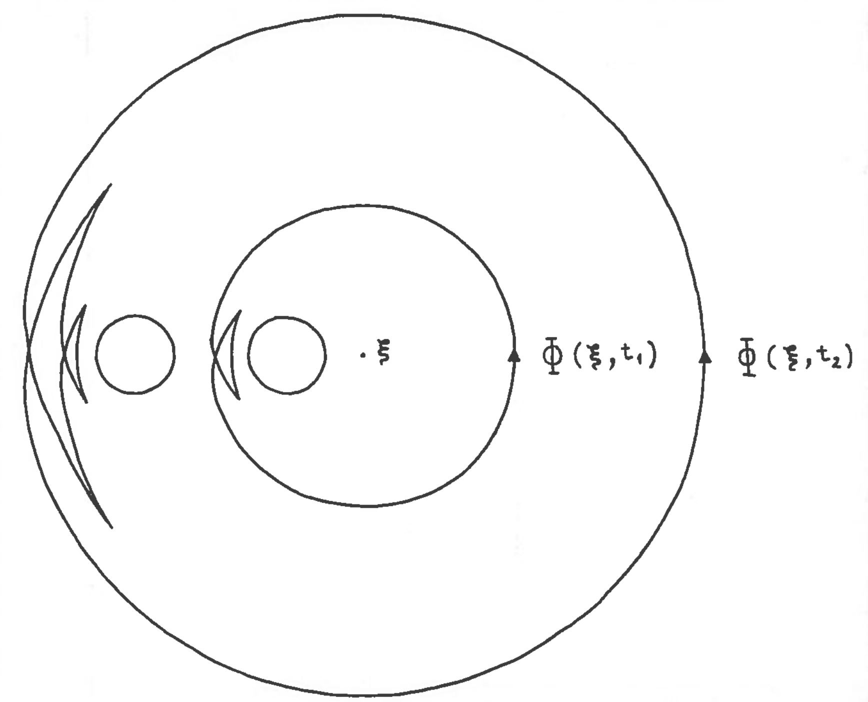

Figure 3.4: Typical wavefront scattered by thickening spheres

Now if one uses *two* thickening spheres with $B > 0$ placed so that their centers are in the same straight line as the origin of the wavefront and allows it to be scattered by both spheres one gets something like figure 3.4, i.e., cusps *within* cusps. This leads to zero linking number of the corresponding skies. Although the linking number is zero, the skies are nonetheless linked; in fact, they form the so-called *Whitehead link*, as is clear from figure 3.5. In this figure, as in figure 3.2, one of the skies is deformed into the wavefront bearing in mind that at self-intersections the segment which goes over is determined by the normal vectors, whereas the other sky is deformed into the z-axis (seen as a curve closing at infinity). This and other examples led Low to conjecture that causally related points correspond to linked (or intersecting) skies (see [L]). We present here a modified version of his conjecture, consistent with what we've done so far:

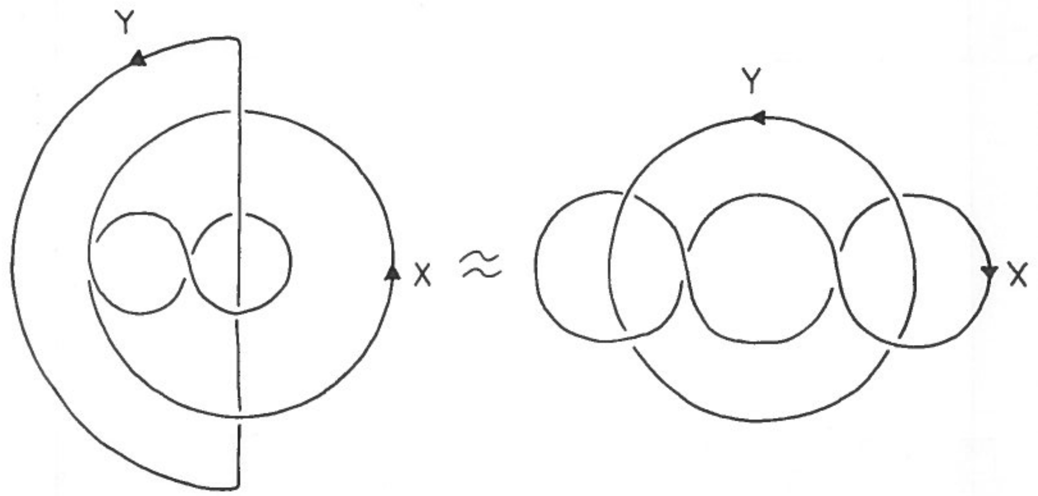

Figure 3.5: Linking with zero linking number

Conjecture 42 *Let (M, g) be a globally hyperbolic (2+1)-spacetime with a Cauchy surface diffeomorphic to an open set of $\mathbb{R}^2$, and $x, y \in M$. Then x and y are causally related if and only if X and Y either intersect or are linked.*

3.7 Einstein static (2+1)-spacetime

Definition 43 *The* Einstein static (2+1)-spacetime *(with scale factor $a = 1$) is $\mathbb{R} \times S^2$ endowed with the Lorentzian metric*

$$ds^2 = dt^2 - \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

(where $\{\theta, \varphi\}$ are spherical polar coordinates on S^2).

This is an interesting example as it is a globally hyperbolic (2+1)-spacetime whose Cauchy surfaces are *not* diffeomorphic to an open subset of B^2, and thus it is not clear wether its manifold of light rays can be embedded in $\mathbb{R}^3$. In fact it cannot, as is clear from

Proposition 44 *The tangent sphere bundle of S^2 is (diffeomorphic to) $SO(3)$.*

Proof: Let us think of S^2 as

$$S^2 = \left\{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\right\}$$

and let $A \in SO(3)$. If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of $\mathbb{R}^3$, it is easily seen that

$$A\mathbf{e}_2 \in T_{A\mathbf{e}_1}S^2$$

and clearly $\|A\mathbf{e}_2\| = 1$, so that $A\mathbf{e}_2$ in fact belongs to the tangent sphere bundle $TS\left(S^2\right)$ of all unit vectors tangent to S^2. It is easy to see that the correspondence between $A \in SO(3)$ and $A\mathbf{e}_2 \in TS\left(S^2\right)$ is in fact a diffeomorphism.□

It is perhaps worthwhile noticing that in this case the manifold of light rays is not even orientable (as is well known, $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$).

CHAPTER 4

Examples in (3+1)-spacetimes

4.1 Orienting skies

Proposition 45 *Any Cauchy surface Σ of an orientable (globally hyperbolic) (3+1)-spacetime (M, g) is necessarily orientable.*

The proof is completely analogous to that of proposition 22. Again we can use this to give the sky of any point $x \in M$ an orientation by noticing that we can orient the fibres of $TS\,(\Sigma)$; one can check that the orientation attributed to the sky of a point $x \in M$ is independent of the particular Cauchy surface $\Sigma \ni x$ used to do so.

4.2 Wavefronts in static (3+1)-spacetimes

Proposition 46 *Let (M, g) be a (globally hyperbolic) (3+1)-spacetime and Σ one of its Cauchy surfaces. If Σ is diffeomorphic to a subset of $\mathbb{R}^3$ then the manifold of light rays of (M, g) is diffeomorphic to a subset of $\mathbb{R}^2 \times (\mathbb{R}^3 \setminus \{(0,0,0)\})$. The skies of spacetime points are certain closed surfaces homotopic to $\{(0,0,0)\} \times S^2$.*

Proof: This arises from the facts that if Σ is diffeomorphic to a subset of $\mathbb{R}^3$ then $TS(\Sigma)$ is a trivial fibre bundle. Since $\mathbb{R} \times S^2$ is diffeomorphic to $\mathbb{R}^3 \setminus \{(0,0,0)\}$ (via spherical polar coordinates, say), we have

$$TS(\Sigma) \approx \mathbb{R}^3 \times S^2 \approx \mathbb{R}^2 \times \left(\mathbb{R}^3 \setminus \{(0,0,0)\}\right).$$

The second statement easily follows from this identification and from the fact that one can use *any* Cauchy surface to build the manifold of light rays.□

Thus one can in this case again use the familiar concepts of linking and linking number as defined for embedded S^2s in $\mathbb{R}^5$ on skies. Once again we must be careful to fix a diffeomorphism beforehand, as these concepts depend on the choice of diffeomorphism; the obvious choice is clear, and we shall not insist on this point any further. Notice that by similar reasons we can define the winding number of a closed oriented surface about a point in Σ.

Obviously definition 35 can be extended to (oriented) wavefronts on the space manifold of static (3+1)-spacetimes:

Definition 47 *Let (M, g) be a static (3+1)-spacetime, (Σ, h) its space manifold and $\xi \in \Sigma$. The* wavefront originating at ξ at time t *is the geodesic sphere $\Phi(\xi, t)$ of radius t for the light metric $\frac{1}{g_{00}}h$, oriented by considering the right-handed orientation on $T_\xi\Sigma$ of the initial condition of each geodesic.*

We still have

Proposition 48 *Let (M, g) be a static (3+1)-spacetime, (Σ, h) its space manifold and t the global time, Σ being diffeomorphic to a subset of $\mathbb{R}^3$. Let $x, y \in M$ be given by $x = (t_1, \xi)$ and $y = (t_2, \eta)$, where $t_1, t_2 \in \mathbb{R}$ and $\xi, \eta \in \Sigma$, and suppose that $t_1 < t_2$ and that x, y do not belong to the same null geodesic. Then the linking number of X and Y is given by the winding number of $\Phi(\xi, t_2 - t_1)$ around η on Σ.*

Proof: By analogy with the (2+1)-dimensional case, once the manifold of light rays is embedded in $\mathbb{R}^5$ the skies of x and y can be deformed into $\Phi(\xi, t_2 - t_1)$ (seen as a subset of $\mathbb{R}^3 \times \{(0,0)\}$) and $\{(0,0,0)\} \times \mathbb{R}^2$ with positive orientation (seen as a surface closing at infinity). Notice that now one has to slide X along the meridians of the S^2 fibres of N, thus getting S^2s at the north pole (say), which we then discard as they do not contribute to the winding number. Let $\Phi(\xi, t_2 - t_1)$ be parametrized by $\mathbf{r}_1(\theta, \varphi)$, and $\{(0,0,0)\} \times \mathbb{R}^2$ with positive orientation by $\mathbf{r}_2(s,t)$; then we have (see [DFN])

$$\begin{aligned}
link(X,Y) &= \frac{3}{8\pi^2} \oint \oint \frac{1}{\|\mathbf{r}_1 - \mathbf{r}_2\|^5} \begin{vmatrix} \mathbf{r}_1 - \mathbf{r}_2 \\ \frac{\partial \mathbf{r}_1}{\partial \theta} \\ \frac{\partial \mathbf{r}_1}{\partial \varphi} \\ \frac{\partial \mathbf{r}_2}{\partial s} \\ \frac{\partial \mathbf{r}_2}{\partial t} \end{vmatrix} dsdtd\theta d\varphi \\
&= \frac{3}{8\pi^2} \oint \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\|\mathbf{r}_1 - s\mathbf{e}_4 - t\mathbf{e}_5\|^5} \begin{vmatrix} \mathbf{r}_1 - s\mathbf{e}_4 - t\mathbf{e}_5 \\ \frac{\partial \mathbf{r}_1}{\partial \theta} \\ \frac{\partial \mathbf{r}_1}{\partial \varphi} \\ \mathbf{e}_4 \\ \mathbf{e}_5 \end{vmatrix} dsdtd\theta d\varphi \\
&= \frac{3}{8\pi^2} \oint \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\left(\|\mathbf{r}_1\|^2 + s^2 + t^2\right)^{\frac{5}{2}}} \begin{vmatrix} \mathbf{r}_1 \\ \frac{\partial \mathbf{r}_1}{\partial \theta} \\ \frac{\partial \mathbf{r}_1}{\partial \varphi} \end{vmatrix} dsdtd\theta d\varphi
\end{aligned}$$

where $\frac{8\pi^2}{3}$ is obviously the volume of S^4 and $\mathbf{e_i}$ is the unit vector along the positive i-direction. Now since

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{ds}{(a^2+s^2+t^2)^{\frac{5}{2}}}=2\pi\int_{0}^{+\infty}\frac{rdr}{(a^2+r^2)^{\frac{5}{2}}}=\frac{2\pi}{3a^3}$$

we see that

$$link(X,Y)=\frac{1}{4\pi}\oint\frac{1}{\|\mathbf{r}_1\|^3}\begin{vmatrix}\mathbf{r}_1\\ \frac{\partial\mathbf{r}_1}{\partial\theta}\\ \frac{\partial\mathbf{r}_1}{\partial\varphi}\end{vmatrix}d\theta d\varphi=\frac{1}{4\pi}\oint\frac{\mathbf{r}_1}{\|\mathbf{r}_1\|^3}\cdot\mathbf{n}dS(\mathbf{r}_1)$$

which is just the winding number of $\Phi(\xi, t_2 - t_1)$ about the origin.□

Notice that clearly $link(X,Y) = -link(Y,X)$, and therefore in (3+1)-spacetimes one *can* use the linking number of skies to tell which point is to the future of which.

Again we have the following

Theorem 49 *Let (M,g) be a static (3+1)-spacetime, (Σ, h) its space manifold and t the global time. Let $x \in M$ be given by $x = (t_1, \xi)$, where $t_1 \in \mathbb{R}$ and $\xi \in \Sigma$. Suppose that $t_1 < t_2 \in \mathbb{R}$ and let $\Xi = t^{-1}(t_2)$. Then the set of points on Ξ causally related to x is $\{t_2\} \times \bigcup_{t_1\leq t\leq t_2} \Phi(\xi, t)$.*

Proof: This can once more be seen to be a consequence of Huygens's principle (see [A]).□

Thus both computing the linking number of skies of two points on a static-(3+1) spacetime and determining their causal relationship amounts to studying wavefronts on the space manifold.

We would like to extend our (2+1)-dimensional examples to their (3+1)-dimensional analogues. In all these examples we notice that the 3-dimensional space manifold is symmetric about an axis, and that the 3-dimensional wavefront originating from a point in this axis is just the surface of revolution generated by the 2-dimensional wavefront of the same point on rotation about the axis. One expects then that the linking numbers of the two wavefronts about points in the axis are related, and in fact we have the following

Proposition 50 *Let γ be a piecewise smooth curve in the Euclidean plane symmetric about a line l, and suppose it has no double points on this line. Let Φ the surface of revolution generated by γ by rotating it about l. If $p \in l \backslash \gamma$ then $wind(\gamma, p) = wind(\Phi, p)$ (provided that γ and Φ are given the right orientations).*

Proof: We assume without loss of generality that l is the y-axis in $\mathbb{R}^3$, p is the origin and $\gamma \subseteq \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. We have

$$wind(\Phi, p) = \frac{1}{4\pi} \oint_{\Phi} \frac{\mathbf{r}}{\|\mathbf{r}\|^3} \cdot \mathbf{n} dS$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{n}$ is the outward-pointing unit normal vector to Φ. We choose orientations such that on the plane $z = 0$ we have $\mathbf{n} = \mathbf{t} \times \mathbf{e}_z$, where $\mathbf{t}$ is the unit tangent vector to γ with the positive orientation. This and the axial symmetry of Φ allow us to write

$$\begin{aligned} wind(\Phi, p) &= \frac{1}{4\pi} \int_0^{\pi} \oint_{\gamma} \frac{\rho}{\|\rho\|^3} \cdot (d\rho \times \mathbf{e}_z) \left|\rho \cdot \mathbf{e}_x\right| d\theta = \\ &= \frac{1}{4} \oint_{\gamma} \frac{\left|\rho \cdot \mathbf{e}_x\right|}{\|\rho\|^3} \mathbf{e}_z \cdot (\rho \times d\rho) \end{aligned}$$

where $\rho = (x, y, 0)$. In other words, we get

$$wind(\Phi, p) = \frac{1}{4} \oint_{\gamma} \frac{|x|}{(x^2 + y^2)^{\frac{3}{2}}} (x dy - y dx)$$

which strangely enough is just $wind(\gamma, p)$. To see this, one introduces polar coordinates $\{r, \varphi\}$ on the plane $z = 0$; since

$$d\varphi = \frac{xdy - ydx}{x^2 + y^2}$$

one gets

$$wind(\Phi, p) = \frac{1}{4}\oint_\gamma |\cos\varphi|\, d\varphi = wind(\gamma, p)$$

as

$$\int_0^{2\pi} |\cos\varphi|\, d\varphi = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi d\varphi = 4.\square$$

Proposition 50 allows us to extend some of our results in (2+1) dimensions to the (3+1) case in a straightforward manner. However, it can only be applied to the case where γ has no double points on the axis of symmetry; as we shall see, when such double points are present one has to be quite careful when determining the correct orientations.

4.3 Minkowski (3+1)-spacetime

For completeness, we introduce the well known

Definition 51 *We define* Minkowski (3+1)-spacetime *to be* $\mathbb{R}^4$ *with Cartesian coordinates* $\{t, x, y, z\}$ *and the flat Lorentzian metric*

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$

From the previous section and our analysis of Minkowski (2+1)-spacetime the following results should be obvious.

Proposition 52 *The manifold of light rays of Minkowski (3+1)-spacetime is (diffeomorphic to)* $\mathbb{R}^3 \times S^2 \approx \mathbb{R}^2 \times (\mathbb{R}^3 \backslash \{(0,0,0)\})$.

Proposition 53 *Two points* x *and* y *in Minkowski (3+1)-spacetime are timelike-separated if and only if their skies* X *and* Y *are linked in* $\mathbb{R}^5$. *Furthermore,* $link(X,Y) = \pm 1$ iff $y \in I^{\pm}(x)$.

Proof: If x and y are timelike-separated and $y \in I^+(x)$, then take the Cauchy surface $\Sigma = t^{-1}(t(y))$ and consider the spherical wavefront Φ on this surface generated by its intersection with the integral line of $\frac{\partial}{\partial t}$ though x after a time $t(y) - t(x)$. Let z be a point in a symmetry axis of this sphere; it is clear that there exists a path $\gamma : [0,1] \to \Sigma$ such that $\gamma(0) = y$, $\gamma(1) = z$ and $\gamma([0,1]) \cap \Phi = \emptyset$. Thus

$$link(X,Y) = wind(\Phi, y) = wind(\Phi, z) = +1$$

as is clear from proposition 50 and our results in (2+1)-dimensions, and X and Y are linked. If x and y are timelike-separated and $y \in I^-(x)$, then $x \in I^+(y)$ and consequently

$$link(X,Y) = -link(Y,X) = -1$$

and X and Y are again linked. If x and y are lightlike-separated then X and Y intersect and therefore are not linked. Finally, if x and y are timelike-separated we use the same ideas as above to show that $link(X,Y) = 0$. The proof is then concluded by the observation that in $\mathbb{R}^5$ two embedded S^2's are linked if and only if their linking number is different from zero (see [RS]).□

4.4 Schwarzschild static (3+1)-spacetime

Definition 54 *We shall define the* Schwarzschild static (3+1)-spacetime of mass parameter $M > 0$ *as* $\mathbb{R}^4 \setminus \{(t, r, \theta, \varphi) : r \leq 2M\}$ *(where* $\{t, r, \theta, \varphi\}$ *are cylindrical coordinates) endowed with the Lorentzian metric*

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2.$$

Again we consider only positive mass parameters since the Schwarzschild static (3+1)-spacetime with $M = 0$ is just Minkowski (3+1)-spacetime and the Schwarzschild (3+1)-spacetime with $M < 0$ is not globally hyperbolic (because of the naked singularity at $r = 0$).

The space manifold is of course diffeomorphic to $\mathbb{R}^3 \setminus \{0\}$; this and proposition 46 make clear that

Proposition 55 *The manifold of light rays of Schwarzschild static (3+1)-spacetime with mass parameter* $M > 0$ *is (diffeomorphic to)* $(\mathbb{R}^3 \setminus \{(0, 0, 0)\}) \times S^2 \approx \mathbb{R} \times S^2 \times S^2$.

Because the wavefronts in the Schwarzschild static (2+1)-dimensional spacetime develop double points on the axis of symmetry, we cannot use a similar argument to that of the proof of proposition 53. In fact, it turns out that it is not true that two points in Schwarzschild static (3+1)-spacetime are causally related if and only if its skies either intersect or are linked. The reasons for this are exactly the reasons why the equivalent statement is not true for an optical (3+1)-spacetime containing *one* thickening sphere, which we now examine.

4.5 Optical (3+1)-spacetimes

Definition 56 *An* optical (3+1)-spacetime *is a static (3+1)-spacetime conformally related to* $\mathbb{R}^4$ *with the Lorentzian metric*

$$ds^2 = dt^2 - A(x, y, z)\left(dx^2 + dy^2 + dz^2\right)$$

where $A > 0$ *is an otherwise arbitrary* C^∞ *function.*

Proposition 57 *The manifold of light rays of an optical (3+1)-spacetime is (diffeomorphic to)* $\mathbb{R}^3 \times S^2 \approx \mathbb{R}^2 \times (\mathbb{R}^3 \backslash \{(0,0,0)\})$.

Again we say that our optical (3+1)-spacetime contains a *thickening sphere* at the origin if

$$A = 1 + BH(R - r)\frac{R - r}{R}$$

where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, $R > 0$, $B > -1$ and H is the Heaviside function. This corresponds to a sphere of radius R centered at the origin of the coordinate system lying in a vacuum and with refraction index going from 1 at its boundary to $(1 + B)^{\frac{1}{2}}$ at its center.

Proposition 50 cannot be used in this example in the case when the wavefront in (2+1) dimensions double points on the axis of symmetry, and hence we must compute the linking numbers directly. Take $B > 0$ and consider the skies X and Y of the points $x = (t_1, \xi)$ and $y = (t_2, \eta)$, where $t_1, t_2 \in \mathbb{R}$ and $\xi, \eta \in \Sigma$. Suppose that $t_1 < t_2$ and that x, y do not belong to the same null geodesic. We know that $link(X, Y)$ is given by the winding number of $\Phi(\xi, t_2 - t_1)$ around η on Σ, where the geodesic sphere $\Phi(\xi, t_2 - t_1)$ must be oriented by considering the right-handed orientation on $T_\xi\Sigma$ of the initial condition of each geodesic. Let $t_2 - t_1$ be big enough so that $\Phi(\xi, t_2 - t_1)$ has already been scattered by the thickening sphere; thus it is the surface of revolution generated by the smaller wavefront on figure 3.4, containing one singular point and a circular cusp edge. While determining the orientation of the outer component of $\Phi(\xi, t_2 - t_1)$ is simple enough (it corresponds to an outward-pointing normal vector), the orientation of the inner surface is not obvious. If however one considers the intersections of the wavefront with orthogonal planes through the symmetry axis and follows through what happens to small displacements on the sphere of initial conditions on $T_\xi\Sigma$ which lead to points in that inner surface along those planes, it is not hard to convince oneself that the orientation of the inner surface corresponds to a normal vector pointing to the *inside* of that surface. This can be thought of as a consequence of the fact that the antipodal map on S^2 is orientation-reversing. Thus if η is inside the inner surface, we have

$$link(X, Y) = wind(\Phi(\xi, t_2 - t_1), \eta) = 0!$$

This is perhaps a bit surprising. We could have instead considered the static metric

$$ds^2 = dt^2 - A(x, y)\left(dx^2 + dy^2\right) - dz^2$$

with the same function A which yields the two thickening spheres in the (2+1)-dimensional case. Then we would have two thickening *cylinders*, and for $\varepsilon > 0$ sufficiently small the wavefront in $\{-\varepsilon < z < \varepsilon\}$ would just be the Cartesian product wavefront for the (2+1)-dimensional case by $(-\varepsilon, \varepsilon)$, and all the winding numbers would coincide (we therefore need two aligned thickening cylinders to get zero linking number). Actually the case of the thickening sphere is a degenerate case of a similar process of generating cusps within cusps, as can be seen from studying the evolution of a ellipsoidal wavefront in Minkowski spacetime backwards in time.

Therefore we have proved

Proposition 58 *The manifold of light rays of an optical (3+1)-spacetime containing* one *thickening sphere contains skies of causally related points whose linking number is zero and which therefore are not linked in* $\mathbb{R}^5$.

This shows that our modified version of Low's conjecture clearly does not apply in the (3+1)-dimensional case. In order to try to prove an appropriate result in this case we must first analyze the concept of linking a little more carefully.

CHAPTER 5

Linking and causality

5.1 Links

At this point it is perhaps useful to recall the following

Definition 59 *Let M be a $(2n+1)$-dimensional manifold ($n \in \mathbb{N}$), and f, g : $S^n \to M$ be embeddings. Then $f(S^n) \amalg g(S^n)$ is said to be a* link *if $f(S^n) \cap g(S^n) = \emptyset$.*

Definition 60 *Two links $X_1 \amalg X_2$ and $Y_1 \amalg Y_2$ are said to be* equivalent *if there exists a one-parameter family of diffeomorphisms $\Phi_t : M \times [0,1] \to M$ such that (i) Φ_0 is the identity map; (ii) $\Phi_1(X_i) = Y_i$ for $i = 1, 2$.*

A one-parameter family of diffeomorphisms as above receives a special designation:

Definition 61 *Let M be a differential manifold. A one-parameter family of diffeomorphisms $\Phi_t : M \times [0,1] \to M$ such that Φ_0 is the identity map is called a* (smooth) isotopy.

In our (2+1)- and (3+1)-dimensional examples we embedded our manifold of light rays in $\mathbb{R}^3$ or $\mathbb{R}^5$ before considering the linking of skies. The following trivial proposition is therefore relevant:

Proposition 62 *Let M be a $(2n+1)$-dimensional manifold ($n \in \mathbb{N}$), and $U \subseteq M$ an open subset. If two links are not equivalent in M, then they are not equivalent in U (seen as a manifold itself).*

5.2 Sky motions

We start by stating a well-known

Theorem 63 *Let M be a $(2n+1)$-dimensional manifold and f_t, $g_t : S^n \times [0,1] \to M$ smooth one-parameter families of embeddings such that $f_t(S^n) \amalg g_t(S^n)$ is a link for all $t \in [0,1]$ Then there exists an isotopy $\Phi_t : M \times [0,1] \to M$ such that $f_t = \Phi_t \circ f_0$ and $g_t = \Phi_t \circ g_0$ for all $t \in [0,1]$.*

Proof: Consider the map $F : S^n \times [0,1] \to M \times [0,1]$ given by $F_t = (f_t, t)$; this yields a vector field on $im(F) \subseteq M \times [0,1]$. Similarly, we can consider a map $G : S^n \times [0,1] \to M \times [0,1]$ given by $G_t = (g_t, t)$, which generates a vector field on $im(G) \subseteq M \times [0,1]$. Since $im(F) \cap im(G) = \emptyset$, it is clear that there exists a vector field on $M \times [0,1]$ which coincides with the two previous vector fields on their domains and whose corresponding one-parameter group of diffeomorphisms $\widetilde{\Phi} : M \times [0,1] \to M \times [0,1]$ at $M \times \{0\}$ is of the form $\widetilde{\Phi}_t = (\Phi_t, t)$. Then Φ_t is the required isotopy.□

One can think of this theorem as stating that links are equivalent *iff* one can find smooth motions of its components taking one link into the other in such a way that the two components never intersect throughout the motion. On the other hand, if (M, g) is a globally hyperbolic $(d+1)$-spacetime ($d = 2$ or 3), any smooth curve $x = x(t)$ on M yields a smooth motion $X = X(t)$ of the skies of the points of the curve on the corresponding manifold of light rays N. This allows us to prove the following

Proposition 64 *Let (M, g) be a globally hyperbolic $(d+1)$-spacetime ($d = 2$ or 3), and N its manifold of light rays. Then the link consisting of the skies of two non-causally related points is equivalent to the link consisting of the skies of any other two non-causally related points.*

Proof: Let x_1, x_2 be a pair of non-causally related points, and y_1, y_2 be another such pair. Let Σ be a Cauchy surface such that all these points belong to $I^+(\Sigma)$. Let t^a be a nowhere vanishing future-pointing timelike vector field. It should be clear that there exist curves $x_1(t)$, $x_2(t)$ ($t \in [0,1]$) such that $x_i(0) = x_i$, $x_i(1) \in \Sigma$ ($i = 1, 2$) and $x_1(t)$ and $x_2(t)$ are never causally related; one can take for instance appropriate parameterizations of the integral curves of t^a joining x_1 and x_2 to Σ. Similarly, there exist curves $y_1(t)$, $y_2(t)$ ($t \in [0,1]$) such that $y_i(0) \in \Sigma$, $y_i(1) = y_i$ ($i = 1, 2$) and $y_1(t)$ and $y_2(t)$ are never causally related. Since Σ is connected (because M is connected and has the topology $\Sigma \times \mathbb{R}$) and open it is pathwise connected and one can therefore find paths $z_1(t)$ and $z_2(t)$ ($t \in [0,1]$) such that $z_i(0) = x_i(1)$ and $z_i(1) = y_i(0)$ ($i = 1, 2$). Since a Cauchy surface is achronal, $z_1(t)$ and $z_2(t)$ are never causally related unless they coincide, and one can certainly choose the parameterizations such that they do not. Therefore, composing each of the three paths $x_i(t)$, $z_i(t)$ and $y_i(t)$ ($t \in [0,1]$) in the usual way, and considering the corresponding skies, one gets a motion of the link $X_1 \amalg X_2$ into the link $Y_1 \amalg Y_2$

on the manifold of light rays.□

This proposition shows the following definition to be a natural one:

Definition 65 *Let* (M, g) *is a globally hyperbolic* $(d+1)$*-spacetime* ($d = 2$ *or* 3)*, and* N *its manifold of light rays. Then the equivalence class of the link consisting of the skies of two non-causally related points is said to be the class of* trivial links*; any two embedded* S^d*s forming a link in this class are said to be* unlinked.

5.3 Linking and causality in (2+1)-spacetimes

Propositions 62 and 64 allow us to restate the results we got from our examples in (2+1)-spacetimes without any reference to any particular embedding of the manifold of light rays in $\mathbb{R}^3$: In all the examples we examined, two spacetime points were causally related *iff* their skies either intersected or were linked (of course that by definition if two points are not causally related their skies are unlinked in N, and it happens to be that skies unlinked in N according to our definition are unlinked in $\mathbb{R}^3$ when one uses the standard diffeomorphism; another way to put the results obtained in chapter 3 would be that no two causally related points had unlinked skies). Our modified version of Low's conjecture implies that this holds for all spacetimes with Cauchy surfaces diffeomorphic to a subset of $\mathbb{R}^2$. We can now state a more accurate version of Low's conjecture:

Conjecture 66 *Let* (M, g) *be a globally hyperbolic (2+1)-spacetime with Cauchy surface diffeomorphic to a subset of* $\mathbb{R}^2$*, and let* N *be its manifold of light rays. Then two spacetime points are causally related in* M iff *their skies either intersect* *or are linked in N.*

Notice that this version of the conjecture is actually weaker; its main advantage is the fact that it does not make any reference to the standard embedding. Notice however that there is no canonical definition of linking on N; definition 65 requires information which is not in the manifold structure of N, namely to which equivalence class of links belongs the link formed by the skies of two non-causally related points in our particular spacetime.

5.4 Linking and causality in (3+1)-spacetimes

Here one can use propositions 62 and 64 to conclude that in Minkowski (3+1)-spacetime two spacetime points are causally related *iff* their skies either intersect or are linked; for the Schwarzschild static (3+1)-spacetime or the optical (3+1)-spacetime containing a thickening sphere, however, one cannot conclude anything from the fact that there exist points whose skies are not linked in $\mathbb{R}^5$. Therefore it is not clear whether this example rules out Low's conjecture for (3+1)-spacetimes. In fact it does, as we will show in the following proposition.

Proposition 67 *The manifold of light rays N of an optical (3+1)-spacetime (M, g) containing a thickening sphere contains skies of causally related points which are not linked.*

Proof: Suppose that the thickening sphere's center is the origin. Consider the wavefront $\Phi \subseteq \mathbb{R}^3$ of a point ξ on the z-axis after a big enough interval of time so that the intersection of Φ with the plane $\{x = 0\}$ has developed two cusps. This is the projection on the Cauchy surface of the sky X of a point x whose future light cone intersects the Cauchy surface along Φ. Consider a point η on the intersection of the z-axis with the region of zero winding number. Clearly it is the projection of the sky Y of a point y on the same Cauchy surface. If one now moves the Cauchy surface towards the past, these projections change into the different intersections of the light cones of x and y with the moving Cauchy surface. Eventually, they turn into the surfaces of revolution $\pi(X)$ and $\pi(Y)$ generated by the curves depicted in figure 5.1 (where $\pi : N \to \Sigma$ is the natural projection). Recall that the intersection of, say, X, with a S^2 fibre is determined by the normal vector to $\pi(X)$ at that point. Because the Cauchy surface we are considering is to the future of x but to the past of y, the relevant normal is the outward-pointing normal for $\pi(X)$ and the inward-pointing normal for $\pi(Y)$. Let (ρ, φ, z) be cylindrical coordinates in $\mathbb{R}^3$, and consider the deformation of X given by

$$f_t(\xi, \mathbf{n}) = \left(\xi, \frac{\mathbf{n} + t\frac{\partial}{\partial\varphi}}{\left\| \mathbf{n} + t\frac{\partial}{\partial\varphi} \right\|} \right)$$

for $\xi \in \pi(X)$ and $\mathbf{n}$ the normal vector to $\pi(X)$ at ξ. It should be clear that one can move X away from Y (by moving $\pi(f_1(X))$ away from $\pi(Y)$ along the z-axis, say) without X ever intersecting Y, and that once one has moved X far enough both can be deformed to skies of points on the Cauchy surface (which by definition are unlinked).□

The same thing happens for the spacetime containing two thickening cylinders. It is instructive to consider in detail how one can unlink skies in this case. The proof of the following proposition is essentially due to Low (see [L]).

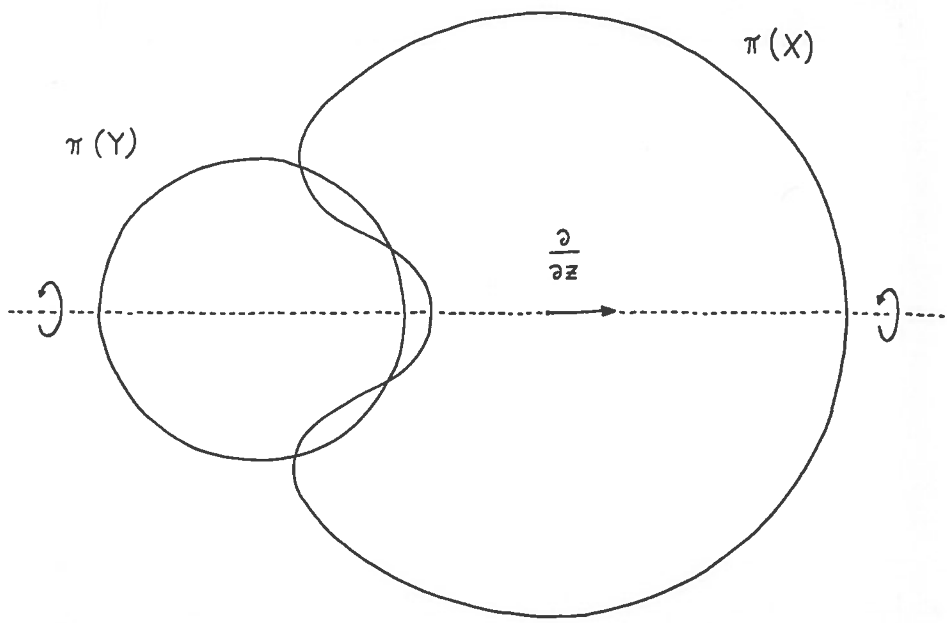

Figure 5.1: Curves generating the projection of the skies of x and y

Proposition 68 *The manifold of light rays N of a static (3+1)-spacetime (M, g) containing two thickening cylinders contains skies of causally related points which are not linked.*

Proof: Suppose that the thickening cylinders are aligned along the z-axis with axes parallel to the x-axis. Consider the wavefront $\Phi \subseteq \mathbb{R}^3$ of a point on the z-axis after a big enough interval of time so that the intersection of Φ with the plane $\{x = 0\}$ has developed two pairs of cusps. Recall that this is the projection on $\mathbb{R}^3$ of a sky $X \subseteq N = \mathbb{R}^3 \times S^2$ where we are using the Cauchy surface of constant t such that its intersection with the light cone of the corresponding point x is Φ. The point on the S^2 fibre above a point $\xi \in \Phi$ is determined by the normal vector to Φ at that point; points $\xi \in \Phi$ at which Φ has self-intersections are projections of points where X intersects the fibre over ξ more than once (X itself, of course, does not have self-intersections). Thus it is clear that if one deforms Φ in such a way that Φ is never self-tangent then there exists a corresponding deformation of X which projects to this deformation of Φ. Now consider the following deformation of Φ: we define

$$\begin{aligned}\gamma &= \left\{(y,z)\in\mathbb{R}^2 : (0,y,z)\in\Phi\right\}\\ \Gamma &= \{(x,y,z)\in\Phi : -\varepsilon < x < \varepsilon\} \approx (-\varepsilon,\varepsilon)\times\gamma\end{aligned}$$

and deform only the subset of X which projects to Γ as follows: if $f : [-\varepsilon,\varepsilon] \longrightarrow [0,+\infty)$ is a smooth function satisfying $f(-\varepsilon) = f(\varepsilon) = 0$ and $f(0) = 1$, and γ_t is the deformation of γ shown if figure 5.2, we deform Γ through

$$\Gamma_t = \left\{(s,\gamma_{f(s)t}) : s\in(-\varepsilon,\varepsilon)\right\}.$$

Let X_t be the deformation of X which projects down to Γ_t; then X_t does not have any self-intersections up to the point when the two vertices of γ_t coincide. To allow them to coincide and cross without any self-intersections, we just have to slightly alter X_t by changing the point on the fibre around one of the intersecting vertices (by adding, say, $f(s)t\frac{\partial}{\partial x}$ and normalizing). Thus we can get the vertices to cross, thereby making it possible for a point originally in the zero winding number region of Φ to be moved to an arbitrarily large distance from X_1. Consequently, the sky of this point is not linked with X on N (although this point is obviously causally related to x).□

Thus we see that for (3+1)-dimensional spacetimes the mere linking of skies is not enough; to understand causal relations on the manifold of light rays we

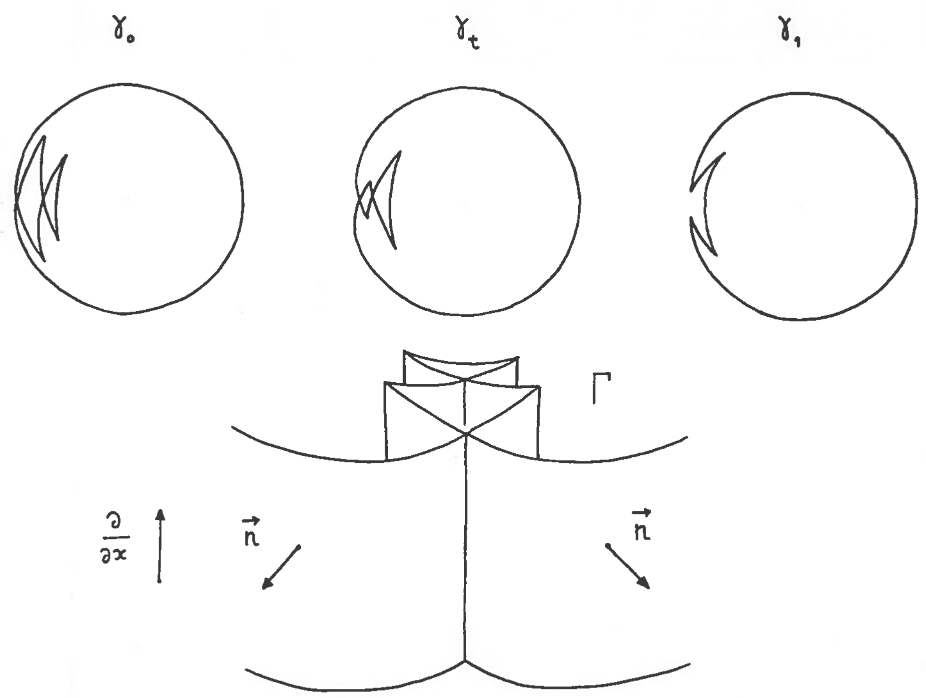

Figure 5.2: Unlinking skies of causally related points

need a new concept of linking. An important clue for finding such concept is the observation that in both the methods above for unlinking skies of causally related points it was not enough deforming its projections on the Cauchy surface keeping the normal vector as the point on the fibres: both deformations involved moving the points in the fibres away from the normal vector at some stage.

CHAPTER 6

Legendrian linking

6.1 What kind of linking?

We would like to introduce a concept of linking in the manifold of light rays such that two spacetime points would be linked if and only if their skies intersected or were linked. As we've seen, the usual concept of linking is too permissive (at least for (3+1)-dimensional spacetimes), as there are examples where the link formed by the skies of two causally related points is equivalent to the link formed by the skies of any two non-causally related points. In order to restrict our notion of linking we must restrict the class of allowed deformations of skies. For instance, one could only allow deforming skies into other skies of a given spacetime. Thus our deformations of skies would *all* arise from motions of the corresponding points on the spacetime manifold, and we would get indeed that the link formed by the skies of two causally related points would never be equivalent to the link formed by the skies of two non-causally related points (to move one of the points out of the other point's light cone one would have to cross it). However, this would be a (not very interesting) direct translation of causality onto the manifold of light rays, originating an extremely restrictive definition of linking. In order to search for a broader concept of linking which would still reflect the causal structure we must first try to identify some further structure on the manifold of light rays.

6.2 The manifold of scaled light rays

Let (M, g) be a (globally hyperbolic) $(d+1)$-dimensional spacetime, and N its manifold of light rays. As we have noted previously, N can be obtained either from the tangent bundle TM or the cotangent bundle T^*M, which has a natural symplectic structure. We shall now start examining in detail a particular way in which this can be accomplished, paying special attention to what happens to the symplectic structure.

If $\{x^{\mathbf{a}}\}$ are local coordinates on M, $\{v^{\mathbf{a}}\}$ are the associated coordinates on the fibres of the tangent bundle TM and $\{p_{\mathbf{a}}\}$ are the corresponding canonical coordinates on the fibres of the cotangent bundle T^*M, we know that the geodesic Lagrangian is

$$L = \frac{1}{2} g_{\mathbf{ab}} v^{\mathbf{a}} v^{\mathbf{b}},$$

the conjugate momenta are

$$p_{\mathbf{a}} = \frac{\partial L}{\partial v^{\mathbf{a}}} = g_{\mathbf{ab}} v^{\mathbf{b}}$$

and the Hamiltonian is

$$H = p_{\mathbf{a}} v^{\mathbf{a}} - L = \frac{1}{2} g^{\mathbf{ab}} p_{\mathbf{a}} p_{\mathbf{b}}.$$

Therefore Hamilton's equations are

$$\begin{aligned} \frac{dp_{\mathbf{a}}}{ds} &= -\frac{\partial H}{\partial x^{\mathbf{a}}} \Leftrightarrow \dot{p}_{\mathbf{a}} = -\frac{1}{2} \nabla_{\mathbf{a}} g^{\mathbf{bc}} p_{\mathbf{b}} p_{\mathbf{c}} \\ \frac{dx^{\mathbf{a}}}{ds} &= \frac{\partial H}{\partial p_{\mathbf{a}}} \Leftrightarrow \dot{x}^{\mathbf{a}} = g^{\mathbf{ab}} p_{\mathbf{b}} \end{aligned}$$

and its solutions yield the lifts to T^*M of the geodesics in M.

The *canonical symplectic potential* on T^*M is given in our local coordinates by

$$\theta = p_{\mathbf{a}} dx^{\mathbf{a}}$$

and the *canonical symplectic form* by

$$\omega = d\theta = dp_{\mathbf{a}} \wedge dx^{\mathbf{a}}.$$

As is well known, Hamilton's equations can be written as

$$\left(\frac{dp_{\mathbf{a}}}{ds}\frac{\partial}{\partial p_{\mathbf{a}}} + \frac{dx^{\mathbf{a}}}{ds}\frac{\partial}{\partial x^{\mathbf{a}}}\right)\rfloor\omega = -dH$$

using this form.

If one is interested in null geodesics, one should consider the $(2d+1)$-dimensional hypersurface of the cotangent bundle whose intersection with our local coordinate patch is given by

$$N^*M = \left\{(p_{\mathbf{a}}, x^{\mathbf{a}}) \in T^*M : g^{\mathbf{ab}}p_{\mathbf{a}}p_{\mathbf{b}} = 0 \text{ and } g^{\mathbf{ab}}p_{\mathbf{a}}\frac{\partial}{\partial x^{\mathbf{b}}} \text{ is future-pointing}\right\}.$$

(We shall from now on stop insisting on the fact that we're using local coordinates, and talk as if our coordinates were global. This does not change the exposition in any way, but makes it considerably simpler). Since

$$\frac{\partial H}{\partial s} = 0$$

it is well known from the usual reduction theorem in Hamiltonian mechanics (see [A]) that the null geodesics will be the so-called *vortex lines* of ω on N^*M, i.e., the integral lines of the distribution given by the zero-eigenvectors of ω (where obviously ω - and θ - are defined on N^*M by restriction). Again these null geodesics define an equivalence relation $\sim$ on N^*M, where one takes two points to be equivalent *iff* they lie on the same null geodesic. Since (M, g) is globally hyperbolic, the set

$$\widetilde{N} = \frac{N^*M}{\sim}$$

is clearly in one-to-one correspondence with the set

$$N^*M\mid_\Sigma = \{(p_{\mathbf{a}}, x^{\mathbf{a}}) \in N^*M : x^{\mathbf{a}} \in \Sigma\}$$

where Σ is an arbitrary Cauchy surface. We use this to endow $\widetilde{N}$ with a differentiable structure.

Definition 69 *We call* $\widetilde{N}$ *the* manifold of scaled light rays.

Note carefully that $\widetilde{N}$ is *not* the manifold of light rays; for one thing,

$$\dim(\widetilde{N}) = \dim(N^*M) - 1 = 2d = \dim(N) + 1.$$

The extra dimension comes from the scaling: obviously, although the null geodesics through $(p_{\mathbf{a}}, x^{\mathbf{a}})$ and $(\lambda p_{\mathbf{a}}, x^{\mathbf{a}})$ $(\lambda \in \mathbb{R}^+ \setminus \{1\})$ have the same projection on M, they are actually different geodesics on T^*M. As parametrized geodesics, their parameters will be related by an affine transformation of the form

$$f(s) = \frac{s}{\lambda} + \kappa$$

for some constant $k \in \mathbb{R}$.

Let $q : N^*M \to \widetilde{N}$ be the quotient map. Our aim will be to prove the following

Theorem 70 *There exist forms* $\widetilde{\theta} \in \Omega^1(\widetilde{N})$ *and* $\widetilde{\omega} \in \Omega^2(\widetilde{N})$ *such that* $\theta = q^*\widetilde{\theta}$, $\omega = q^*\widetilde{\omega}$.

Notice that $\omega = d\theta$ on N^*M (by restriction of the same equality in T^*M); therefore theorem 70 yields

$$q^*\widetilde{\omega} = dq^*\widetilde{\theta} \Leftrightarrow q^*\widetilde{\omega} = q^*d\widetilde{\theta} \Leftrightarrow q^*\left(\widetilde{\omega} - d\widetilde{\theta}\right) = 0.$$

Since q is a surjective map, this implies that we will have $\widetilde{\omega} = d\widetilde{\theta}$ on $\widetilde{N}$.

In order to prove theorem 70, we must consider forms on $\widetilde{N}$; it is therefore convenient that one understands the tangent space $T_\gamma\widetilde{N}$ to $\widetilde{N}$ at a given scaled geodesic γ. Any tangent vector on $T_\gamma\widetilde{N}$ arises from a one-parameter family of scaled geodesics. Let $\gamma : \mathbb{R}^2 \to M$ be such a family, where we take $\gamma(s, \alpha)$ to be the point corresponding to the value s of the affine parameter of the geodesic indexed by α (we may, of course, assume our geodesics to be affinely parametrized). Let us define

$$\begin{aligned} p^a &= \gamma_* \frac{\partial}{\partial s} \\ X^a &= \gamma_* \frac{\partial}{\partial \alpha}. \end{aligned}$$

We have

$$\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial \alpha}\right] = 0 \Rightarrow [p^a, X^a] = 0 \Leftrightarrow p^b\nabla_b X^a - X^b\nabla_b p^a = 0$$

or, defining the operator

$$D = p^a\nabla_a$$

we have

$$DX^a = X^b\nabla_b p^a.$$

The fact that the curves indexed by α are affinely parametrized geodesics translates as

$$Dp^a = 0. \tag{6.1}$$

As is well known, these equations imply that

$$\begin{aligned} D^2X^a &= D\left(X^b\nabla_b p^a\right) = \left(DX^b\right)\nabla_b p^a + p^c X^b \nabla_c \nabla_b p^a \\ &= p^c X^b \left(\nabla_b \nabla_c p^a + R_{cbd}{}^a p^d\right) + \left(X^c \nabla_c p^b\right)\nabla_b p^a \\ &= R_{bcd}{}^a p^b X^c p^d + X^c \nabla_c \left(p^b \nabla_b p^a\right) \\ &= R_{bcd}{}^a p^b X^c p^d \end{aligned}$$

which is the Jacobi propagation equation; in other words, X^a is a Jacobi field. Notice that since we are dealing with a family of null geodesics we have

$$X^a \nabla_a \left(p_b p^b\right) = 0 \Leftrightarrow p_b X^a \nabla_a p^b = 0 \Leftrightarrow p_a DX^a = 0 \tag{6.2}$$

and therefore we can only take as initial conditions for the Jacobi equation vectors DX^a in the three-dimensional orthogonal complement of p^a. Thus the vector space of Jacobi fields X^a we are considering on any null geodesic has dimension $2d-1$.

If we choose

$$X^a = f(s)p^a \tag{6.3}$$

we have

$$R_{bcd}{}^a p^b X^c p^d = f R_{bcd}{}^a p^b p^c p^d = 0$$

and

$$D^2 X^a = f''(s) p^a.$$

Thus if $f''(s) = 0$, i.e., if $f(s) = as + b$ for some $a, b \in \mathbb{R}$, then X^a is a Jacobi field. Also,

$$p_a DX^a = f'(s) p_a p^a = 0,$$

and thus X^a connects null geodesics. Thus Jacobi fields of the form 6.3 account for two of the $2d - 1$ dimensions. Notice that fields proportional to p^a, $X^a = \lambda p^a$, correspond to one-parameter families of null geodesics which comprise only one parametrized geodesic, $\gamma(s, \alpha) = \gamma(s + \lambda\alpha, 0)$, and therefore correspond to the zero vector in $T_\gamma \widetilde{N}$. Thus we have

Proposition 71 *Let (M, g) be a (globally hyperbolic) spacetime, $\widetilde{N}$ its manifold of scaled light rays and $\gamma \in \widetilde{N}$. Then $T_\gamma \widetilde{N}$ is isomorphic to the vector space of Jacobi fields X^a on γ which connect null geodesics quotiented by its (1-dimensional) vector subspace spanned by its affine tangent vector field p^a.*

Next we must understand the map

$$q_* : T_{(p_\mathbf{a}, x^\mathbf{a})} N^* M \to T\gamma \widetilde{N}.$$

If

$$V_\mathbf{a} \frac{\partial}{\partial p_\mathbf{a}} + X^\mathbf{a} \frac{\partial}{\partial x^\mathbf{a}} \in T_{(p_\mathbf{a}, x^\mathbf{a})} N^* M$$

then consider the one-parameter family of scaled geodesics with initial conditions given by any curve tangent to this vector in N^*M (affinely parameterized). This yields a Jacobi field X^a on the scaled geodesic γ with initial conditions on N^*M given by $(p_\mathbf{a}, x^\mathbf{a})$ which at the point in M with coordinates $x^\mathbf{a}$ satisfies

$$X^a = X^\mathbf{a} \frac{\partial}{\partial x^\mathbf{a}}$$

$$DX_a = X^b \nabla_b p_a = \left(V_\mathbf{a} - \Gamma^\mathbf{b}_\mathbf{ca} p_\mathbf{b} X^\mathbf{c}\right) dx^\mathbf{a},$$

which one can then identify with an element of $T_\gamma \widetilde{N}$.

If $\theta = q^* \widetilde{\theta}$, then one must have

$$\begin{aligned} \widetilde{\theta}\,(X^a) &= \widetilde{\theta}\left(q_*\left(V_\mathbf{a}\frac{\partial}{\partial p_\mathbf{a}} + X^\mathbf{a}\frac{\partial}{\partial x^\mathbf{a}}\right)\right) \\ &= \theta\left(V_\mathbf{a}\frac{\partial}{\partial p_\mathbf{a}} + X^\mathbf{a}\frac{\partial}{\partial x^\mathbf{a}}\right) \\ &= p_\mathbf{a} X^\mathbf{a} = p_a X^a \end{aligned}$$

(where we've used the identification in proposition 71). Notice that this does not depend on the representative of the equivalence class of Jacobi fields chosen:

$$\widetilde{\theta}\,(X^a + \lambda p^a) = p_a X^a + \lambda p_a p^a = p_a X^a = \widetilde{\theta}\,(X^a)$$

for all $\lambda \in \mathbb{R}$.

Similarly, if $\omega = q^* \widetilde{\omega}$ then one must have

$$\begin{aligned} \widetilde{\omega}\,(X^a, Y^a) &= \widetilde{\omega}\left(q_*\left(V_\mathbf{a}\frac{\partial}{\partial p_\mathbf{a}} + X^\mathbf{a}\frac{\partial}{\partial x^\mathbf{a}}\right), q_*\left(W_\mathbf{a}\frac{\partial}{\partial p_\mathbf{a}} + Y^\mathbf{a}\frac{\partial}{\partial x^\mathbf{a}}\right)\right) \\ &= \omega\left(V_\mathbf{a}\frac{\partial}{\partial p_\mathbf{a}} + X^\mathbf{a}\frac{\partial}{\partial x^\mathbf{a}}, W_\mathbf{a}\frac{\partial}{\partial p_\mathbf{a}} + Y^\mathbf{a}\frac{\partial}{\partial x^\mathbf{a}}\right) \\ &= V_\mathbf{a} Y^\mathbf{a} - W_\mathbf{a} X^\mathbf{a} \\ &= \left(V_\mathbf{a} - \Gamma^\mathbf{b}_\mathbf{ca} p_\mathbf{b} X^\mathbf{c}\right) Y^\mathbf{a} - \left(W_\mathbf{a} - \Gamma^\mathbf{b}_\mathbf{ca} p_\mathbf{b} Y^\mathbf{c}\right) X^\mathbf{a} \\ &= DX_a Y^a - DY_a X^a. \end{aligned}$$

Again this is independent of the representatives of the equivalence class of Jacobi fields used to perform the computation:

$$\widetilde{\omega}\,(X^a + \lambda p^a, Y^a + \mu p^a) = D\,(X_a + \lambda p_a)\,(Y^a + \mu p^a) - D\,(Y_a + \mu p_a)\,(X^a + \lambda p^a)$$

$$= DX_a (Y^a + \mu p^a) - DY_a (X^a + \lambda p^a)$$

$$= DX_a Y^a - DY_a X^a$$

$$= \widetilde{\omega} (X^a, Y^a)$$

for all $\lambda, \mu \in \mathbb{R}$, where we have used equations 6.1 and 6.2.

To complete the proof of theorem 70, all that remains to prove is that $\widetilde{\theta}$ and $\widetilde{\omega}$ as defined above are well-defined, i.e., that $\widetilde{\theta}(X^a)$ and $\widetilde{\omega}(X^a, Y^a)$ are independent of the point of the null geodesic γ we choose to perform the computation.

Proposition 72 *Let (M, g) be a spacetime, γ a null geodesic with affine tangent vector field p^a and X^a, Y^a Jacobi fields on γ connecting null geodesics. Then*

$$\widetilde{\theta}(X^a) = p_a X^a$$

and

$$\widetilde{\omega}(X^a, Y^a) = DX_a Y^a - DY_a X^a$$

are constant throughout γ.

Proof: The invariance of $\widetilde{\theta}(X^a)$ along γ is a simple consequence of equations 6.1 and 6.2:

$$D(p_a X^a) = p_a DX^a = 0.$$

The invariance of $\widetilde{\omega}(X^a, Y^a)$ along γ is a consequence of Jacobi's equation and the interchange symmetry of the Riemann tensor with respect to its two pairs of indices:

$$\begin{aligned} D(DX_a Y^a - DY_a X^a) &= D^2 X_a Y^a - D^2 Y_a X^a \\ &= Y_a R_{bcd}{}^a p^b X^c p^d - X_a R_{bcd}{}^a p^b Y^c p^d = 0. \end{aligned}$$

(this equation is sometimes referred to as *Lagrange's identity*).□

To compute $\widetilde{\theta}$ and $\widetilde{\omega}$ we must introduce local coordinates in $\widetilde{N}$; to this end the following result is useful:

Proposition 73 *Let (M, g) be a globally hyperbolic $(d+1)$-dimensional spacetime and Σ an arbitrary Cauchy surface; then $\widetilde{N}$ is diffeomorphic to $T^*\Sigma \backslash (\Sigma \times \{0\})$.*

Proof: Choose an arbitrary nowhere vanishing future-pointing timelike vector field t^a on M and use it to extend any local chart $\{x^{\mathbf{i}}\}$ on Σ to a local chart $\{t, x^{\mathbf{i}}\}$ on M and consequently to a local chart $\{p_0, p_{\mathbf{i}}, t, x^{\mathbf{i}}\}$ on T^*M. Since

$$p_{\mathbf{a}} dx^{\mathbf{a}} \left(\frac{\partial}{\partial t} \right) = p_0,$$

we see that the intersection of $\widetilde{N} \approx N^*M\mid_\Sigma$ with the domain of the chart is clearly the pre-image of

$$\left\{ (p_0, p_{\mathbf{i}}, 0, x^{\mathbf{i}}) : g^{\mathbf{ab}} p_{\mathbf{a}} p_{\mathbf{b}} = 0 \text{ and } p_0 > 0 \right\}$$

whereas the intersection of $T^*\Sigma \backslash (\Sigma \times \{0\}) \subseteq T^*M$ with the domain of the same chart is the pre-image of

$$\left\{ (0, p_{\mathbf{i}}, 0, x^{\mathbf{i}}) : p_{\mathbf{i}} dx^{\mathbf{i}} \neq 0 \right\}.$$

Provided that $p_{\mathbf{i}} dx^{\mathbf{i}} \neq 0$, the equation

$$\begin{aligned} g^{\mathbf{ab}} p_{\mathbf{a}} p_{\mathbf{b}} &= 0 \Leftrightarrow g^{00} (p_0)^2 + 2 g^{0\mathbf{i}} p_{\mathbf{i}} p_0 + g^{\mathbf{ij}} p_{\mathbf{i}} p_{\mathbf{j}} = 0 \\ &\Leftrightarrow g^{00} \left(p_0 + \frac{g^{0\mathbf{i}}}{g^{00}} p_{\mathbf{i}} \right)^2 + \left(g^{\mathbf{ij}} - \frac{g^{0\mathbf{i}} g^{0\mathbf{j}}}{g^{00}} \right) p_{\mathbf{i}} p_{\mathbf{j}} = 0 \end{aligned}$$

can always be solved for a unique $p_0 > 0$, as

$$h^{\mathbf{ij}} = -g^{\mathbf{ij}} + \frac{g^{0\mathbf{i}} g^{0\mathbf{j}}}{g^{00}}$$

is the Euclidean metric induced on $T^*_{(x^i)}\Sigma$ by taking the orthogonal projection on the orthogonal complement of dt (we shall call such metric the *space metric*, as it coincides with our previous definition for static spacetimes). Indeed, the tangent vectors $\left\{\frac{\partial}{\partial x^i}\right\}$ span the tangent hyperplanes to the Cauchy surfaces $\{dt = 0\}$. Since $dt\left(\frac{\partial}{\partial x^i}\right) = 0$ and $dt\left(\frac{\partial}{\partial t}\right) = 1$, we see that dt corresponds to a future-pointing timelike vector. Therefore its orthogonal complement is a spacelike hyperplane, whose induced metric is minus a Euclidean metric. Because

$$\langle dt, q_a dx^a\rangle = g^{00}q_0 + g^{0i}q_i$$

we have

$$\left\langle dt, -\frac{g^{0i}q_i}{g^{00}}dt + q_i dx^i\right\rangle = 0$$

and consequently the orthogonal projection of $q_a dx^a$ on the orthogonal complement of dt is seen to be

$$-\frac{g^{0i}q_i}{g^{00}}dt + q_i dx^i.$$

Now

$$\begin{aligned}
\left\langle -\frac{g^{0i}q_i}{g^{00}}dt + q_i dx^i, -\frac{g^{0i}q_i}{g^{00}}dt + q_i dx^i\right\rangle &= \frac{g^{0i}g^{0j}}{g^{00}}q_i q_j - 2g^{0i}\frac{g^{0j}q_j}{g^{00}}q_i + g^{ij}q_i q_j \\
&= -h^{ij}q_i q_j
\end{aligned}$$

which shows that h^{ij} is in fact an Euclidean metric on $T^*_{(x^i)}\Sigma$.

The solution

$$p_0 = p_0(p_i, x^i)$$

provides the diffeomorphism between $T^*\Sigma\backslash(\Sigma\times\{0\})$ and $\widetilde{N}$: one identifies the point

$$(p_i, x^i) \in T^*\Sigma \backslash (\Sigma \times \{0\})$$

with the scaled null geodesic with initial condition

$$\left(p_0(p_i, x^i), p_i, 0, x^i\right) \in N^*M \mid_\Sigma . \square$$

Notice that the zero section must be removed from $T^*\Sigma$ as $p_0(0, x^i) = 0$.

In particular, this proposition implies that one can use $\left\{p_i, x^i\right\}$ as local coordinates on $\widetilde{N}$. We can now prove

Theorem 74 $(\widetilde{N}, \widetilde{\theta}, \widetilde{\omega})$ *is a symplectic manifold symplectomorphic to* $T^*\Sigma \backslash (\Sigma \times \{0\})$ *endowed with the canonical symplectic structure, where* Σ *is an arbitrary Cauchy surface on* M.

Proof: We already know that $\widetilde{N}$ is diffeomorphic to $T^*\Sigma \backslash (\Sigma \times \{0\})$. A tangent vector on $T_{(p_0, p_i, 0, x^i)}N^*M$ given in our standard local coordinates by

$$\left(\frac{\partial p_0}{\partial p_i} V_i + \frac{\partial p_0}{\partial x^i} X^i\right) \frac{\partial}{\partial p_0} + V_i \frac{\partial}{\partial p_i} + X^i \frac{\partial}{\partial x^i}$$

clearly satisfies

$$q_* \left(\left(\frac{\partial p_0}{\partial p_i} V_i + \frac{\partial p_0}{\partial x^i} X^i\right) \frac{\partial}{\partial p_0} + V_i \frac{\partial}{\partial p_i} + X^i \frac{\partial}{\partial x^i}\right) = V_i \frac{\partial}{\partial p_i} + X^i \frac{\partial}{\partial x^i}.$$

Consequently,

$$\begin{aligned} \widetilde{\theta} \left(V_i \frac{\partial}{\partial p_i} + X^i \frac{\partial}{\partial x^i}\right) &= \theta \left(\left(\frac{\partial p_0}{\partial p_i} V_i + \frac{\partial p_0}{\partial x^i} X^i\right) \frac{\partial}{\partial p_0} + V_i \frac{\partial}{\partial p_i} + X^i \frac{\partial}{\partial x^i}\right) \\ &= p_0 . 0 + p_i X^i = p_i dx^i \left(V_i \frac{\partial}{\partial p_i} + X^i \frac{\partial}{\partial x^i}\right). \end{aligned}$$

Thus

$$\tilde{\theta} = p_i dx^i$$

and consequently

$$\tilde{\omega} = d\tilde{\theta} = dp_i \wedge dx^i$$

which are the canonical symplectic forms in $T^*\Sigma \backslash (\Sigma \times \{0\})$.□

6.3 Contact structure on the manifold of light rays

Proposition 75 *Let (M,g) be a (globally hyperbolic) $(d+1)$-dimensional space-time and Σ an arbitrary Cauchy surface. Then its manifold of light rays N is diffeomorphic to $\frac{T^*\Sigma \backslash (\Sigma \times \{0\})}{\sim}$, where "$\sim$" is the equivalence relation induced on $T^*\Sigma \backslash (\Sigma \times \{0\})$ by the action of $\mathbb{R}^+$ on its fibres.*

Proof: As we have noted before, null geodesics through (p_a, x^a) and $(\lambda p_a, x^a)$ $(\lambda \in \mathbb{R}^+)$, while being different geodesics on T^*M, yield the same null geodesic on M (with a different affine parametrization). Thus we get N from $\widetilde{N} \approx N^*M\,|_\Sigma$ by identifying the points given on our coordinate system by

$$\left(p_0(p_i, x^i), p_i, 0, x^i\right)$$

and

$$\left(\lambda p_0(p_i, x^i), \lambda p_i, 0, x^i\right) = \left(p_0(\lambda p_i, x^i), \lambda p_i, 0, x^i\right)$$

for all $\lambda > 0$ (notice that p_0 is homogeneous of degree one in p_i). This corresponds to identifying

$$(p_i, x^i) \sim (\lambda p_i, x^i)$$

on $T^*\Sigma\backslash(\Sigma\times\{0\})$ for all $\lambda>0$.□

Obviously N cannot be a symplectic manifold, as it has odd dimension, $\dim(N)=2d-1$. However, the fact that it can be obtained from a cotangent bundle by quotienting the fibres by the $\mathbb{R}^+$-action allows us to introduce a so-called *contact structure* on N. In order to do this we start with

Proposition 76 *Let $\pi:\widetilde{N}\to N$ be the projection map. Then*

$$W=\pi_*\left(\ker\left(\widetilde{\theta}\right)\right)$$

is a $(2d-2)$-dimensional distribution on N.

Proof: Take the usual local coordinates $\left\{p_i,x^i\right\}$ on $\widetilde{N}$. The paths on $\widetilde{N}$ given in these local coordinates by

$$(p_i(t),x^i(t))\text{ and }(\lambda(t)p_i(t),x^i(t))$$

project down to the same path on N, where λ is any positive function; thus, is easy to see that the vectors which project down to the same vector on $T_{\pi(p_i,x^i)}N$ are of the form

$$(\alpha p_i+\lambda V_i)\frac{\partial}{\partial p_i}+X^i\frac{\partial}{\partial x^i}\in T_{(\lambda p_i,x^i)}\widetilde{N}$$

for $\alpha\in\mathbb{R}$ and $\lambda\in\mathbb{R}^+$ arbitrary. Now

$$\widetilde{\theta}\left((\alpha p_i+\lambda V_i)\frac{\partial}{\partial p_i}+X^i\frac{\partial}{\partial x^i}\right)=\lambda p_iX^i$$

and thus one sees that if $X\in\ker\left(\widetilde{\theta}\right)$ and $\pi_*Y=\pi_*X$ then $Y\in\ker\left(\widetilde{\theta}\right)$ (notice that X and Y do not have to belong to the same tangent space). This shows that W is well-defined.

To get a better understanding of this distribution, we introduce an arbitrary Riemannian metric h on Σ and identify N with $TS^*\Sigma$:

$$N \approx TS^*\Sigma = \left\{(p_i, x^i) \in T^*\Sigma : h^{ij}p_ip_j = 1\right\}.$$

This way we get N as a submanifold of $\widetilde{N}$, and the map $\pi : \widetilde{N} \to N$ becomes the projection on this submanifold along the integral lines of the so-called *Euler vector field*

$$E = p_i\frac{\partial}{\partial p_i}.$$

The advantage of this approach is that now obviously π is the identity when restricted to N; therefore, we get $W = \ker(\theta)$, where $\theta = \widetilde{\theta}\mid_N$. Since locally $\theta = p_idx^i$ and $\left\{x^i\right\}$ can be extended to local coordinates on N, θ never vanishes identically and W is in fact a distribution of hyperplanes of codimension 1, i.e., of dimension $2d-2$.□

Proposition 77 *The 1-form $\theta \in \Omega^1(N)$ satisfies the nondegeneracy condition*

$$\theta \wedge (d\theta)^{d-1} \neq 0.$$

Proof: It is easy to guess from

$$d\left(\widetilde{\theta} \wedge \left(d\widetilde{\theta}\right)^{d-1}\right) = \left(d\widetilde{\theta}\right)^d = \widetilde{\omega}^d = d!dp_1 \wedge ... \wedge dp_d \wedge dx^1 \wedge ... \wedge dx^d$$

that

$$\widetilde{\theta} \wedge \left(d\widetilde{\theta}\right)^{d-1} = (d-1)!\sum_{i=1}^{d}(-1)^ip_idp_1 \wedge ... \wedge \widehat{dp_i} \wedge ... \wedge dp_d \wedge dx^1 \wedge ... \wedge dx^d.$$

(where the hat means omission). Consequently, if

$$X_1 = V_i^1\frac{\partial}{\partial p_i} + X_1^i\frac{\partial}{\partial x^i}, ..., X_{2d-1} = V_i^{2d-1}\frac{\partial}{\partial p_i} + X_{2d-1}^i\frac{\partial}{\partial x^i} \in T\widetilde{N}$$

we have

$$\begin{aligned}
&\widetilde{\theta}\wedge\left(d\widetilde{\theta}\right)^{d-1}(X_1,...,X_{2d-1})\\
&= (d-1)!\sum_{\mathrm{i}=1}^{d}(-1)^{\mathrm{i}}p_{\mathrm{i}}\begin{vmatrix} V_1^1 & \dots & \widehat{V_{\mathrm{i}}^1} & \dots & V_d^1 & X_1^1 & \dots & X_1^d \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ V_1^{2d-1} & \dots & \widehat{V_{\mathrm{i}}^{2d-1}} & \dots & V_d^{2d-1} & X_{2d-1}^1 & \dots & X_{2d-1}^d \end{vmatrix}\\
&= (d-1)!\begin{vmatrix} p_1 & \dots & p_d & 0 & \dots & 0 \\ V_1^1 & \dots & V_d^1 & X_1^1 & \dots & X_1^d \\ \dots & \dots & \dots & \dots & \dots & \dots \\ V_1^{2d-1} & \dots & V_d^{2d-1} & X_{2d-1}^1 & \dots & X_{2d-1}^d \end{vmatrix}.
\end{aligned}$$

Since clearly

$$E = p_{\mathrm{i}}\frac{\partial}{\partial p_{\mathrm{i}}} \in (TN)^{\perp}$$

we see that if we take $X_1, ..., X_{2d-1}$ to be linearly independent vectors on $TN \subseteq T\widetilde{N}$ then

$$\theta \wedge (d\theta)^{d-1}(X_1, ..., X_{2d-1}) \neq 0$$

as required.□

Notice that if $\varphi \in \Omega^1(U)$ is any other 1-form defined on an open set $U \subseteq N$ such that on U we have $W = \ker(\varphi)$ then necessarily $\varphi = \lambda\theta$ for some nonvanishing function $\lambda \in C^\infty(U)$; therefore,

$$\begin{aligned}
\varphi \wedge (d\varphi)^{d-1} &= \lambda\theta \wedge (d\lambda \wedge \theta + \lambda d\theta)^{d-1} = \\
&= \lambda^d\theta \wedge (d\theta)^{d-1}
\end{aligned}$$

and thus the nondegeneracy condition above is actually independent of the particular 1-form whose kernel yields W. It is easy to see that this condition is just the condition that $d\theta$ be nondegenerate on each of the planes of the distribution W.

Definition 78 *A* $(2d-1)$*-dimensional manifold* N *endowed with a* $(2d-2)$*-hyperplane distribution* W *given locally by a 1-form satisfying the nondegeneracy condition above is called a* contact manifold.

Thus the manifold of light rays of a globally hyperbolic $(d+1)$-spacetime endowed with the distribution W arising from the canonical symplectic potential $\widetilde{\theta}$ on $\widetilde{N}$ is a contact manifold. This fact is hinted in [PR] and [P], and fully expressed in [L1] (where a considerably more compact abstract derivation of the contact structure can also be found). We shall now discuss a few relevant issues about contact manifolds concentrating on this particular example; a more general discussion can be found in [A].

Definition 79 *Let* (N, W) *be a contact manifold and* $\gamma \in N$*. Then the hyperplanes* $W_\gamma \subseteq T_\gamma N$ *are called the* contact planes*; any covector* $\alpha \in T^*_\gamma N$ *such that* $\ker(\alpha) = W_\gamma$ *is called a* contact covector*; and any (locally defined) 1-form* $\theta \in \Omega^1(U)$ *($U \subseteq N$ an open set) such that* $\theta\mid_\gamma$ *is a contact covector for all* $\gamma \in U$ *is called a* local contact 1-form.

Because in the case of the manifold of light rays one has a globally defined contact 1-form θ, at each point $\gamma \in N$ the set of contact covectors splits in two disjoint sets, namely the set of positive multiples of $\theta\mid_\gamma$ and the set of negative multiples of $\theta\mid_\gamma$. We use this fact to make the (slightly nonstandard)

Definition 80 *The* symplectification $\widehat{N}$ *of the (contact) manifold of light rays* (N, W) *is the fibre bundle of all contact covectors which are positive multiples of the canonical contact 1-form* θ.

Clearly $\widehat{N}$ is a fibre bundle over N with fibre $\mathbb{R}^+$. One can define a canonical 1-form $\widehat{\theta}$ on $\widehat{N}$ as follows: let $\pi : \widehat{N} \to N$ be the canonical projection, $\alpha \in \widehat{N}$ and $\xi \in T_\alpha \widehat{N}$; then we define

$$\widehat{\theta}(\xi) = \alpha\left(\pi_* \xi\right).$$

Obviously $\widehat{\theta}$ is globally defined, and one can define $\widehat{\omega} = d\widehat{\theta}$.

Proposition 81 $(\widehat{N}, \widehat{\omega})$ *is a symplectic manifold symplectomorphic to* $(\widetilde{N}, \widetilde{\omega})$.

Proof: Think of N as a submanifold of $\widetilde{N}$. Given $\alpha \in \widehat{N}$, there exists a unique $\lambda > 0$ such that $\alpha = \lambda\theta$. Use this to define the map $f : \widehat{N} \to \widetilde{N}$ given in the canonical local coordinates of $\widetilde{N}$ by

$$f(\alpha) = (\lambda p_i, x^i).$$

Clearly f is a diffeomorphism. We can use this to introduce local coordinates $\left\{p_i, x^i\right\}$ on $\widehat{N}$. Now if $\xi \in T_\alpha \widehat{N}$, it is easy to see that

$$\xi = V_i \frac{\partial}{\partial p_i} + X^i \frac{\partial}{\partial x^i} \Rightarrow \widehat{\theta}(\xi) = \alpha\left(\pi_* \xi\right) = \lambda p_i dx^i \left(X^i \frac{\partial}{\partial x^i}\right) = \widetilde{\theta}(\xi). \square$$

Definition 82 *Let* (N, W) *be a contact manifold. A diffeomorphism* $f : N \to N$ *is said to be a* contact diffeomorphism *(or* contactomorphism*) if* $f_* W = W$, *i.e., if* $f_* W_\gamma = W_{f(\gamma)}$ *for all* $\gamma \in N$. *A vector field* ξ *on* N *is said to be a* contact vector field *if the one-parameter group of diffeomorphisms generated by* ξ *is a group of contact diffeomorphisms.*

A nice characterization of contact vector fields is as follows:

Proposition 83 *Let (N, W) be a contact manifold and ξ a vector field on N. Then ξ is a contact vector field* iff $\theta \wedge \pounds_\xi \theta = 0$ *for all local contact 1-forms θ.*

Proof: Let Φ_t be the one-parameter group of diffeomorphisms generated by ξ, i.e., $\xi = \frac{d}{dt}\Phi_t$. Then ξ is a contact vector field *iff*

$$(\Phi_t)_* W_\gamma = W_{\Phi_t(\gamma)}$$

for all $\gamma \in N$, i.e., *iff*

$$\theta(\Phi_t)((\Phi_t)_* V) = 0$$

for all $\gamma \in N$, $V \in T_\gamma N$ and local contact 1-forms θ. This is of course equivalent to saying that at each point

$$(\Phi_t)^* \theta(\Phi_t) = f(t)\theta$$

for some function f, or to saying that at each point

$$\frac{d}{dt}\left[(\Phi_t)^*\theta(\Phi_t)\right]|_{t=0} = f'(0)\theta \Leftrightarrow \pounds_\xi \theta = f'(0)\theta \Leftrightarrow \theta \wedge \pounds_\xi \theta = 0. \square$$

This characterization is especially useful to prove

Theorem 84 *The set of all contact vector fields on a contact manifold is a Lie subalgebra of the Lie algebra of smooth vector fields.*

Proof: If ξ and η are contact vector fields and θ is a local contact 1-form then we have

$$\begin{aligned} \pounds_\xi \theta &= f\theta \\ \pounds_\eta \theta &= g\theta \end{aligned}$$

for some locally defined functions f and g. Therefore,

$$\pounds_{[\xi,\eta]}\theta = [\pounds_\xi, \pounds_\eta]\,\theta = \pounds_\xi(g\theta) - \pounds_\eta(f\theta) = (\xi g - \eta f)\theta.\square$$

We now return to the case when N is the manifold of light rays of a globally hyperbolic $(d+1)$-dimensional spacetime (M, g) (or, equivalently, the tangent sphere bundle of a d-dimensional Riemannian manifold (Σ, h):

Definition 85 *Let (N, W) be a manifold of light rays and $(\widetilde{N}, \widetilde{\omega})$ its symplectification. If $f : N \to N$ is a contactomorphism such that $f^*\theta = \beta\theta$ for some smooth function $\beta > 0$, we define its* symplectification *$\widetilde{f} : \widetilde{N} \to \widetilde{N}$ by assigning to each contact covector $\alpha \in \widehat{N} \approx \widetilde{N}$ the contact covector*

$$\widetilde{f}(\alpha) = (f^*)^{-1}\,\alpha.$$

Notice that the fact that f is a contactomorphism implies that $f^*\theta = \beta\theta$ for some function $\beta \neq 0$; in the above definition we discard the possibility $\beta < 0$ due to our nonstandard definition of symplectification of the manifold of light rays. Since we'll be interested in one-parameter families of contactomorphisms through the identity map, there is no loss of generality (for our purposes) in using only the connected component of the identity map.

Theorem 86 *Let (N, W) be a manifold of light rays and $f : N \to N$ a contactomorphism; then its symplectification $\widetilde{f} : \widetilde{N} \to \widetilde{N}$ is a symplectomorphism which commutes with the action of $\mathbb{R}^+$ and preserves the canonical symplectic potential $\widetilde{\theta}$.*

Proof: It is obvious from the definition that

$$\widetilde{f}(\lambda\alpha) = (f^*)^{-1}(\lambda\alpha) = \lambda (f^*)^{-1}\alpha = \lambda \widetilde{f}(\alpha)$$

for all $\lambda > 0$. Thus all we have to prove is that $\widetilde{f}^*\widetilde{\theta} = \widetilde{\theta}$ (obviously if $\widetilde{f}$ preserves $\widetilde{\theta}$ it also preserves $\widetilde{\omega} = d\widetilde{\theta}$). If $(f^*)^{-1}\theta = \beta\theta$ for some function $\beta > 0$ (note that *this* function β is the multiplicative inverse of the function β in definition 85), we have

$$\left(\widetilde{f}^*\widetilde{\theta}\right)|_{\lambda\theta}(\xi) = \left(\widetilde{\theta}\right)|_{\lambda\beta\theta}\left(\widetilde{f}_*\xi\right) = \lambda\beta\theta\left(\pi_*\widetilde{f}_*\xi\right) = \lambda\beta\theta\left(f_*\pi_*\xi\right) = \lambda\beta f^*\theta\left(\pi_*\xi\right)$$

where we've used the fact that $\widetilde{f}$ commutes with the action of $\mathbb{R}^+$ on $\widetilde{N}$ (and thus $\pi_*\widetilde{f}_* = f_*\pi_*$, as π corresponds precisely to taking the quotient by this action). But

$$\widetilde{f}(\theta) = \beta\theta \Leftrightarrow (f^*)^{-1}\theta = \beta\theta \Leftrightarrow \theta = \beta f^*\theta$$

and therefore

$$\left(\widetilde{f}^*\widetilde{\theta}\right)|_{\lambda\theta}(\xi) = \lambda\theta\left(\pi_*\xi\right) = \widetilde{\theta}|_{\lambda\theta}(\xi)$$

i.e.,

$$\widetilde{f}^*\widetilde{\theta} = \widetilde{\theta}$$

as we needed to show.□

This theorem admits the following converse:

Theorem 87 *Every symplectomorphism of the symplectification $(\widetilde{N}, \widetilde{\omega})$ of the manifold of light rays (N, W) which commutes with the action of $\mathbb{R}^+$ projects onto N as a contactomorphism and preserves the canonical symplectic potential $\widetilde{\theta}$.*

Proof: Let $F : \widetilde{N} \to \widetilde{N}$ be a symplectomorphism which commutes with the action of $\mathbb{R}^+$ on $\widetilde{N}$; then clearly it projects to a diffeomorphism $f : N \to N$. Now if $F^*\widetilde{\theta} = \widetilde{\theta}$ then for all $\xi \in T\widetilde{N}$

$$F^*\widetilde{\theta}\,(\xi) = \widetilde{\theta}\,(\xi) \Leftrightarrow \widetilde{\theta}\,(F_*\xi) = \widetilde{\theta}\,(\xi)$$

or, using the usual coordinate λ on $\widetilde{N}$, and setting $F(\theta) = \beta\theta$,

$$\beta\theta\,(\pi_* F_* \xi) = \theta\,(\pi_*\xi) \Leftrightarrow \beta\theta\,(f_*\pi_*\xi) = \theta\,(\pi_*\xi)$$

Since $\pi : \widetilde{N} \to N$ is surjective, this is equivalent to saying that

$$f^*\theta = \frac{1}{\beta}\theta$$

i.e., that f is a contactomorphism.

Thus all we have to prove is that F preserves $\widetilde{\theta}$. Let Γ be a path on $\widetilde{N}$, and consider the surface $\Sigma(\varepsilon)$ obtained from Γ by multiplying each covector on Γ by all numbers in the interval $[\varepsilon, 1]$. Since the boundary of $\Sigma(\varepsilon)$ tends to Γ and two integral lines of the Euler vector field E (where $\widetilde{\theta}$ vanishes), we get from Stokes' theorem that

$$\int_\Gamma \widetilde{\theta} = \lim_{\varepsilon \to 0} \int_{\Sigma(\varepsilon)} \widetilde{\omega}.$$

Now since F preserves $\widetilde{\omega}$ and commutes with multiplication by elements of $\mathbb{R}^+$, it follows that

$$\int_{F(\Gamma)} \widetilde{\theta} = \int_\Gamma \widetilde{\theta}$$

for arbitrary paths Γ, and thus F indeed preserves $\widetilde{\theta}$.□

Definition 88 *The* symplectification *of a contact vector field ξ is the vector field $\widetilde{\xi}$ generating the one-parameter group of symplectomorphisms $\widetilde{\Phi}_t$ which are the symplectifications of the one-parameter group of contactomorphisms Φ_t generated by ξ.*

At this point it is useful to recall

Proposition 89 *Let* $(\widetilde{N}, \widetilde{\omega})$ *be a symplectic manifold and* ξ *a vector field on* $\widetilde{N}$*; then the one-parameter group of diffeomorphisms* Φ_t *generated by* ξ *is a group of symplectomorphisms* iff ξ *is locally Hamiltonian.*

Proof: We have

$$(\Phi_t)^* \widetilde{\omega} (\Phi_t) = \widetilde{\omega} \Leftrightarrow \frac{d}{dt} |_{t=0} (\Phi_t)^* \widetilde{\omega} (\Phi_t) = 0 \Leftrightarrow \pounds_\xi \widetilde{\omega} = 0.$$

Now

$$\pounds_\xi \widetilde{\omega} = d (\xi \rfloor \widetilde{\omega}) + \xi \rfloor d\widetilde{\omega} = d (\xi \rfloor \widetilde{\omega})$$

as $d\widetilde{\omega} = 0$, and thus

$$d (\xi \rfloor \widetilde{\omega}) = 0 \Leftrightarrow \xi \rfloor \widetilde{\omega} = -dH$$

for some local function H, i.e., ξ is a locally Hamiltonian vector field.□

Theorem 90 *The symplectification of a contact vector field is a globally Hamiltonian vector field. The Hamiltonian can be chosen to be homogeneous of degree one with respect to the multiplicative action of* $\mathbb{R}^+$*:*

$$H(\lambda \alpha) = \lambda H(\alpha)$$

for all $\lambda > 0$*. Conversely, every globally Hamiltonian field on* $\widetilde{N}$ *having a Hamiltonian homogeneous of degree one with respect to the multiplicative action of* $\mathbb{R}^+$ *projects onto* N *as a contact vector field.*

Proof: If ξ is the symplectification of a contact vector field, then the one-parameter group of symplectomorphisms Φ_t generated by ξ preserves the canonical symplectic potential $\widetilde{\theta}$, and thus

$$\begin{aligned}(\Phi_t)^* \widetilde{\theta}(\Phi_t) &= \widetilde{\theta} \Leftrightarrow \frac{d}{dt}|_{t=0} (\Phi_t)^* \widetilde{\theta}(\Phi_t) = 0 \Leftrightarrow \pounds_\xi \widetilde{\theta} = 0 \Leftrightarrow \\ &\Leftrightarrow d\left(\xi \rfloor \widetilde{\theta}\right) + \xi \rfloor d\widetilde{\theta} = 0 \Leftrightarrow \xi \rfloor \widetilde{\omega} = -dH\end{aligned}$$

where $H = \xi \rfloor \widetilde{\theta}$ is clearly homogeneous of degree 1:

$$H(\lambda\alpha) = \lambda\alpha(\pi_* \xi) = \lambda H(\alpha).$$

The converse statement is a trivial consequence of theorem 87 once we have proven that the elements of the one-parameter group of diffeomorphisms Φ_t generated by the symplectic vector field ξ arising from a homogeneous of degree 1 Hamiltonian H commute with the action of $\mathbb{R}^+$, i.e., once we've proven that

$$[\xi, E] = 0$$

where

$$E = p_i \frac{\partial}{\partial p_i}$$

is the Euler vector field. Saying that H is homogeneous of degree 1 is the same as saying that $EH = H$. Now

$$\pounds_E (\xi \rfloor \widetilde{\omega}) = \pounds_E(-dH) = -d(EH) = -dH$$

whereas also

$$\pounds_E (\xi \rfloor \widetilde{\omega}) = (\pounds_E \xi) \rfloor \widetilde{\omega} + \xi \rfloor (\pounds_E \widetilde{\omega})$$

and

$$\pounds_E\widetilde{\omega} = d\left(E\rfloor\widetilde{\omega}\right) = d\left(p_i dx^i\right) = d\widetilde{\theta} = \widetilde{\omega}.$$

Thus,

$$(\pounds_E\xi)\rfloor\widetilde{\omega} + \xi\rfloor\widetilde{\omega} = -dH \Leftrightarrow (\pounds_E\xi)\rfloor\widetilde{\omega} = 0$$

and consequently

$$\pounds_E\xi = 0$$

as we needed to show.□

We can state this relationship between contact vector fields on N and globally Hamiltonian vector fields on $\widetilde{N}$ with homogeneous degree 1 Hamiltonians in the following

Theorem 91 *Symplectification contact vector fields provides a isomorphism between the Lie algebra of all contact vector fields on N and the Lie algebra of globally Hamiltonian vector fields on $\widetilde{N}$ with homogeneous degree 1 Hamiltonians.*

Proof: This is obvious from the above and the fact that if ξ is a contact vector field and $\widetilde{\xi}$ its symplectification, then

$$\xi = \pi_*\widetilde{\xi}$$

and thus if ξ, η are two contact vector fields then

$$[\xi, \eta] = \left[\pi_*\widetilde{\xi}, \pi_*\widetilde{\eta}\right] = \pi_*\left[\widetilde{\xi}, \widetilde{\eta}\right].$$

Recall that if

$$\begin{aligned} \widetilde{\xi}\rfloor\widetilde{\omega} &= -dH \\ \widetilde{\eta}\rfloor\widetilde{\omega} &= -dK \end{aligned}$$

then

$$\left(\left[\tilde{\xi},\tilde{\eta}\right]\rfloor\tilde{\omega}\right)=\left(\pounds_{\tilde{\xi}}\tilde{\eta}\right)\rfloor\tilde{\omega}=\pounds_{\tilde{\xi}}(\tilde{\eta}\rfloor\tilde{\omega})-\tilde{\eta}\rfloor\pounds_{\tilde{\xi}}\tilde{\omega}=\pounds_{\tilde{\xi}}(-dK)=-d\left(\tilde{\xi}\rfloor dK\right)=-d\left(\tilde{\xi}K\right).$$

In our standard local coordinates

$$\tilde{\xi}K=-\frac{\partial H}{\partial x^i}\frac{\partial K}{\partial p_i}+\frac{\partial H}{\partial p_i}\frac{\partial K}{\partial x^i}$$

is easily seen to be homogeneous of degree 1; therefore the Lie bracket of two Hamiltonian vector fields with Hamiltonian homogeneous of degree 1 is again a Hamiltonian vector field with Hamiltonian homogeneous of degree 1, i.e., such vector fields do form a Lie algebra.□

6.4 Legendrian submanifolds and Legendrian linking

Definition 92 *Let (N,W) be a contact manifold and $X\subseteq N$ a submanifold. Then X is said to be an* integral submanifold *of W iff $T_\gamma X\subseteq W_\gamma$ for all $\gamma\in X$.*

If N is the manifold of light rays and X is the sky of a point x, it is easy to see that X is an integral submanifold. This is most easily done by considering a Cauchy surface Σ through x: X is then a fibre of $TS^*\Sigma$ and is certainly an integral submanifold of $\theta=p_i dx^i$.

Proposition 93 *Let (N,W) be a contact manifold of dimension $2d-1$ and $X\subseteq N$ an integral submanifold of W; then*

$$\dim(X)\leq d-1.$$

Proof: Let θ be a local contact 1-form and ξ, η two vector fields tangent to X. Then $[\xi, \eta]$ is also tangent to X. Now

$$d\theta\left(\xi, \eta\right) = \xi\theta\left(\eta\right) - \eta\theta\left(\xi\right) - \theta\left(\left[\xi, \eta\right]\right) = 0$$

and, since $d\theta$ is nondegenerate in W_γ for all $\gamma \in X$,

$$\dim(X) = \dim\left(T_\gamma X\right) \leq \frac{\dim\left(W_\gamma\right)}{2} = \frac{2d-2}{2} = d - 1. \square$$

Definition 94 *Let* (N, W) *be a contact manifold with* $\dim(N) = 2d - 1$ *and* $X \subseteq N$ *an integral submanifold of* W*. Then* X *is said to be a* Legendrian submanifold *iff* $\dim(X) = d - 1$.

Thus the sky of a point is a Legendrian submanifold of the manifold of light rays. In fact, it is a special kind of Legendrian submanifold:

Definition 95 *Let* (N, W) *be a contact manifold with* $\dim(N) = 2d - 1$ *and* $f : S^{d-1} \to N$ *an embedding such that* $X = f\left(S^{d-1}\right)$ *is a Legendrian submanifold; then* X *is called a* Legendrian knot.

Definition 96 *Let* (N, W) *be a contact manifold and* X, Y *Legendrian knots such that* $X \cap Y = \emptyset$*; then* $X \amalg Y$ *is called a* Legendrian link.

Thus the disjoint union of the skies of two point which do not lie in the same null geodesic is a Legendrian link.

Definition 97 *Let* (N, W) *be a contact manifold. A* contact isotopy *is a one-parameter family of contactomorphisms* $\Phi_t : N \times [0, 1] \to N$ *such that* Φ_0 *is the identity map.*

Definition 98 *Let (N,W) be a contact manifold. Two Legendrian links $X_1 \amalg X_2$ and $Y_1 \amalg Y_2$ are said to be* equivalent *if there exists a contact isotopy $\Phi_t : N \times [0,1] \to N$ such that $\Phi_1(X_i) = Y_i$ for $i = 1,2$.*

We've seen that one-parameter *groups* of contactomorphisms are generated by contact vector fields. However, the concept of Legendrian linking forces us to consider contact isotopies, i.e., one-parameter *families* of contactomorphisms through the identity map. In order to understand these we first consider one-parameter families of symplectomorphisms through the identity map.

Definition 99 *Let $(\widetilde{N},\widetilde{\omega})$ be a symplectic manifold. A* symplectic isotopy *is a one-parameter family of symplectomorphisms $\Phi_t : \widetilde{N} \times [0,1] \to \widetilde{N}$ such that Φ_0 is the identity map.*

Proposition 100 *Let $(\widetilde{N},\widetilde{\omega})$ be a symplectic manifold and $\Phi_t : \widetilde{N} \times [0,1] \to \widetilde{N}$ a symplectic isotopy. Then for each $t \in [0,1]$ the vector field*

$$\xi_t = \frac{d}{ds}\Phi_s \mid_{s=t}$$

is a locally Hamiltonian vector field.

Proof: The proof is completely analogous to the case when Φ_t is a one-parameter group of symplectomorphisms:

$$0 = \frac{d}{ds} \mid_{s=t} (\Phi_s)^* \widetilde{\omega}(\Phi_s) = \pounds_{\xi_t}\widetilde{\omega} = d(\xi_t \rfloor \widetilde{\omega}) + \xi_t \rfloor d\widetilde{\omega} = d(\xi_t \rfloor \widetilde{\omega}) . \square$$

Thus we have

Theorem 101 *Let (N, W) be a contact manifold and $\Phi_t : N \times [0,1] \to N$ a contact isotopy. Then for each $t \in [0,1]$ the vector field*

$$\xi_t = \frac{d}{ds}\Phi_s \mid_{s=t}$$

is a contact vector field.

Proof: One has but to notice that the symplectification of a contact isotopy is a symplectic isotopy generated by a homogeneous of degree one Hamiltonian.□

This is a nice characterization of contact isotopies. However, an even nicer characterization of Legendrian linking is provided by the so-called *Legendrian isotopy extension theorem:*

Theorem 102 *Let (N, W) be a contact manifold and L a (not necessarily connected) closed Legendrian submanifold of N. Let $\varphi_t : L \times [0,1] \to N$ a smooth injective map such that φ_0 is the inclusion map and $\varphi_t(L)$ is Legendrian. Then there exists a contact isotopy $\Phi_t : N \times [0,1] \to N$ such that $\Phi_t \mid_L = \varphi_t$.*

Proof: See [T].

One can think of this theorem as stating that Legendrian links are equivalent *iff* one can find smooth motions of its components taking one link into the other in such a way that the two components remain Legendrian submanifolds throughout the motion (and never intersect). This allows us to prove the following

Proposition 103 *Let (M, g) be a globally hyperbolic $(d+1)$-spacetime ($d = 2$ or 3), and N its manifold of light rays. Then the Legendrian link consisting of* *the skies of two non-causally related points is equivalent to the Legendrian link consisting of the skies of any other two non-causally related points.*

Proof: The proof is completely analogous to that of proposition 64, as moving through skies is a particular case of moving through Legendrian submanifolds.□

Again the following definition is seen to be a natural one:

Definition 104 *Let (M, g) is a globally hyperbolic $(d+1)$-spacetime ($d = 2$ or 3), and N its manifold of light rays. Then the equivalence class of the Legendrian link consisting of the skies of two non-causally related points is said to be the class of* trivial Legendrian links*; any two Legendrian knots forming a link in this class are said to be* unlinked.

Legendrian linking provides us with a more restrictive concept of linking which fits nicely with the natural contact structure of the manifold of light rays. An illustration of this restrictiveness is given by the following theorem by Lisa Traynor:

Theorem 105 *Let (M, g) be Minkowski (2+1)-spacetime and (N, W) the corresponding (contact) manifold of light rays. Let $x, y \in M$ with $y \in I^+(x)$. Then $X \amalg Y$ and $Y \amalg X$ are* not *equivalent Legendrian links.*

Proof: See [T].□

Corollary 106 *Let (M, g) be Minkowski (2+1)-spacetime and (N, W) the corresponding (contact) manifold of light rays. Let $x, y, z \in M$ with $y \in I^+(x)$ and $z \notin J^-(x) \cup J^+(x)$. Then $X \amalg Y$, $Y \amalg X$ and $X \amalg Z$ yield the different equivalence classes of Legendrian links formed by pairs of skies.*

Proof: This is an immediate consequence of theorem 105 and the fact that any two points on M not lying on the same null geodesic can be moved into x and y, y and x or x and z in such a way that they never are on the same null geodesic.□

Note that if $y \in I^+(x)$ then $X \amalg Y$ and $Y \amalg X$ are clearly equivalent links on N: one can always find coordinates on M such that on $N = \mathbb{R}^2 \times S^1$ we have

$$\begin{aligned} X &= \{(R\cos\theta, R\sin\theta, \theta) : 0 \leq \theta \leq 2\pi\} \\ Y &= \{(R\cos(\theta+\pi), R\sin(\theta+\pi), \theta) : 0 \leq \theta \leq 2\pi\} \end{aligned}$$

for some $R > 0$; thus we see that

$$\Phi_t\left(x^1, x^2, \theta\right) = \left(\cos(\pi t)\,x^1 - \sin(\pi t)\,x^2, \sin(\pi t)\,x^1 + \cos(\pi t)\,x^2, \theta\right)$$

is an isotopy which carries $X \amalg Y$ to $Y \amalg X$. Thus there are only two distinct equivalence classes of links formed by pairs of skies in Minkowski (2+1)-spacetime. This is related to the fact that in the (2+1)-dimensional case $link(X,Y) = link(Y,X)$.

It is instructive to see how the above isotopy fails to be a contact isotopy: it is generated by the vector field

$$\xi = -\pi x^2 \frac{\partial}{\partial x^1} + \pi x^1 \frac{\partial}{\partial x^2}$$

which does not preserve the contact structure:

$$\begin{aligned} & \pounds_\xi\left(\cos\theta dx^1 + \sin\theta dx^2\right) \\ = {} & \xi\rfloor\left(-\sin\theta d\theta \wedge dx^1 + \cos\theta d\theta \wedge dx^2\right) + d\left[\xi\rfloor\left(\cos\theta dx^1 + \sin\theta dx^2\right)\right] \\ = {} & \left(-\pi x^2 \sin\theta - \pi x^1\cos\theta\right) d\theta + d\left(-\pi x^2\cos\theta + \pi x^1 \sin\theta\right) \\ = {} & \pi\sin\theta dx^1 - \pi\cos\theta dx^2 \neq 0 \end{aligned}$$

Thus Legendrian linking allows us to distinguish between past and future in Minkowski (2+1)-spacetime, whereas ordinary linking does not. This hints that it could be the concept of linking we are looking for to express causal relations in the manifold of light rays for $d = 3$. Even for $d = 2$, the knowledge that skies are Legendrian submanifolds of the manifold of light rays will be quite useful, as we shall see.

CHAPTER 7

General spacetimes

7.1 Fermat's principle

So far we've dealt with static spacetimes only. Although general spacetimes are naturally more complicated, we shall see that the main points of importance to us do not suffer any significant complication. In particular, a version of Fermat's principle *still* applies in these general spacetimes.

In what follows, (M, g) is a globally hyperbolic $(d+1)$-spacetime with Cauchy surface Σ diffeomorphic to a subset of $\mathbb{R}^d$. Thus we can assume that Σ possesses global coordinates $\left\{x^{\mathbf{i}}\right\}$. As is well known, one can always choose a global time function t such that the hypersurfaces $\{dt = 0\}$ are Cauchy surfaces. To build up a coordinate system on our spacetime we choose an arbitrary timelike vector field t^a such that $t^a \nabla_a t = 1$ and use its integral lines as the coordinate lines $\left\{dx^{\mathbf{i}} = 0\right\}$. Thus these coordinate lines are timelike.

Definition 107 *A coordinate system as described above is said to be a* standard coordinate system*, and the observers whose worldlines are the integral lines of* t^a *are said to be* coordinate observers*.*

Theorem 108 *Let* O *be an event,* o *a coordinate observer,* γ *a future-directed null curve joining* O *to* o *and* $t(\gamma)$ *the time coordinate of the intersection of* γ *with* o*.*

Then the light rays joining O to o are the stationary points of $t(\gamma)$ (and one of them is the global minimum).

Proof: The proof is surprisingly simple. Recall from the Hamiltonian reduction theorem (see [A]) that the projections of the light rays on the Cauchy surface parametrized by the coordinate time are the solutions of Hamilton's equation for the (time-dependent) Hamiltonian

$$K = -p_0$$

where $p_0 = p_0\left(p_i, t, x^i\right)$ is obtained by solving the equation

$$g^{00}\left(p_0 + \frac{g^{0i}}{g^{00}}p_i\right)^2 + \left(g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}\right)p_ip_j = 0 \Leftrightarrow p_0 = \pm\left(\gamma^{ij}p_ip_j\right)^{\frac{1}{2}} - \beta^ip_i$$

Here we've set

$$\begin{aligned}\beta^i &= \frac{g^{0i}}{g^{00}} \\ \gamma^{ij} &= -\frac{1}{g_{00}}g^{ij} + \beta^i\beta^j\end{aligned}$$

and recall that

$$h^{ij} = g^{00}\gamma^{ij}$$

is the Riemannian metric induced on the Cauchy surface by orthogonally projecting on the orthogonal complement of $\frac{\partial}{\partial t}$. The two possible signs refer to past-oriented or future-oriented null geodesics, which clearly yield the same projection as their time orientation is irrelevant in the coordinate time parametrization. For convenience, and to fix ideas, let us choose the sign corresponding to past-oriented null geodesics; thus our Hamiltonian is

$$K = \left(\gamma^{ij}p_ip_j\right)^{\frac{1}{2}} + \beta^ip_i$$

and is, of course, strictly positive. At any given event the possible values of the coordinate velocities for null geodesics are

$$v^i = \frac{\partial K}{\partial p_i} = \frac{\gamma^{ij} p_j}{(\gamma^{ij} p_i p_j)^{\frac{1}{2}}} + \beta^i$$

and these are of course also the possible values of coordinate velocities for any null curve.

One way to prove the stationary property of null geodesics is to set up the following problem of optimal control theory (a so-called Mayer problem; see for instance [La]): assume that the coordinates of O are (t_0, x_0^i) and that the space coordinates of o are (x_1^i). Regard the x^i as state variables and the p_i as the control variables. It is clear that one can choose the p_i to be defined on an interval $[t_0, t_1]$ so that the x^i determined by the Cauchy problem

$$\begin{aligned} x^i(t_0) &= x_0^i \\ \frac{dx^i}{dt} &= \frac{\partial K}{\partial p_i} \end{aligned}$$

satisfy

$$x^i(t_1) = x_1^i.$$

Optimal control theory then tells you that in order to find the choices of the p_i which yield stationary values of t_1 one must set up the Hamiltonian function

$$H = \lambda_i \frac{\partial K}{\partial p_i}$$

(where the λ_i are auxiliary momentum variables) and solve the minimum conditions

$$\frac{\partial H}{\partial p_i} = 0$$

$$\begin{aligned}
\frac{d\lambda_i}{dt} &= -\frac{\partial H}{\partial x^i} \\
H(t_1) &= 1
\end{aligned}$$

(see [La]).

Now using only the facts that K is homogeneous of degree 1 in the momenta and strictly positive it is easy to see that these equations are equivalent to

$$\begin{aligned}
\lambda_i &= \alpha p_i \\
\frac{dp_i}{dt} &= -\frac{\partial K}{\partial x^i} \\
\alpha &= \frac{1}{K(t_1)}
\end{aligned}$$

(where $\alpha > 0$ is a constant; notice that it is important that the coordinate system be a standard on in order to guarantee that K is always positive and that therefore α can always be computed). In other words, the stationary points are the light rays. Since we know that $t_1 \geq t_0$ must have a global minimum, it is clear that this must be a light ray. (See [SEF] for a more direct proof).□

This is a generalization of theorem 18, and can be conceived as a Fermat's principle for the propagation of light on our Cauchy surface: a light ray will travel between two points of the Cauchy surface (i.e., two coordinate observers) using the path that allows it to do so in minimum (coordinate) time, the allowed speeds of light at time t being given at each point by

$$v^i = \frac{\gamma^{ij}p_j}{(\gamma^{ij}p_ip_j)^{\frac{1}{2}}} + \beta^i$$

which, of course, are just the solutions of

$$g_{00} + 2g_{0i}v^i + g_{ij}v^iv^j = 0.$$

Definition 109 *Let $\xi \in \Sigma$ be the point with coordinates $\xi^{\rm i}$ and let $\tau \in \mathbb{R}$. The* wavefront originating at (τ, ξ) at time t *is defined to be the set*

$$\Phi(\tau, \xi, t) = \left\{ x^{\rm i}(t) : \dot{x}^{\rm i} = \frac{\partial K}{\partial p_{\rm i}} \quad \textit{and} \quad \dot{p}_{\rm i} = -\frac{\partial H}{\partial x^{\rm i}} \quad \textit{and} \quad x^{\rm i}(\tau) = \xi^{\rm i} \right\}.$$

oriented by considering the right-handed orientation on $T_\xi\Sigma$ of the initial conditions.

Note that this wavefront is clearly the projection on Σ of the sky of (τ, ξ). Note also that if $t < \tau$ ($t > \tau$) this is the intersection of the past (future) light cone of (τ, ξ) with Σ.

Recall that if n^a is the future-pointing normal unit vector to the Cauchy surfaces and t^a a timelike vector field generating a standard coordinate system, then one can always write $t^a = Nn^a + N^a$, with $N > 0$ a smooth function and $n^a N_a = 0$ (obviously N^a is tangent to the Cauchy surface).

Definition 110 *N is called the* lapse function *and N^a the* shift vector.

Obviously one can always replace t^a by Nn^a and thus get coordinate observers whose worldlines are orthogonal to the Cauchy surfaces.

Definition 111 *A standard coordinate system whose coordinate observers' worldlines are orthogonal to the Cauchy surfaces is said to be a* synchronized *coordinate system.*

Synchronized coordinate systems are important to us because of the following

Proposition 112 *On a synchronized coordinate system the set of all light rays emanating from point at a given time in our Cauchy surface is orthogonal to each of the rays (with respect to the Riemannian metric induced in the Cauchy surface by the spacetime metric).*

Proof: Because this set is just the projection of the sky of the event consisting of the point at the instant of emission on the Cauchy surface, which is a Legendrian submanifold of the manifold of light rays, we know that

$$p_i t^i = 0$$

for any vector $t^i \frac{\partial}{\partial x^i}$ tangent to the set; but for a synchronized coordinate system

$$p_i = \left(\gamma^{ij} p_i p_j\right)^{\frac{1}{2}} \gamma_{ij} v^j.$$

Thus we get

$$\gamma_{ij} t^i v^j = 0$$

and the observation that γ_{ij} is clearly conformally related to the Riemannian metric $-g_{ij}$ induced on Σ by the spacetime metric concludes the proof.□

As we shall see, this proposition teaches us how to reconstruct a sky from the wavefront. Note that the sky itself does not depend on the particular choice of coordinate observers we happen to make, and thus we can always assume to be dealing with a synchronized coordinate system.

7.2 Wavefronts, skies and linking numbers

Proposition 113 *Let $x, y \in M$ be given by $x = (t_1, \xi)$ and $y = (t_2, \eta)$, where $t_1, t_2 \in \mathbb{R}$ and $\xi, \eta \in \Sigma$, and suppose that $t_1 < t_2$ and that x, y do not belong to the* *same null geodesic. Then the linking number of X and Y is given by the winding number of $\Phi(t_1, \xi, t_2)$ around η on Σ.*

Proof: The proof is completely analogous to the static case.□

In particular, we see that the winding number of $\Phi(t_1, \xi, t_2)$ around η is equal to $(-1)^d$ times the winding number of $\Phi(t_2, \eta, t_1)$ around ξ. This is a by no means trivial statement regarding the intersections of the light cones of x and y with Cauchy surfaces through y and x.

Theorem 114 *Let $x \in M$ be given by $x = (t_1, \xi)$, where $t_1 \in \mathbb{R}$ and $\xi \in \Sigma$. Suppose that $t_1 < t_2 \in \mathbb{R}$ and let $\Xi = t^{-1}(t_2)$. Then the set of points on Ξ causally related to x is $\{t_2\} \times \bigcup_{t_1 \leq t \leq t_2} \Phi(t_1, \xi, t)$.*

Proof: Again this can be seen to be a consequence of Huygens's principle (see [A]).□

Thus we have now generalized the tools to compute the linking number of the skies of two points, as well as to determine their causal relationship, to a general $(d+1)$-spacetime. In both cases this amounts to studying wavefronts on the Cauchy surface.

CHAPTER 8

Skies in (2+1)-spacetimes

8.1 Legendrian links

We will attempt to prove Low's conjecture. For this we will rely heavily on the fact that skies are Legendrian knots, whose projections on Σ have known kinds of singularities. We thus start by taking a closer look at Legendrian knots and links.

Let (M, g) be a globally hyperbolic (2+1)-dimensional spacetime with Cauchy surface Σ diffeomorphic to a subset of $\mathbb{R}^2$, and N its manifold of light rays. Let (t, x, y) be a synchronized coordinate system on M with (x, y) global coordinates on Σ. Recall that by identifying Σ with the level surface $t^{-1}(\tau)$ one has on Σ the Riemannian metric $h = h(\tau)$ induced by (minus) the spacetime metric for each $t \in \mathbb{R}$. This metric has the particularity that, as is easily seen in the proof of proposition 112, the velocity of the light ray determined by $(p_{\mathrm{i}}, x^{\mathrm{i}})$ is

$$v^{\mathrm{i}} = \frac{h^{\mathrm{ij}} p_{\mathrm{j}}}{(h^{\mathrm{ij}} p_{\mathrm{i}} p_{\mathrm{j}})^{\frac{1}{2}}}.$$

Since one can clearly identify N with the submanifold of $T^*\Sigma$ defined by the condition $\left(h^{\mathrm{ij}} p_{\mathrm{i}} p_{\mathrm{j}}\right)^{\frac{1}{2}} = 1$, we see that

Proposition 115 *The Riemannian metric h provides a diffeomorphism between N and $TS(\Sigma)$ carrying fibres to fibres. The sky of a point is carried by this diffeomorphism to the set of all unit normal vectors on its projection on Σ pointing in the direction of propagation of the light ray.*

Proof: This is an immediate consequence of proposition 112.□

Thus we may as well think of N as $TS(\Sigma)$ (with respect to h). To introduce coordinates in N we notice that any vector $v^{\mathrm{i}}\frac{\partial}{\partial x^{\mathrm{i}}} \in TS(\Sigma)$ is completely determined by the angle φ it forms with, say, $\frac{\partial}{\partial x}$. For such a vector, one has

$$\frac{v_x}{(h_{xx})^{\frac{1}{2}}} = \cos\varphi$$

and

$$h^{xx}(v_x)^2 + 2h^{xy}v_xv_y + h^{yy}(v_y)^2 = 1$$

yielding

$$h^{yy}(v_y)^2 + 2h^{xy}(h_{xx})^{\frac{1}{2}}\cos\varphi v_y + h^{xx}h_{xx}\cos^2\varphi - 1 = 0.$$

Remembering that

$$\left(h^{\mathrm{ij}}\right) = \left(h_{\mathrm{ij}}\right)^{-1} = \frac{1}{h}\begin{pmatrix} h_{yy} & -h_{xy} \\ -h_{xy} & h_{xx} \end{pmatrix}$$

$(h = \det(h_{\mathrm{ij}}))$, and that consequently

$$h_{xx} = hh^{yy},$$

one gets, after some algebra,

$$v_y = \frac{-h^{xy}(h_{xx})^{\frac{1}{2}}\cos\varphi \pm (h^{yy})^{\frac{1}{2}}\sin\varphi}{h^{yy}}$$

the two signs corresponding to the two possible orientations. We choose the positive one, corresponding to the plus sign.

Proposition 116 *A contact 1-form θ for N can be written in the coordinates (x, y, φ) as*

$$\theta = v_xdx + v_ydy = (h_{xx})^{\frac{1}{2}}\cos\varphi dx + \frac{(h^{yy})^{\frac{1}{2}}\sin\varphi - h^{xy}(h_{xx})^{\frac{1}{2}}\cos\varphi}{h^{yy}}dy$$

or better yet as

$$\theta = \cos\varphi dx + \left(\frac{\sin\varphi}{h^{\frac{1}{2}}h^{yy}} - \frac{h^{xy}\cos\varphi}{h^{yy}}\right) dy$$

Proof: This obviously follows from the previous remarks and the fact that a contact 1-form is always defined up to multiplication by a nonvanishing smooth function.□

It is a consequence of the generalized Riemann map theorem that one can always choose global coordinates for $\Sigma \approx \mathbb{R}^2$ in which $h_{\mathbf{ij}} = \Omega^2\delta_{\mathbf{ij}}$; in this case one obviously gets

$$\theta = \cos\varphi dx + \sin\varphi dy$$

which is the standard contact 1-form for $TS^*(\mathbb{R}^2)$. This 1-form can also be written as

$$\begin{aligned} \theta &= d(x\cos\varphi + y\sin\varphi) + (x\sin\varphi - y\cos\varphi)\, d\varphi \\ &= dq - pd\varphi \end{aligned}$$

with the new coordinates (p, q) given by the (volume-preserving) coordinate transformation

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\sin\varphi & \cos\varphi \\ \cos\varphi & \sin\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In particular we've proved the following

Proposition 117 *The 1-jet manifold $J^1(S^1)$ is a contact manifold contactomorphic to $TS^*(\mathbb{R}^2)$.*

Definition 118 *Let $\pi : N \to \Sigma$ be the natural projection map and $L \subseteq N$ a Legendrian link. Then $\Phi = \pi(L) \subseteq \Sigma$ is called a* wavefront*. If L has an orientation, given by a tangent vector v, its wavefront has the orientation given by $\pi_* v$. If L is a sky, we shall call its wavefront also a* sky.

Notice that obviously wavefronts are defined up to orientation-preserving diffeomorphisms of Σ (corresponding to our freedom to change coordinates on the Cauchy surface). We have

Theorem 119 *Wavefronts are formed by piecewise immersions of S^1 on Σ whose generic singularities are (a finite number of) cusps.*

Proof: See [EN].□

We only need to consider *generic* (or *stable*) singularities of skies as we can always move the Cauchy surface slightly so as to remove non-generic singularities.

In general, a wavefront defines several Legendrian links, corresponding to the two possible orientations of its unit normal vector field in each component; the fact that a wavefront is the projection of a Legendrian link means that either of these unit normal vector fields vary continuously along the wavefront (with respect to a parametrization coming from the Legendrian link).

Definition 120 *A* coorientation *on a wavefront is simply a choice of a unit normal vector field.*

From this point on we will tend to not distinguish between Legendrian links and cooriented wavefronts.

A wavefront is reminiscent of another kind of plane curve:

Definition 121 *A* link diagram *is formed by immersions of S^1 on $\mathbb{R}^2$ with only simple or double points allowed and an indication of over-and-under at each double point.*

A link diagram defines a link in $\mathbb{R}^3$. In figure 8.1 we show a link diagram for a link with just one component (i.e., a knot). The particular knot defined by this diagram is known as the *trefoil knot.*

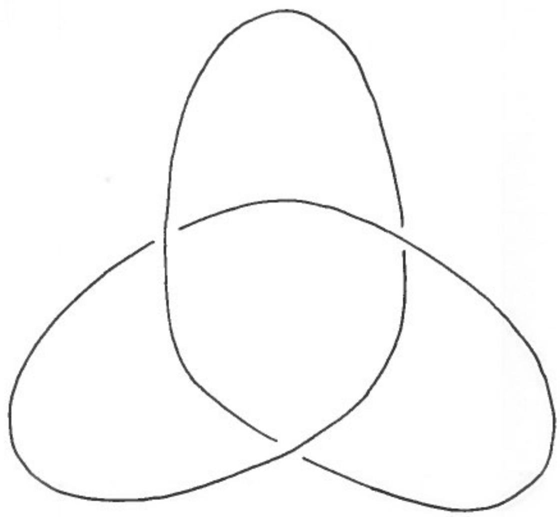

Figure 8.1: Knot diagram

Similarly, a wavefront with a coorientation defines a Legendrian link, as shown in figure 8.2 (again for a one component Legendrian link, i.e., a Legendrian knot).

To avoid unnecessary complications, assume from this point on that $\Sigma \approx \mathbb{R}^2$. It is then clear that any link diagram can be reinterpreted as a wavefront. If one chooses a coorientation, it will then define a Legendrian link $L \subseteq N$. Notice

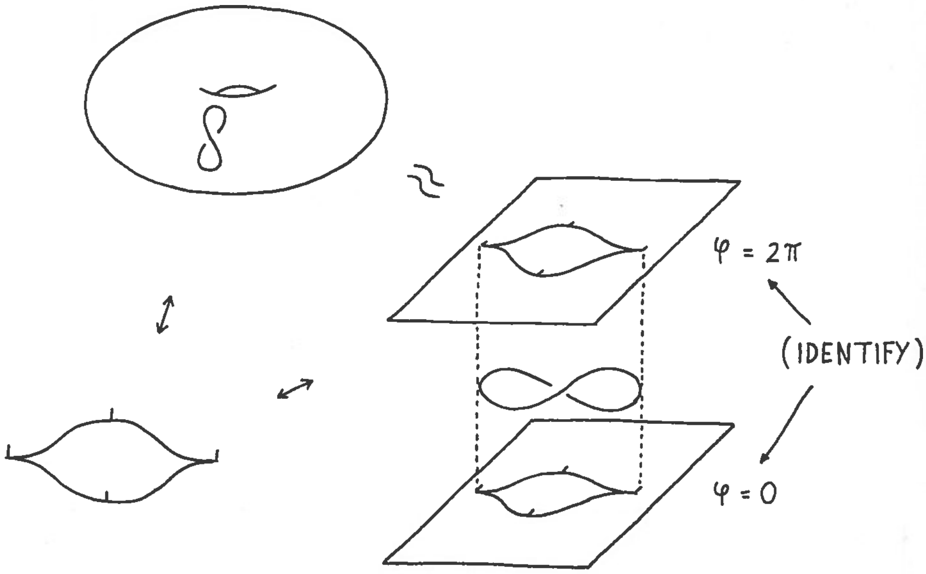

Figure 8.2: Cooriented wavefront and its associated Legendrian knot

that for knots, the two possible choices of Legendrian knots are isotopic (but not necessarily Legendrian isotopic), the isotopy being generated by $\frac{\partial}{\partial\varphi}$ (clearly the transformation $\varphi \mapsto \varphi + \pi$ carries one into the other; recall, however, that this particular isotopy is not a Legendrian isotopy, as we showed in the comments following theorem 105). Since N is diffeomorphic to the standard torus, L can be seen as a link in $\mathbb{R}^3$. However, as is clear from figure 8.3 (where one sees that the Legendrian knot corresponding to the trefoil knot diagram is really an unknot), it is *not* the link corresponding to the link diagram we started with.

With relation to this, we have the following

Theorem 122 *Any knot in* $\mathbb{R}^2 \times S^1$ *has a Legendrian representative; however, its wavefront may not be immersed in* $\Sigma \approx \mathbb{R}^2$ *(because of cusps).*

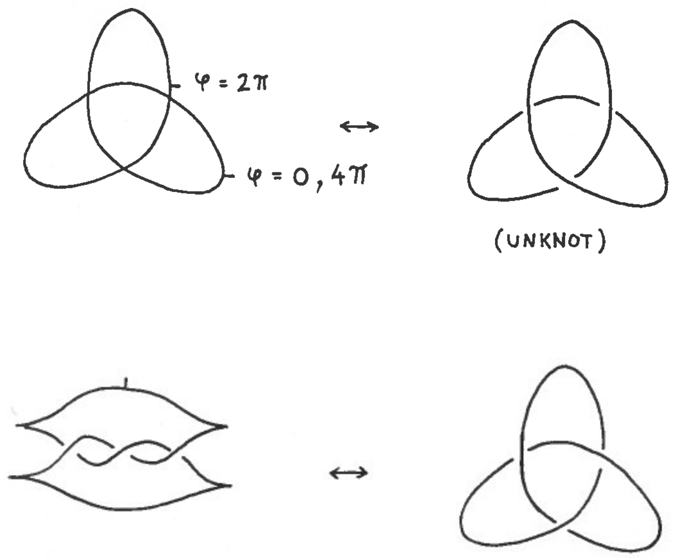

Figure 8.3: Legendrian knot and wavefront for the trefoil

Proof: See [A1].□

In figure 8.3 we show the four-cusped wavefront for the Legendrian representative of the trefoil knot.

8.2 Evolvability

When discussing linking or Legendrian linking one is interested in isotopies or Legendrian isotopies. Two links are considered equivalent if there exists an isotopy of the appropriate kind deforming one of the links into the other. Because of the isotopy extension theorems, one only has to worry about deforming the link itself (making sure it remains a link of the appropriate kind throughout the deformation). In the case of Legendrian links, one can easily see which deformations of the wavefront are allowed: these are exactly the ones which deform wavefronts into other wavefronts so that whenever there exists a tangency the coorientation of the tangent curves is opposite (this is called a *safe tangency*; a tangency where the tangent curves have the same coorientation is called a *dangerous tangency* and is obviously forbidden as it corresponds to an intersection of the corresponding Legendrians. This applies to any links, including knots; in this case all tangencies are self-tangencies).

We now wish to discuss a different relation between Legendrian links (which, unlike isotopy or Legendrian isotopy is *not* an equivalence relation). First, though, we introduce *another* equivalence relation following up on the fact wavefronts are defined up to orientation-preserving diffeomorphisms of Σ (corresponding to our freedom to change coordinates on the Cauchy surface).

Definition 123 *Let $L \subseteq N$ be a Legendrian link with wavefront $\Phi = \pi(L)$. We say that the Legendrian link $K \subseteq N$ is* equivalent *to L if there exists an orientation-preserving diffeomorphism $f : \Sigma \to \Sigma$ such that $\pi(K) = f(\Phi)$.*

By abuse of language we shall also speak of equivalent wavefronts. From this point on we shall consider all equivalent wavefronts to be basically the same wavefront (i.e., we shall deal with equivalence classes of wavefronts instead of the wavefronts themselves). That we are justified in doing so is shown in the following

Theorem 124 *If two Legendrian links are equivalent, then they are Legendrian isotopic.*

Proof: It is well known that the set of orientation-preserving diffeomorphisms of $\mathbb{R}^n$ ($n \in \mathbb{N}$) is pathwise connected (see [HM]). We can easily use a path connecting the identity map to f to generate a Legendrian isotopy carrying L to K.□

Definition 125 *Let L_1, L_2 be two Legendrian links. We say that L_1 is* evolvable *into L_2, and write $L_1 \leq L_2$, if there exists a Lorentzian metric in which (a link equivalent to) L_1 is carried to (a link equivalent to) L_2 by the Legendrian isotopy generated by the homogeneous degree one Hamiltonian corresponding to the null geodesic equations.*

Again we will also say that a wavefront Φ_1 is evolvable to a wavefront Φ_2 if the corresponding Legendrians L_1 and L_2 satisfy $L_1 \leq L_2$. We then see that a wavefront is evolvable into another if there exists a spacetime (globally hyperbolic, with Cauchy surface Σ diffeomorphic to $\mathbb{R}^2$, equipped with a synchronized coordinate system) such that light rays on Σ through the first wavefront (moving in the normal direction specified by the coorientation) form the second wavefront after a certain (non-negative) amount of coordinate time (all this modulo equivalence of wavefronts).

At this point, it is convenient to make the following

Definition 126 *The wavefront corresponding to a single point (and the corresponding Legendrian knot) is called the* canonical circle *(CC).*

The projection map $\pi : CC \subseteq N \to \Sigma$ is singular at all points. This is a good example of an unstable singularity: moving the Cauchy surface slightly (i.e., evolving the CC slightly) one removes the singularity. The wavefronts thus obtained also receive names:

Definition 127 *The wavefront corresponding to a circle with the outward coorientation (and the corresponding Legendrian knot) is called the* future canonical circle *(FCC); the wavefront corresponding to a circle with the inward coorientation (and the corresponding Legendrian knot) is called the* past canonical circle *(PCC).*

Proposition 128 *Evolvability is a partial order but not an equivalence relation, i.e., is reflexive and transitive but not symmetric.*

Proof: It is clear that one can evolve a wavefront into itself by doing so in zero coordinate time; hence evolvability is a reflexive relation. Also, it is clearly transitive. However, it is not symmetric: because of Huygens's principle (which stems directly from Fermat's principle), it should be clear that while we can evolve a PCC into a FCC, it is impossible to evolve a FCC into a PCC.□

This immediately suggests the following

Definition 129 *If* Φ *is a cooriented wavefront, we shall define* $-\Phi$ *to be the same wavefront with the opposite coorientation.*

Proposition 130 *Let* Φ *be a cooriented wavefront corresponding to a Legendrian knot,* Φ_1, Φ_2 *be general wavefronts. Then (i)* $\Phi = -\Phi$ iff $\Phi = CC$; *(ii)* $PCC = -FCC$; *(iii)* $\Phi_1 \leq \Phi_2$ iff $-\Phi_2 \leq -\Phi_1$.

Proof: (i) Clearly the canonical circle is the only wavefront corresponding to a Legendrian knot which does not require a coorientation; (ii) obvious; (iii) to evolve $-\Phi_2$ into $-\Phi_1$ simply consider the spacetime used to evolve Φ_1 into Φ_2 but with the time direction reversed.□

Definition 131 *A* past sky *is any cooriented wavefront* Φ *satifying* $\Phi \leq CC$. *A* future sky *is any cooriented wavefront* Φ *satifying* $CC \leq \Phi$. *A* sky *is either a past or a future sky.*

We would like to be able to characterize skies. Clearly any sky is Legendrian isotopic (and hence isotopic) to a CC, the evolution providing the Legendrian isotopy; however, not all wavefronts corresponding to Legendrian knots Legendrian isotopic to the CC are skies. To see this, we start by making the following

Definition 132 *Let* Φ *be a given a wavefront. The set* $E(\Phi)$ *defined as the unbounded pathwise connected component of* $\Sigma \backslash \Phi$ *is called the* absolute exterior *of* Φ, *and* $B(\Phi) = \partial E(\Phi)$ *is called the* outer boundary *of* Φ.

Proposition 133 *Let* Φ *be a sky; then the coorientation of* Φ *must point either towards or away from* $E(\Phi)$ *along the outer boundary (depending on wether* Φ *is a future or past sky).*

Proof: Suppose Φ is a future sky; then Φ is the intersection of the future light cone of some point $x \in M$ with a Cauchy surface Σ. It should be clear that $B(\Phi) \subseteq \Phi \subseteq J^+(x)$ and that $E(\Phi) \cap J^+(x) = \emptyset$. Thus $B(\Phi) \subseteq J^+(x) \backslash I^+(x)$, and all null geodesics through $B(\Phi)$ must be moving towards $E(\Phi)$, i.e., the coorientation must point towards $E(\Phi)$. If Φ is a past sky, then $-\Phi$ is a future sky and hence the opposite coorientation to that of Φ must point towards $E(\Phi)$.□

Proposition 133 is enough to show that the wavefront in figure 8.4 is *not* a sky (though it is clearly Legendrian isotopic to the CC).

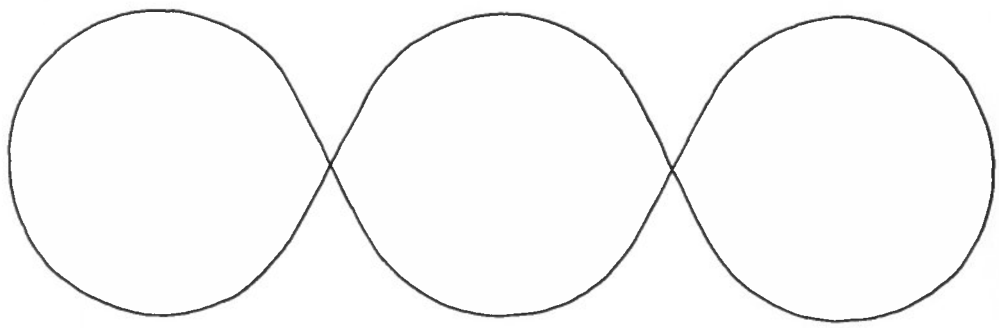

Figure 8.4: Wavefront Legendrian isotopic to the CC but not a sky

If Φ is, say, a future sky, there will exist a region $I(\Phi)$ which is the intersection of the causal future of some spacetime point with the Cauchy surface Σ. Clearly $B(\Phi) \subseteq \partial I(\Phi) \subseteq \Phi$, and the same argument used in the proof of proposition 133 implies that the coorientation must point away from $I(\Phi)$ along $\partial I(\Phi)$.

Definition 134 *We shall call the above condition the* P condition.

Given an arbitrary wavefront, it is not clear how one should identify $I(\Phi)$ (if it is a sky). Proposition 133 gives us a necessary (but not sufficient) condition for the P condition to hold, but the P condition itself is in general difficult to check. It is none-the-less a condition which all skies must satisfy. Similar comments (with obvious changes) apply when Φ is a past sky.

8.3 Reidemeister moves

The Reidemeister moves are 3 kinds of local changes in ordinary link diagrams (usually referred to as Ω_1, Ω_2 and Ω_3 - see [PS]), not allowed by plane isotopies, which preserve the corresponding links. They are listed in figure 8.5.

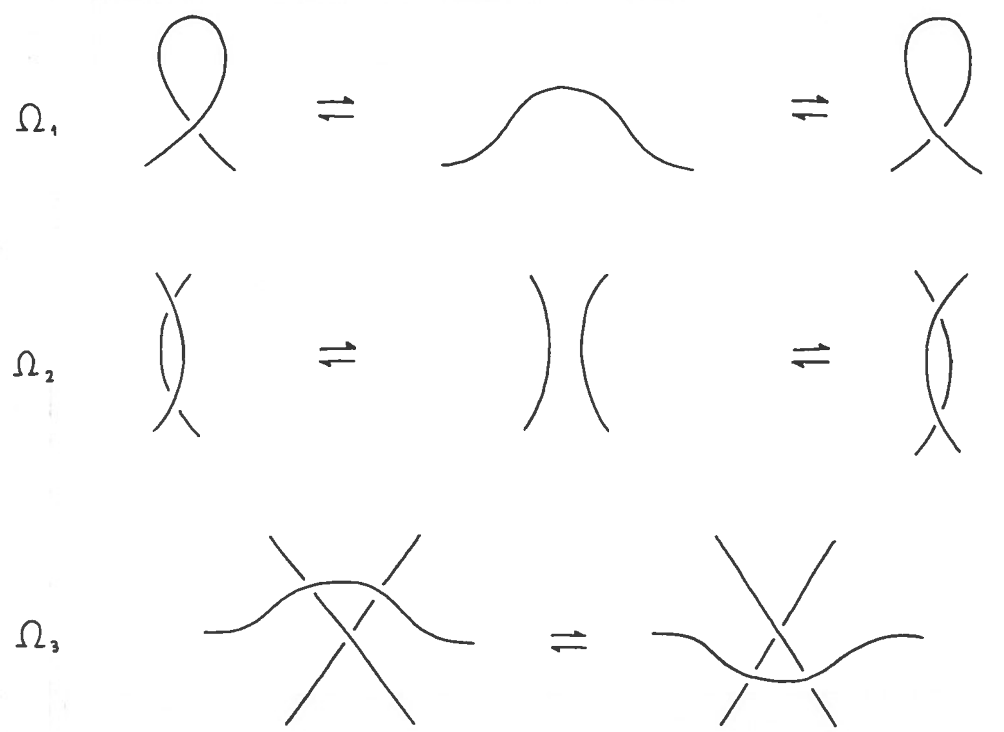

Figure 8.5: Reidemeister moves

As is well known we have the following

Theorem 135 *Two link diagrams correspond to isotopic links* iff *one can be obtained from the other by a finite sequence of Reidemeister moves and plane isotopies.*

Proof: See, for instance, [PS].□

Interestingly, the same theorem holds true for Legendrian links, as long as we redefine the allowed Reidemeister moves of the wavefronts (see [CG]): in fact, we must take into account that not only are the allowed deformations of Legendrian links more restrictive, but also wavefronts may have cusps. Hence we have the following changes:

Ω_1: Here two cusps must appear in order to keep the winding number constant (see next section);

Ω_2: Here dangerous self-tangencies are obviously forbidden;

Ω_3: These are basically the same as the corresponding ordinary moves;

Ω_4: This new set of moves arises due to the existence of cusps; it can be seen as a limiting case of Ω_2.

The Legendrian Reidemeister moves are listed in figure 8.6.

Of these moves, only some actually occur when one evolves a wavefront, those which we've signaled in the figure with an 'E'. About the allowed Ω_1, Ω_4 moves in evolution, it is useful to bear in mind the following

Proposition 136 *In a wavefront being evolved, cusps move at the speed of light (in the direction specified by the coorientation).*

Proof: Let φ be the parameter describing the initial Legendrian knot. As one evolves the wavefront, one therefore has a set of functions $x^{\mathbf{i}} = x^{\mathbf{i}}(t, \varphi)$, with t being the coordinate time of the spacetime where we're carrying out the evolution and each value of φ yielding a different light ray. Whenever a cusp develops it is

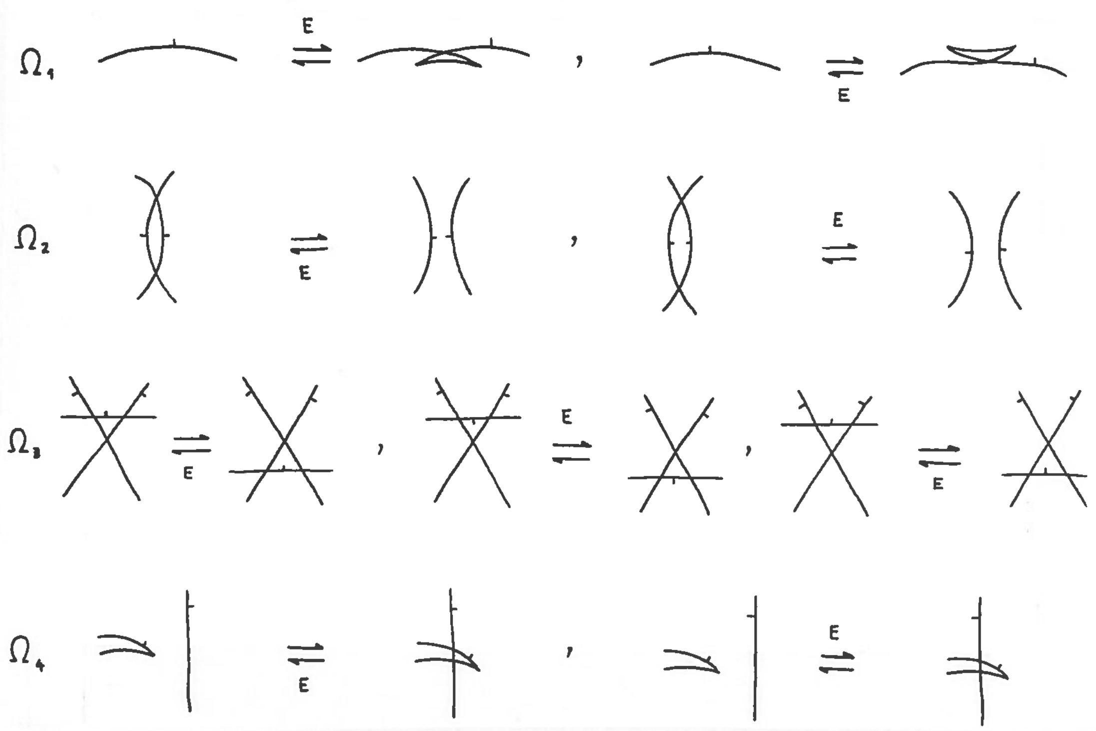

Figure 8.6: Legendrian Reidemeister moves

characterized by the condition

$$\frac{\partial x^{\mathrm{i}}}{\partial \varphi}(t, \varphi) = 0 \tag{8.1}$$

which locally yields $\varphi = \varphi(t)$. The velocity of the cusp across Σ is then given by

$$\begin{aligned} \frac{d}{dt}\left[x^{\mathrm{i}}(t, \varphi(t))\right] &= \frac{\partial x^{\mathrm{i}}}{\partial t}(t, \varphi(t)) + \frac{\partial x^{\mathrm{i}}}{\partial \varphi}(t, \varphi(t))\frac{d\varphi}{dt} \\ &= \frac{\partial x^{\mathrm{i}}}{\partial t}(t, \varphi(t)) \end{aligned}$$

(where we've used 8.1), which is exactly the velocity of the light ray through the cusp at that instant.□

It is initially a bit puzzling to realize that although the velocity of the cusp is exactly the same as the velocity of the light ray through it, it is *not* a light ray: it has only contact of order 1 with the light ray (i.e., it has only the same velocity, not the same acceleration); the light ray through the cusp keeps changing. This, of course, is only possible due to 8.1.

Proposition 136 combined with Huygens's principle and the fact that an evolving wavefront always moves in the direction defined by its coorientation easily explain why some of the Legendrian Reidemeister moves do not occur in the course of evolution.

8.4 Legendrian invariants of Legendrian knots

Given that skies are Legendrian knots, it is important to consider Legendrian invariants of knots in order to identify which Legendrian knots are skies.

An obvious invariant can be obtained by counting the number of times the coorienting normal vector winds around itself as one transverses the knot (see [A1]). Notice that in order for this to work it is important to have an orientation defined on the knot (recall that for skies this is the usual orientation defined by the right-handed orientation of the circle of initial velocities on the Cauchy surface containing the point). More formally, let $L \subseteq N$ be an oriented Legendrian knot. Then L defines an equivalence class $[L] \in \pi_1(N)$. Since N is diffeomorphic to the standard torus, $\pi_1(N) \approx \mathbb{Z}$. Clearly $[CC]$ is a generator of $\pi_1(N)$. Thus there exists an integer $i = i(L)$ such that $[L] = i\,[CC]$, which for topological reasons alone must be a Legendrian invariant.

Definition 137 *The invariant $i(L)$ is called the* winding number *of L.*

Notice that $i(CC) = 1$.

A not so obvious invariant can be obtained by counting the cusps of the knot's wavefront with appropriate signs.

Definition 138 *A cusp is called* positive *if the inner product of the orienting and coorienting vectors is positive in a neighbourhood of the cusp (at which the orienting vector vanishes); cusp is called* negative *if the inner product of the orienting and coorienting vectors is negative in a neighbourhood of the cusp (see figure 8.7).*

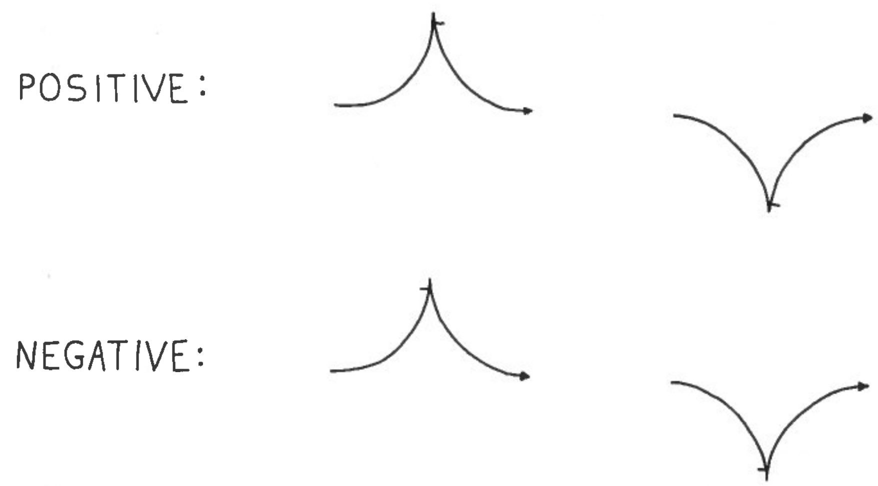

Figure 8.7: Positive and negative cusps

Notice that the inner product of the orienting and coorienting vectors in a neighbourhood of a cusp is just the value of the contact 1-form on the knot's orienting vector in a neighbourhood of the cusp's inverse image under the natural projection (see [A1]). Let $\mu_+(L)$ be the number of positive cusps of L's wavefront, and $\mu_-(L)$ the number of negative cusps.

Definition 139 *The invariant* $\mu(L) = \mu_+(L) - \mu_-(L)$ *is called the* Maslov index *of the oriented Legendrian knot* L.

It is easy to see that the Maslov index is indeed an invariant: the only Legendrian Reidemeister move in which cusps are created or destroyed is Ω_1, and clearly the cusps occurring in this move are of opposite signs. Also, note that each cusp rotates the coorienting vector by π. Since it must rotate $2i(L)\pi$, it follows that the total number of cusps $\mu_+(L) + \mu_-(L)$ must be even; thus, so must $\mu(L) = \mu_+(L) + \mu_-(L) - 2\mu_-(L)$.

Notice that $\mu(CC) = \mu(FCC) = 0$.

Definition 140 *A dangerous self-tangency is said to be* positive *if the orientation of both tangent branches coincides, and* negative *otherwise. A* positive crossing *occurs when after a* positive *dangerous self-tangency the number of double points* increases *or after a* negative *dangerous self-tangency the number of double points* decreases. *Obviously, a* negative crossing *occurs when the inverse happens.*

Definition 141 *The* standard wavefronts $\Phi_{j,k}$ *are the wavefronts with winding number* $\pm j$ *and Maslov index* $\pm k$ *(depending on the choice of orientation an coorientation) described in figure 8.8 for* $j, k \in \mathbb{N}_0$.

Definition 142 *The so-called* Arnold J^+ invariant *is defined as follows: (i) For all* $i, k \in \mathbb{N}_0$, $J^+(\Phi_{0,k}) = -k$, $J^+(\Phi_{i+1,k}) = -2i - k$; *(ii)* J^+ *increases by 2 in any positive crossing (and thus decreases by 2 in any negative crossing).*

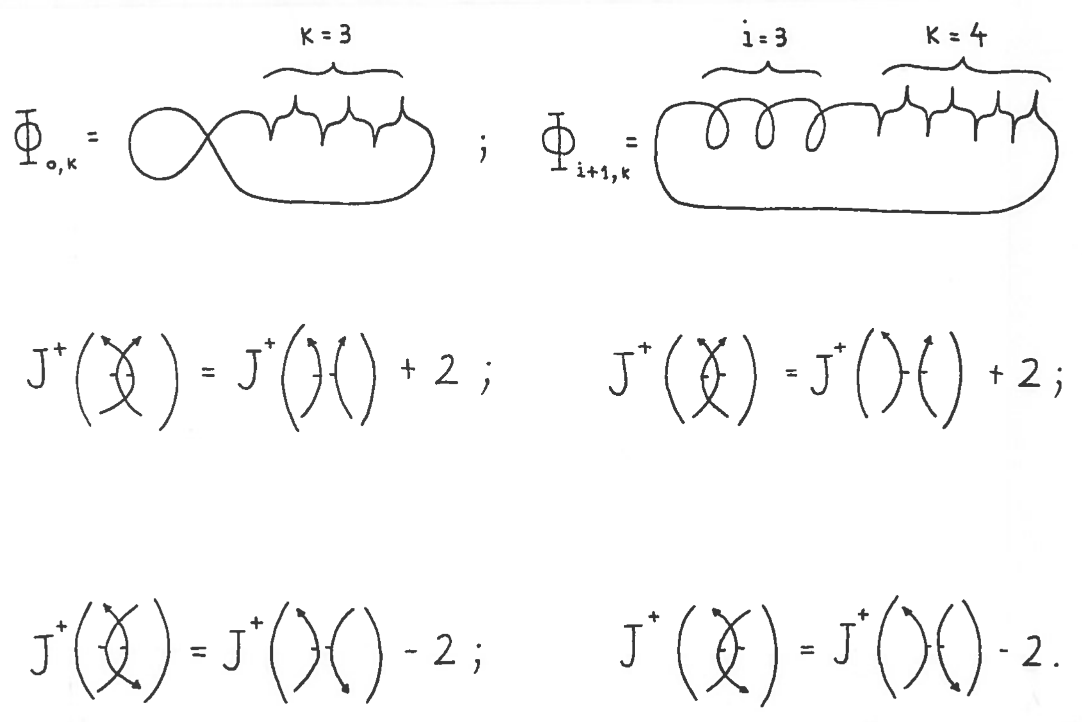

Figure 8.8: Standard wavefronts and Arnold's invariant

Theorem 143 *The Arnold J^+ invariant is a Legendrian invariant of Legendrian knots which does not depend either on the orientation or on the coorientation of the corresponding wavefront.*

Proof: See [A1].□

Thus $J^+(CC) = J^+(FCC) = 0$.

Definition 144 *If Φ is an oriented cooriented wavefront, $\overline{\Phi}$ is the same wavefront with the same coorientation but opposite orientation.*

Proposition 145 *If Φ is an oriented cooriented wavefront, then (i) $i(-\Phi) = i(\Phi)$; (ii) $i(\overline{\Phi}) = -i(\Phi)$; (iii) $\mu(-\Phi) = -\mu(\Phi)$; (iv) $\mu(\overline{\Phi}) = -\mu(\Phi)$; (v) $J^+(-\Phi) = J^+(\Phi)$; (vi) $J^+(\overline{\Phi}) = J^+(\Phi)$.*

Proof: All of these are trivial consequences of the definitions.□

Definition 146 *If Φ_1, Φ_2 are oriented cooriented wavefronts, we define their* connected sum $\Phi_1 \# \Phi_2$ *in the same way as is done for ordinary knot diagrams, making sure the orientations and coorientations coincide at the junctions (see figure 8.9).*

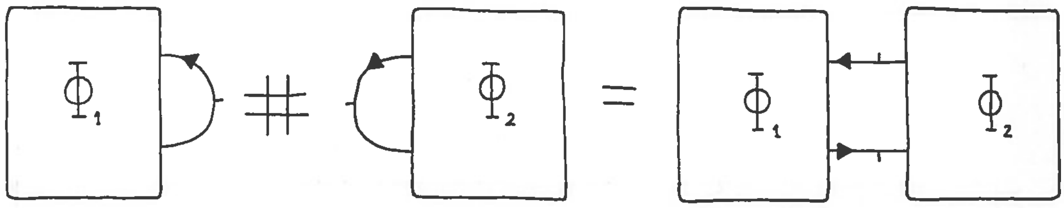

Figure 8.9: Connected sum of wavefronts

Theorem 147 *Let Φ_1, Φ_2 be oriented cooriented wavefronts; then (i) $i(\Phi_1 \# \Phi_2) = i(\Phi_1) + i(\Phi_2) - 1$; (ii) $\mu(\Phi_1 \# \Phi_2) = \mu(\Phi_1) + \mu(\Phi_2)$; (iii) $J^+(\Phi_1 \# \Phi_2) = J^+(\Phi_1) + J^+(\Phi_2)$.*

Proof: (i) and (ii) are obvious; for the proof of (iii), see [A1].□

In addition to these integer invariants, it is also possible to define polynomial invariants, namely the *Kauffman polynomial K* and the *HOMFLY polynomial P*. These are actually defined for any Legendrian link, and their definitions are quite similar, the main difference being that the Kauffman polynomial does not depend on the wavefront's orientation, whereas the HOMFLY polynomial does.

Definition 148 *The wavefront* Z_i *(*$i \in \mathbb{Z} \backslash \{0\}$*) is the wavefront with* $2|i| - 2$ *cusps and winding number* $i \neq 0$ *shown in figure 8.10. (Its winding number depends on its orientation).*

Definition 149 *The* Kauffman polynomial *is the polynomial in the variables* $x, y^{\pm 1}, z_1, z_2, \ldots$ *defined by the following rules: (i) If* Φ_1*,* Φ_2 *are wavefronts corresponding to unlinked Legendrian knots,* $K(\Phi_1 \amalg \Phi_2) = K(\Phi_1) K(\Phi_2)$*; (ii)* $K(Z_i) = z_{|i|}$*; (iii) The local rules shown in figure 8.10.*

Thus $K(CC) = z_1$.

Definition 150 *The* HOMFLY polynomial *is the polynomial in the variables* $x, y^{\pm 1}, z_{\pm 1}, z_{\pm 2},$ *defined by the following rules: (i) If* Φ_1*,* Φ_2 *are wavefronts corresponding to unlinked Legendrian knots,* $P(\Phi_1 \amalg \Phi_2) = P(\Phi_1) P(\Phi_2)$*; (ii)* $P(Z_i) = z_i$*; (iii) The local rules shown in figure 8.10.*

Hence $P(CC) = z_1$.

It can be proved that these polynomials are well-defined and are Legendrian invariants (see [CG]). It is not known whether they suffice to distinguish all Legendrian knots, but they certainly don't distinguish all Legendrian links, as we shall see.

Thus we see that a sky must satisfy $(i, \mu, J^+) = (1, 0, 0)$ plus the P condition. In figure 8.11 we present a list of examples of wavefronts showing that all these

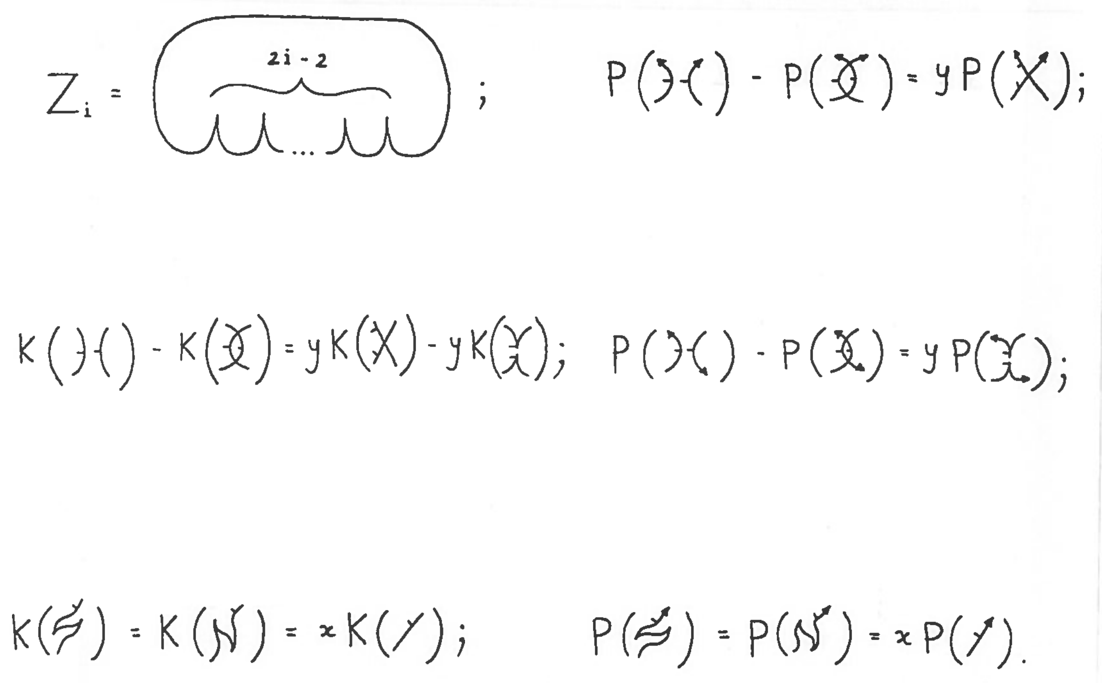

Figure 8.10: Definition of the Kauffman and the HOMFLY polynomials

conditions are independent. In each of them we have indicated which condition is not satisfied (if any).

However, these invariants plus the P condition are not enough to completely characterize a sky: in figure 8.12 we show a wavefront with all the right invariants satisfying the P condition which is *not* a sky (as we will show in theorem 152). Worse still, the invariants are not enough to characterize wavefronts which are Legendrian isotopic to skies: in the same figure 8.12 we present a wavefront Φ satisfying $i(\Phi) = 1$, $\mu(\Phi) = 0$, $J^+(\Phi) = 0$ but whose Kauffman and HOMFLY polynomials are respectively

$$K(\Phi) = z_1\left(x^2 + x^2y^2\right) + z_3\left(xy - y^2\right) - yz_1z_2$$

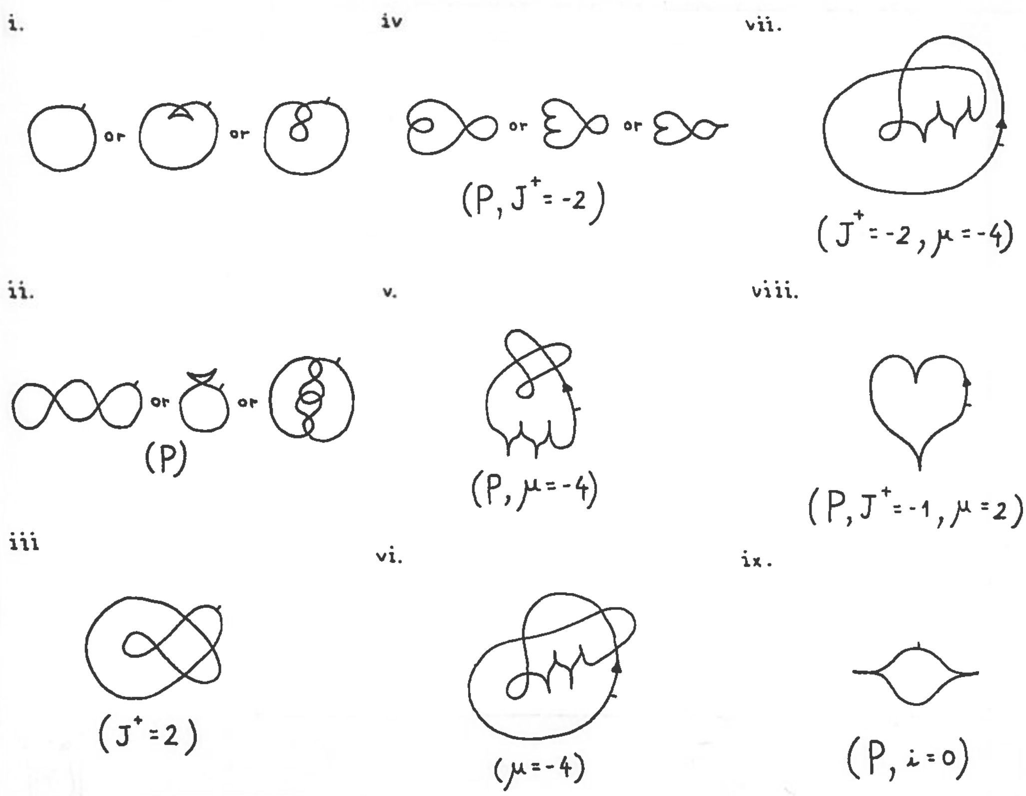

Figure 8.11: Examples of wavefronts

and

$$P(\Phi) = z_1 \left(x^2 + x^2 y^2\right) + y z_1 z_2$$

instead of z_1 (and therefore is *not* Legendrian isotopic to a sky).

A better way to proceed is then to analyze carefully the formation of skies.

8.5 Formation of skies

As we have seen previously, the problem of deciding whether a given wavefront is a sky is highly nontrivial. Instead of dwelling on this problem, let us assume that we are given a sky, i.e., a wavefront which has evolved from the CC in some Lorentzian metric (here we consider only future skies without any loss of generality). Let us then consider in detail how this happens. Aside from moving

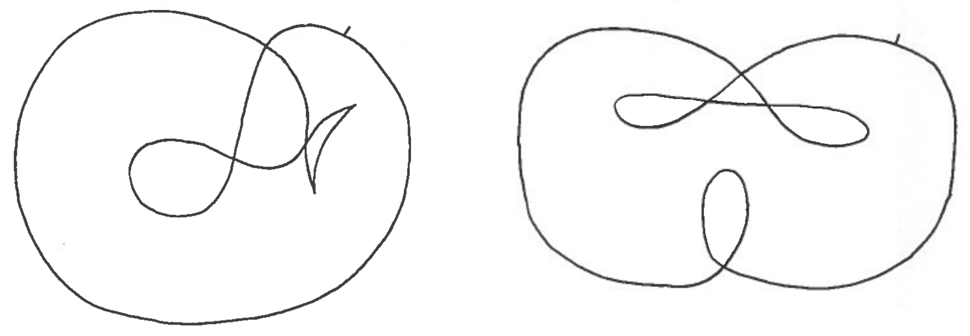

Figure 8.12: Counterexamples

(eventually crossing itself), the only other change the evolving wavefront can go through is the Legendrian Reidemeister move Ω_1, of which evolution allows only two kinds (indicated in figure 8.6), corresponding either to the creation or the destruction of two cusps.

Definition 151 *The segment of wavefront occurring in the Legendrian Reidemeister move* Ω_1 *allowed by evolution which creates two cusps will be called a* left twist*; the segment in the move which destroys two cusps will be called a* right twist.

The reason for these names is shown in figure 8.13: these segments of wavefront are the projections of left or right twists of the corresponding Legendrian knots.

We emphasize the point that only two of the four possible Legendrian Reidemeister moves Ω_1 actually occur through evolution in the following

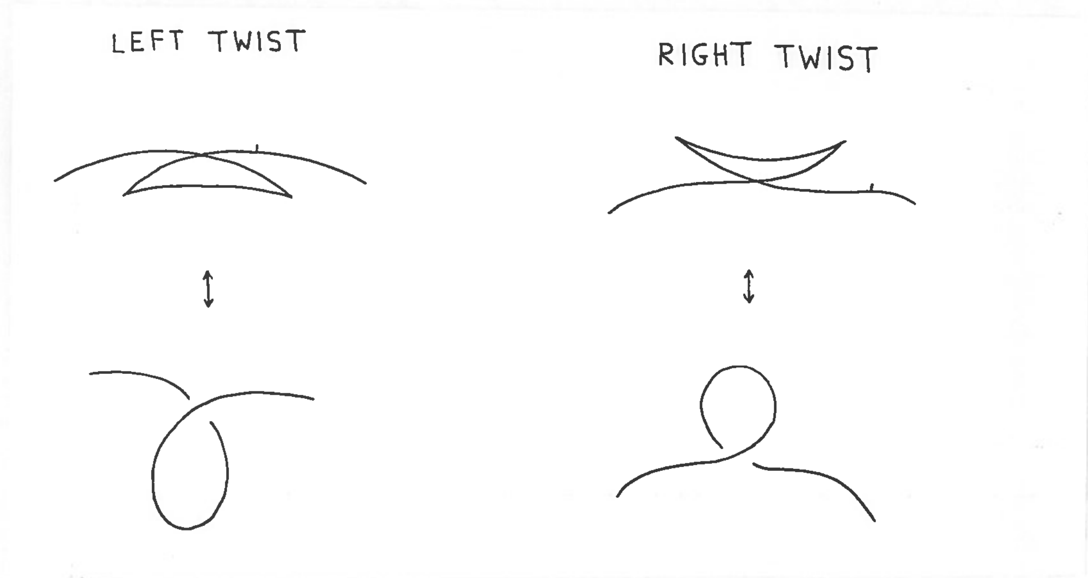

Figure 8.13: Left and right twists

Theorem 152 *The only Legendrian Reidemeister moves Ω_1 occurring in evolution are creation of left twists and destruction of right twists.*

Proof: Huygens's principle clearly implies that right twists cannot be created on the portions of a future sky which are part of the boundary of the causal future of the point corresponding to the sky. Because the creation of a right twist is a local issue, we can always assume that the segment of wavefront where it is created is actually part of such a boundary. Thus evolution cannot create right twists. Because a right twist is minus a left twist, this is equivalent to saying that evolution cannot destroy left twists.□

A generic future sky is a circle (the outer boundary) with a finite number of double points. Each double point is the projection of two points in the sky, which split the sky into two subsets, one of them containing the outer boundary.

Definition 153 *We shall call the subset of a future sky arising from a double point in the outer boundary in this way and not containing the outer boundary a* pendant.

The two basic mechanisms for forming pendants are overlaps (e.g., as in Schwarzschild (2+1)-spacetime) and growing cusps (e.g., the optical (2+1)-spacetime corresponding to a thickening sphere). Clearly pendants are never in the absolute exterior of the sky. Whenever a cusp forms, both sides of it are in the causal future of the spacetime point; whenever an overlap forms a region not in the causal future will appear inside the outer boundary. Obviously one can use the same reasoning to divide a pendant into subpendants; in that respect a pendant is like a tree. See figure 8.14 for an example.

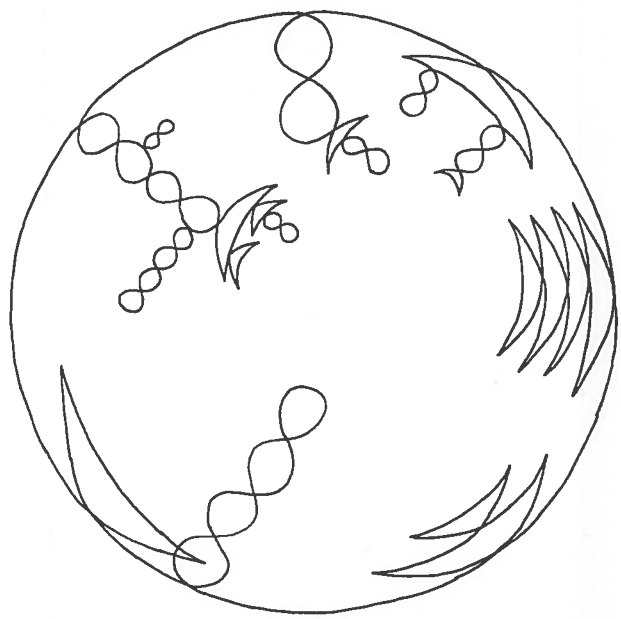

Figure 8.14: Example of a sky as a tree of pendants

CHAPTER 9

Results for (2+1)-spacetimes

9.1 Low's conjecture revisited

Recall that Low's conjecture states that if (M, g) is a globally hyperbolic (2+1)-spacetime with Cauchy surface diffeomorphic to a subset of $\mathbb{R}^2$, and N is its manifold of light rays, then two spacetime points x and y are causally related in M *iff* their skies X and Y either intersect or are linked in N.

We already know that if X and Y either intersect or are linked in N then x and y must be causally related. Suppose that x and y are causally related and that their skies do not intersect. Then one has, say, $t(x) < t(y)$. Let Σ be a Cauchy surface through y so that X projects to Σ as a future sky (which we will also call X) and Y as a point $\{\eta\} = CC$. Then η lies inside the outer boundary of X. If it does not lie in any pendant, the winding number of X around η is 1, and X and Y are linked. Suppose that η lies in one or more pendants, and consider how such pendants were formed. If all pendants arose from the wavefront crossing itself, then it is easy to see that the winding number of X around η is n, where n is the number of times that η was hit by the wavefront evolving from a CC into X (and thus by hypothesis $n \geq 1$). Thus X and Y can only be unlinked if some of the pendants were generated by creating left twists or destroying right twists. Notice that both these operations involve left twisting; never in the formation of any pendant is one allowed to do right twisting. It is because of this that one suspects that Low's conjecture must be true.

9.2 Results using the Kauffman polynomial

Unfortunately, it is not easy to turn this insight into the relationship between left twisting and Low's conjecture into a proof of the latter. However it can be used to prove Low's conjecture for a large class of examples.

Definition 154 *Let* $n_{11},...,n_{1k_1},n_{21},...,n_{2k_2},...,n_{m1},...,n_{mk_m} \in \mathbb{N}$. *Let* $x, y \in M$ *with* $y \in I^+(x)$. *Then* (X, Y) *is said to be a* pair of skies of type

$$\begin{pmatrix} n_{11} & n_{21} & \dots & n_{m1} \\ \dots & \dots & \dots & \dots \\ n_{1k_1} & n_{2k_2} & \dots & n_{mk_m} \end{pmatrix}$$

if $X \amalg Y$ *admits a link diagram as shown if figure 9.1 (when embedded in* $\mathbb{R}^3$ *through the standard diffeomorphism).*

It is easy to see how such skies may form, each double point of X corresponding to two cusps in its projection on a Cauchy surface Σ through y. Thus for instance the pair of skies with linking number 0 we considered in the optical (2+1)-dimensional spacetime containing two thickening spheres is a pair of skies of type (2).

We now prove that such skies are always linked. In order to do this we must recall the following

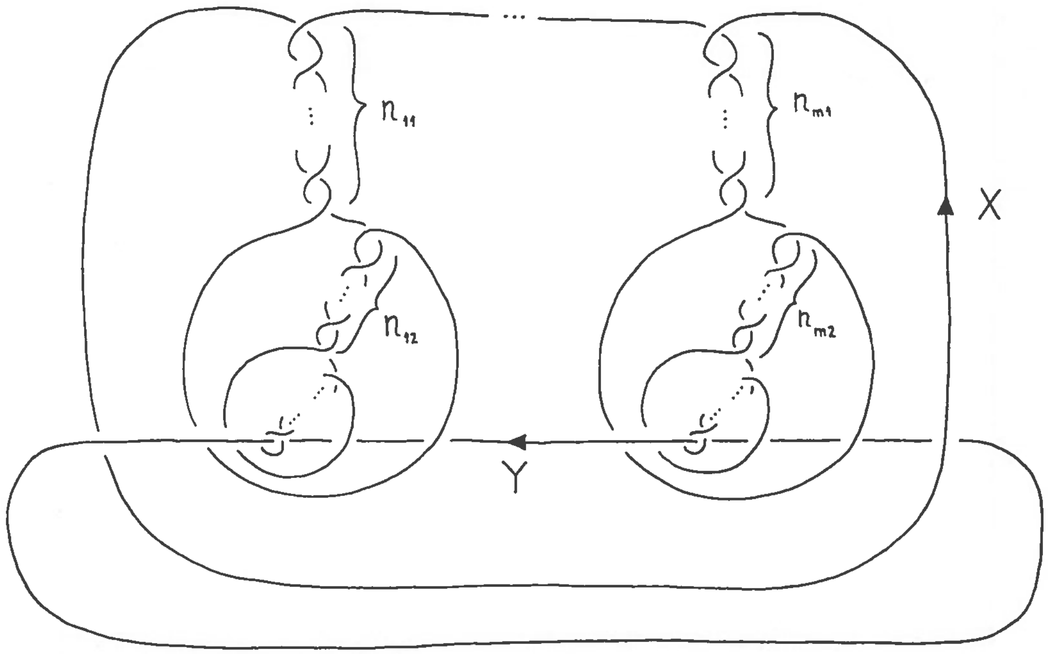

Figure 9.1: Skies of a given type

Definition 155 *The* bracket polynomial $<\Lambda>$ *of a link diagram* Λ *is the polynomial in the variables* $a^{\pm 1}$ *defined by the rules shown in figure 9.2.*

Notice that the bracket polynomial does not depend on the orientations of the link, a fact that we will use to our advantage. The bracket polynomial is *not* a link invariant, but can be used to build a link invariant. In order to do so we must define

Definition 156 *A crossing in a link diagram is said to be* positive *or* negative *according to whether the branch going rightward goes over or under the branch going leftward (right and left being defined with respect to the orientations of the branches; see figure 9.3).*

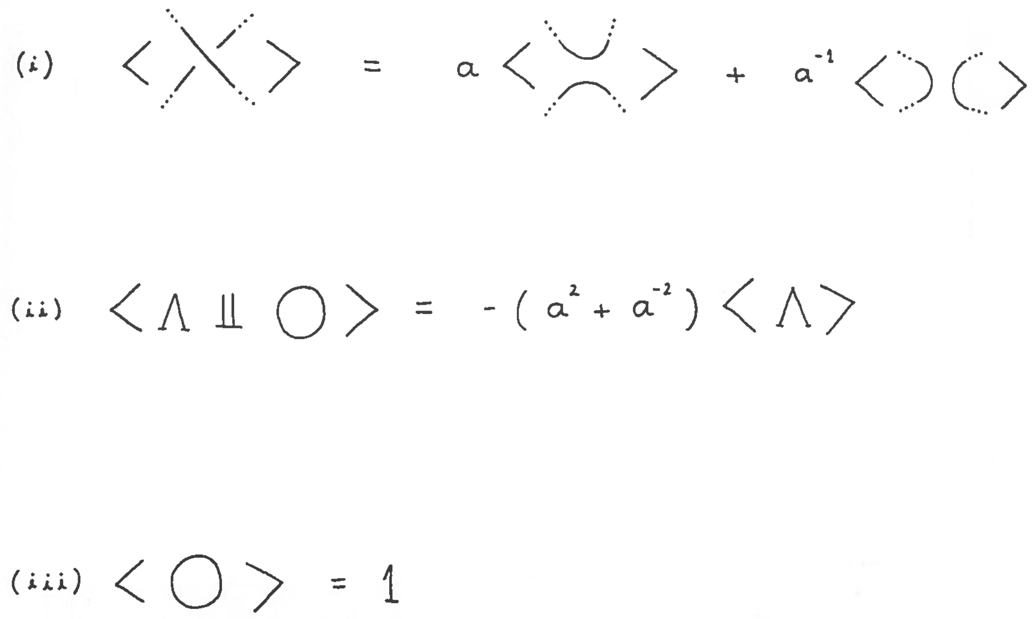

Figure 9.2: Definition of the bracket polynomial

Definition 157 *The* writhe number $w(\Lambda)$ *of a link diagram* Λ *is the sum of the signs of all crossings (where the sign of a crossing is* ± 1 *according to whether it is positive or negative).*

Definition 158 *The* Kauffman polynomial *of a link* L *is*

$$K(L) = (-a)^{-3w(\Lambda)} < \Lambda >$$

where Λ *is any link diagram for* L.

Theorem 159 *The Kauffman polynomial is a link invariant.*

Proof: See [PS].

The Kauffman polynomial is related to the Jones polynomial in one variable through the variable change $a = q^{-\frac{1}{4}}$.

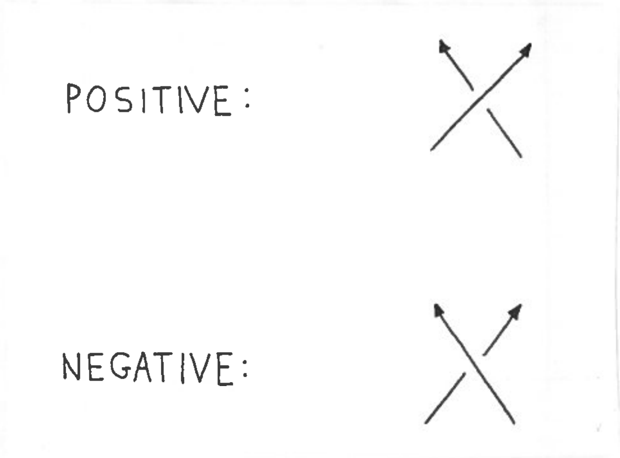

Figure 9.3: Positive and negative crossings

To prove that skies of the type we are considering are always linked we compute certain invariants obtained from the Kauffman polynomial. In order to do so we shall need the following

Theorem 160 *Let Λ_n, Λ_0, Λ_{-1} be as in figure 9.4. Then*

$$\langle \Lambda_n \rangle = a^n \langle \Lambda_{-1} \rangle - \left(a^4 + a^{-4}\right) \sum_{i=1}^{n} (-1)^{-3(n-i)} a^{-3n+4i-2} \langle \Lambda_0 \rangle .$$

Proof: We prove this result by induction. For $n = 1$ all there is to prove is

$$\langle \Lambda_1 \rangle = a \langle \Lambda_{-1} \rangle - \left(a^4 + a^{-4}\right) a^{-1} \langle \Lambda_0 \rangle .$$

This is a simple application of the rules defining the bracket polynomial and is done in figure 9.5.

As for the inductive step, our formula yields

$$\langle \Lambda_n \rangle \quad = \quad a a^{n-1} \langle \Lambda_{-1} \rangle - \left(a^4 + a^{-4}\right) \sum_{i=2}^{n} (-1)^{-3(n-i)} a^{-3n+4i-2} \langle \Lambda_0 \rangle$$

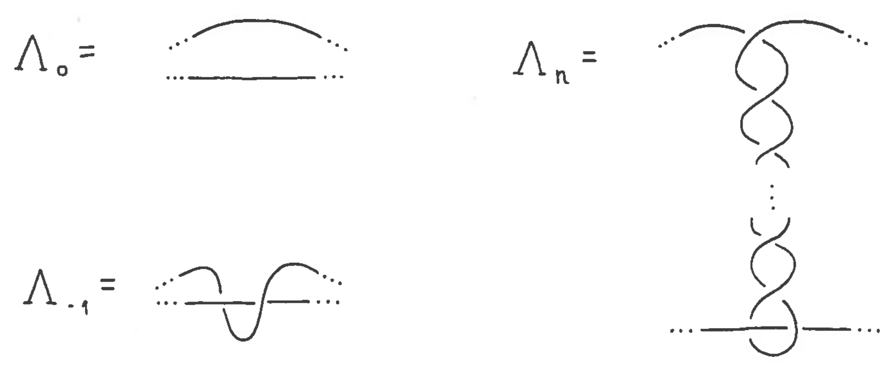

Figure 9.4: Basic link diagrams

$$
\begin{aligned}
& -\left(a^4+a^{-4}\right)(-1)^{-3(n-1)}a^{-3n+2}\langle\Lambda_0\rangle \\
= \;& a\left[a^{n-1}\langle\Lambda_{-1}\rangle-\left(a^4+a^{-4}\right)\sum_{j=1}^{n-1}(-1)^{-3(n-1-j)}a^{-3(n-1)+4j-2}\langle\Lambda_0\rangle\right] \\
& -a^{-1}\left(a^4+a^{-4}\right)(-a)^{-3(n-1)}\langle\Lambda_0\rangle \\
= \;& a\langle\Lambda_{n-1}\rangle-a^{-1}\left(a^4+a^{-4}\right)(-a)^{-3(n-1)}\langle\Lambda_0\rangle
\end{aligned}
$$

and hence all that must be proved is

$$\langle\Lambda_n\rangle = a\langle\Lambda_{n-1}\rangle-a^{-1}\left(a^4+a^{-4}\right)(-a)^{-3(n-1)}\langle\Lambda_0\rangle.$$

This can be done much as the $n=1$ case; we do so in figure 9.6. The last step in the proof is justified in lemma 161.□

Lemma 161 *Let K_m be the link diagram shown in figure 9.7. Then* $\langle K_m\rangle = (-a)^{-3m}\langle K_0\rangle$.

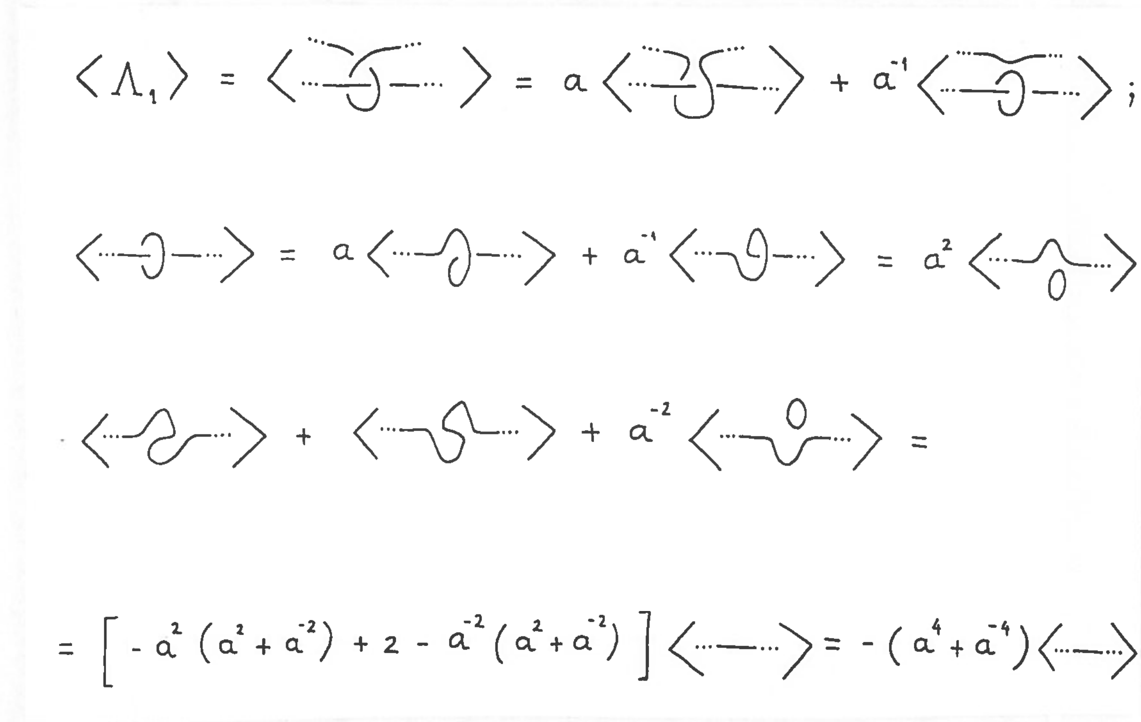

Figure 9.5: Proof for $n = 1$

Proof: Choose any orientation on K_m (and hence on K_0). Clearly $w(K_m) = w(K_0) - m$. Since K_m and K_0 are link diagrams for the same link (going from one to the other involves only Reidemeister moves Ω_1), they must have the same Kauffman polynomial, and hence

$$\begin{aligned} K(K_m) &= K(K_0) \Leftrightarrow (-a)^{-3w(K_m)} \langle K_m \rangle = (-a)^{-3w(K_0)} \langle K_0 \rangle \\ &\Leftrightarrow (-a)^{-3w(K_0)+3m} \langle K_m \rangle = (-a)^{-3w(K_0)} \langle K_0 \rangle \\ &\Leftrightarrow \langle K_m \rangle = (-a)^{-3m} \langle K_0 \rangle . \square \end{aligned}$$

Notice that it is in this lemma that the left twisting comes into the proof.

We will be only interested in the terms of higher and lower order of bracket polynomials. It is therefore useful to keep in mind the slightly simpler (if less

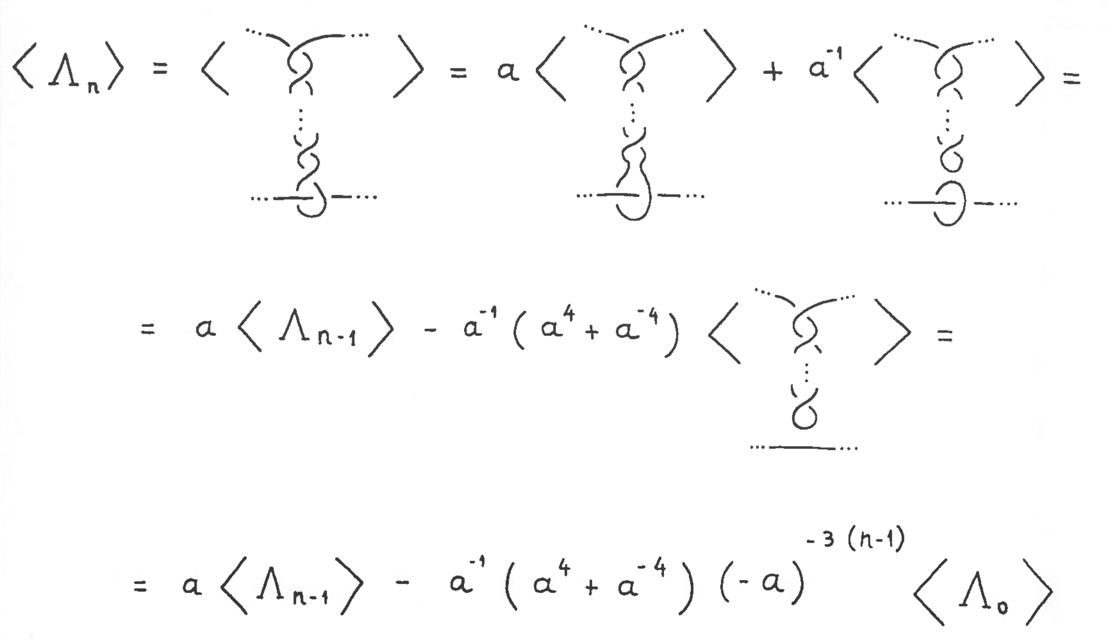

Figure 9.6: Proof for the inductive step

accurate) formula given in theorem 160:

$$\langle \Lambda_n \rangle = a^n \langle \Lambda_{-1} \rangle \pm a^{n+2} \langle \Lambda_0 \rangle \pm ... \pm a^{-3n-2} \langle \Lambda_0 \rangle \tag{9.1}$$

Definition 162 *Let $n_{11}, ..., n_{1k_1}, n_{21}, ..., n_{2k_2}, ..., n_{m1}, ..., n_{mk_m} \in \mathbb{N}$. We shall denote by*

$$\begin{pmatrix} n_{11} & n_{21} & \cdots & n_{m1} \\ \cdots & \cdots & \cdots & \cdots \\ n_{1k_1} & n_{2k_2} & \cdots & n_{mk_m} \end{pmatrix}$$

the standard link diagram for the link of such type (depicted in figure 9.1). Also, we shall also allow n_{ik_i} to assume the values 0,-1 with the meaning described in figure 9.4.

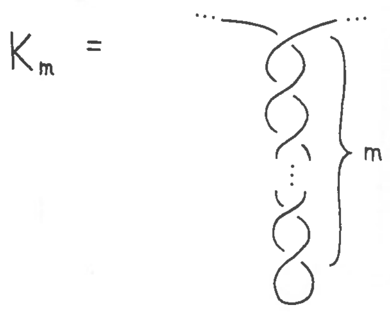

Figure 9.7: Definition of K_m

Theorem 163 *Let* $n_{11}, ..., n_{1k_1}, n_{21}, ..., n_{2k_2}, ..., n_{m1}, ..., n_{mk_m} \in \mathbb{N}$. *Then*

$$\left\langle \begin{pmatrix} n_{11} & n_{21} & \dots & n_{m1} \\ \dots & \dots & \dots & \dots \\ n_{1k_1} & n_{2k_2} & \dots & n_{mk_m} \end{pmatrix} \right\rangle = \pm a^{N+2k+4} \pm \dots \pm a^{-3N-2k-4}$$

where

$$N = \sum_{i=1}^{m} \sum_{j=1}^{k_i} n_{ij}$$

is the total number of double points and

$$k = \sum_{i=1}^{m} k_i$$

is the total number of subpendants of the knot diagram of X.

Proof: It should be clear that (0) is a link diagram for the Hopf link and (-1) is a link diagram for the unlink, and hence

$$\langle (0) \rangle = -\left(a^4 + a^{-4}\right);$$

$$\langle(-1)\rangle = -\left(a^2 + a^{-2}\right).$$

Consequently formula 9.1 yields

$$\begin{aligned}\langle(n)\rangle &= a^n \langle(-1)\rangle \pm a^{n+2} \langle(0)\rangle \pm ... \pm a^{-3n-2} \langle(0)\rangle \\ &= a^{n+6} \pm ... \pm a^{-3n-6}.\end{aligned}$$

and the theorem's conclusion holds in this case.

Next, we notice that

$$\begin{pmatrix} n_1 \\ 0 \end{pmatrix} = (n_1) \Rightarrow \left\langle \begin{pmatrix} n_1 \\ 0 \end{pmatrix} \right\rangle = a^{n_1+6} \pm ... \pm a^{-3n_1-6}$$

and that lemma 161 yields

$$\begin{aligned}\left\langle \begin{pmatrix} n_1 \\ -1 \end{pmatrix} \right\rangle &= (-a)^{-3n_1} \langle(0)\rangle \\ &= -(-a)^{-3n_1}\left(a^4 + a^{-4}\right) \\ &= \pm a^{-3n_1+4} \pm a^{-3n_1-4}.\end{aligned}$$

Consequently, one can ignore the corresponding term in formula 9.1,

$$\left\langle \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \right\rangle = \pm a^{n_1+n_2+8} \pm ... \pm a^{-3n_1-3n_2-8}.$$

and again the theorem's conclusion holds. In general, one has

$$\begin{pmatrix} n_1 \\ ... \\ n_{k-1} \\ 0 \end{pmatrix} = \begin{pmatrix} n_1 \\ ... \\ n_{k-1} \end{pmatrix}$$

and consequently

$$\left\langle \begin{pmatrix} n_1 \\ \ldots \\ n_{k-1} \\ 0 \end{pmatrix} \right\rangle = \pm a^{n_1+\ldots+n_{k-1}+2k+2} \pm \ldots \pm a^{-3n_1-\ldots-3n_{k-1}-2k-2}.$$

From lemma 161, on the other hand, one gets

$$\begin{aligned} \left\langle \begin{pmatrix} n_1 \\ \ldots \\ n_{k-1} \\ -1 \end{pmatrix} \right\rangle &= (-a)^{-3n_{k-1}} \left\langle \begin{pmatrix} n_1 \\ \ldots \\ n_{k-2} \end{pmatrix} \right\rangle \\ &= \pm a^{n_1+\ldots+n_{k-2}-3n_{k-1}+2k} \pm \ldots \pm a^{-3n_1-\ldots-3n_{k-1}-2k} \end{aligned}$$

and again one can ignore the corresponding term in formula 9.1, thus getting

$$\left\langle \begin{pmatrix} n_1 \\ \ldots \\ n_{k-1} \end{pmatrix} \right\rangle = \pm a^{n_1+\ldots+n_k+2k+4} \pm \ldots \pm a^{-3n_1-\ldots-3n_k-2k-4}.$$

This proves the theorem for $m = 1$. For $m > 1$ the proof easily follows (by using the exact same reasoning) from the observation that

$$\begin{pmatrix} n_{11} & \ldots & n_{m1} & 0 \\ \ldots & \ldots & \ldots & \\ n_{1k_1} & \ldots & n_{mk_m} & \end{pmatrix} = \begin{pmatrix} n_{11} & \ldots & n_{m1} & -1 \\ \ldots & \ldots & \ldots & \\ n_{1k_1} & \ldots & n_{mk_m} & \end{pmatrix} = \begin{pmatrix} n_{11} & \ldots & n_{m1} \\ \ldots & \ldots & \ldots \\ n_{1k_1} & \ldots & n_{mk_m} \end{pmatrix}$$

and that consequently one can go on ignoring the first term in formula 9.1.□

Theorem 164 *Let* $n_{11}, ..., n_{1k_1}, n_{21}, ..., n_{2k_2}, ..., n_{m1}, ..., n_{mk_m} \in \mathbb{N}$, *and* (X, Y) *be of type*

$$\begin{pmatrix} n_{11} & n_{21} & ... & n_{m1} \\ ... & ... & ... & ... \\ n_{1k_1} & n_{2k_2} & ... & n_{mk_m} \end{pmatrix}.$$

Then

$$K(X \amalg Y) = \pm a^{4N+2k-6l+4} \pm ... \pm a^{-2k-6l-4}$$

where

$$l = link(X, Y).$$

Proof: We just have to compute the writhe number of the standard link diagram for the link $X \amalg Y$ (depicted in figure 9.1). There are two kinds of crossings in this link diagram: those involving only the knot diagram of X (N of them, all corresponding to negative crossings) and those involving the two knot diagrams. It is a well known fact that the sum of the signs of the latter crossings is equal to twice the linking number of X and Y. Consequently, the writhe number is

$$w = -N + 2l$$

and hence

$$\begin{aligned} K(X \amalg Y) &= (-a)^{-3w} \left\langle \begin{pmatrix} n_{11} & n_{21} & ... & n_{m1} \\ ... & ... & ... & ... \\ n_{1k_1} & n_{2k_2} & ... & n_{mk_m} \end{pmatrix} \right\rangle \\ &= (-a)^{3N-6l} \left(\pm a^{N+2k+4} \pm ... \pm a^{-3N-2k-4} \right) \\ &= \pm a^{4N+2k-6l+4} \pm ... \pm a^{-2k-6l-4}. \square \end{aligned}$$

Corollary 165 *Let* $n_{11}, ..., n_{1k_1}, n_{21}, ..., n_{2k_2}, ..., n_{m1}, ..., n_{mk_m} \in \mathbb{N}$, *and* (X, Y) *be of type*

$$\begin{pmatrix} n_{11} & n_{21} & \dots & n_{m1} \\ \dots & \dots & \dots & \dots \\ n_{1k_1} & n_{2k_2} & \dots & n_{mk_m} \end{pmatrix}.$$

Then X *and* Y *are linked.*

Proof: If $l \neq 0$ then obviously X and Y are linked. If $l = 0$ then

$$K(X \amalg Y) = \pm a^{4N+2k+4} \pm \dots \pm a^{-2k-4} \neq -a^2 - a^{-2}$$

and consequently $X \amalg Y$ is not the unlink.□

Since both $link(X, Y)$ and the exponents of the terms of higher and lower order of $K(X \amalg Y)$ are clearly invariants of $X \amalg Y$, one can in fact conclude that

Corollary 166 *Let* $n_{11}, ..., n_{1k_1}, n_{21}, ..., n_{2k_2}, ..., n_{m1}, ..., n_{mk_m} \in \mathbb{N}$, *and* (X, Y) *be of type*

$$\begin{pmatrix} n_{11} & n_{21} & \dots & n_{m1} \\ \dots & \dots & \dots & \dots \\ n_{1k_1} & n_{2k_2} & \dots & n_{mk_m} \end{pmatrix}.$$

Then

$$N = \sum_{i=1}^{m} \sum_{j=1}^{k_i} n_{ij}$$

and

$$k = \sum_{i=1}^{m} k_i$$

are invariants of $X \amalg Y$.

Thus not only are all skies of the type we have considered linked, but also pairs of skies with different values of N and k form non-equivalent links. See figure 9.8 for some examples with zero linking number.

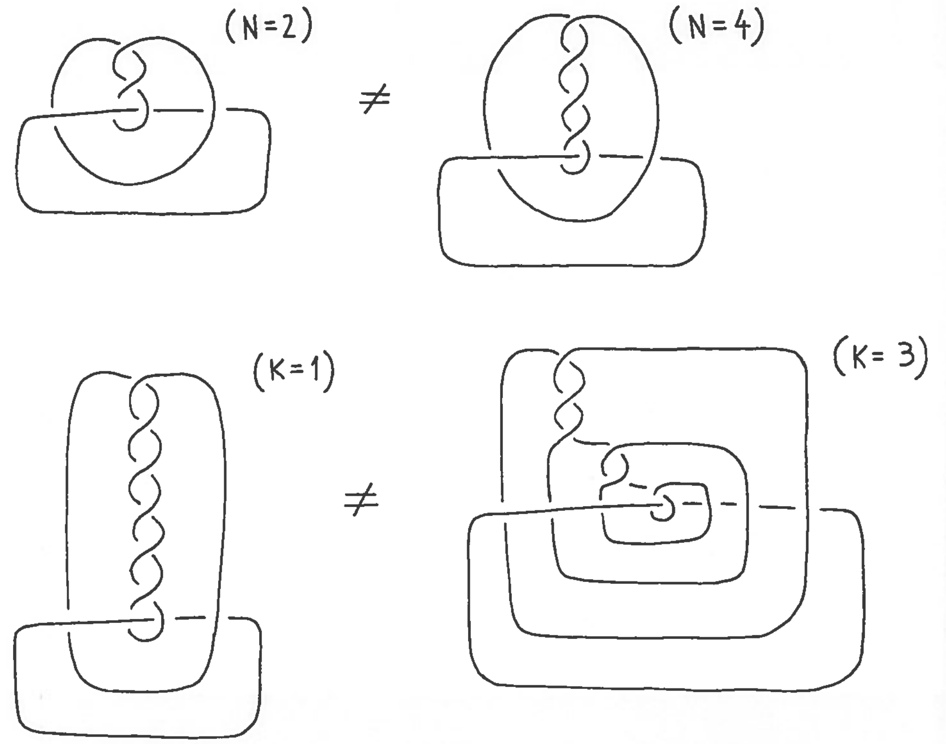

Figure 9.8: Pairs of skies with zero linking number forming non-equivalent links

This method can be successfully employed in some generalizations of the skies we have considered. For instance, it can handle multiple subpendants and a pendant overlapping itself. Unfortunately, it is hard to see how it can handle *generic* skies.

As we said, the bracket polynomial (and hence the Kauffman polynomial when calculated through it) is especially amenable to the kind of computations we have to do, because it is local and orientation-independent. Using the usual definitions of the Kauffman (or other) polynomial in terms of skein relations turns out to be less fruitful, as one must continuously worry about related knots and orientations. See [L] for such computations using the Conway polynomial.

9.3 Results for static spacetimes

Let (M, g) be a globally hyperbolic (2+1)-dimensional spacetime with manifold of light rays N and $x \in M$. Part of the problem one encounters when trying to prove Low's conjecture is how to identify the intersection of the causal future of x, say, with a Cauchy surface Σ. As with so many other properties of skies, there doesn't seem to be any way to do this just by examining X (or its projection on Σ, i.e., the intersection of the null cone of x with Σ); one has to take into account the formation of the sky. In the static case and subject to a single simplifying assumption it is then possible to make progress, as we shall see in theorem **??**.

Let $\left\{t, x^{\mathbf{i}}\right\}$ be a standard coordinate system such that $t(x) = 0$ and $\Sigma = t^{-1}(0)$. Let Σ_τ be the (Cauchy) level surface $\{t = \tau\}$, and let us employ our standard trick of using the integral lines of $\frac{\partial}{\partial t}$ to identify all such surfaces (hence obtaining a projection function $p : M \to \Sigma$). If Φ_τ is the intersection of the null cone of x with Σ_τ, define $\Psi_\tau = p(\Phi_\tau)$. Each curve $\Psi_\tau \subseteq \Sigma$ defines (and can be thought of as) a (Legendrian) curve on the manifold of light rays.

Definition 167 *The* causality surface *of x with respect to this standard coordinate system is the set*

$$\mathcal{C}(x) = \bigcup_{0 \le \tau \le 1} \Psi_\tau \subseteq N$$

Notice that this surface is really a (in general self-intersecting) band, whose boundary are two closed curves. One of them is $\Psi_0 = p(x)$, which is an integral line of $\frac{\partial}{\partial\varphi}$; the other one is $\Psi_1 = X$. It is interesting to see that this surface is much like a blow-up of the portion of the future null cone of x to the past of Σ. Indeed, any null geodesic through x defines a point in each of the curves Ψ_τ and hence defines a curve on $\mathcal{C}(x)$. This curve is simply the lift to N of the projection of the null geodesic on Σ.

The importance of the causality surface lies in the following

Proposition 168 *Let $y \in \Sigma$; then $y \in J^+(x)$ iff $Y \cap \mathcal{C}(x) \neq \emptyset$.*

Proof: This is an immediate consequence of our definitions and the fact that we used a standard coordinate system.□

Thus the causality surface provides us with a geometric structure on N which allows us to tell whether points on Σ are causally related to x. However, this is not without its problems: first of all, it depends quite strongly on the standard coordinate system used to construct it (although the conclusion of proposition 168 is itself coordinate-independent); and secondly it is in general a complicated surface, with self-intersections (which generally depend on the choice of standard coordinates).

Interestingly, both these problems are avoided when one considers static spacetimes only. Indeed, such spacetimes clearly have a canonical system of standard coordinates, namely the one in which $\frac{\partial}{\partial t}$ is the timelike Killing vector field.

Definition 169 *If (M, g) is a static (2+1)-dimensional spacetime, then the causality surfaces constructed using this canonical standard coordinate system will be* *called* canonical causality surfaces.

These canonical causality surfaces turn out to have quite nice properties:

Proposition 170 *Canonical causality surfaces can only self-intersect along spatially closed null geodesics.*

Proof: It is easily seen that a self-intersection of any causality surface $\mathcal{C}(x)$ arises from two null geodesics through x whose projections on Σ are tangent at some point. But for static spacetimes in the canonical standard coordinate system, the projections of null geodesics on Σ are geodesics of the light metric on Σ, and hence projections of different null geodesics cannot be tangent. Notice however that the projection of the *same* null geodesic can be tangent to itself at different points, corresponding to a closed geodesic of the light metric. These are the only kind of allowed self-intersections of canonical causality surfaces.□

For instance, in the Schwarzschild static (2+1)-dimensional spacetimes the canonical causality surfaces of points in the surface $\{r = 3M\}$ all develop self-intersections if sufficiently extended. In fact, only the canonical causality surfaces of these points do develop self-intersections, as they are the only points on a spatially closed null geodesic.

Definition 171 *A static (2+1)-dimensional spacetime such that its light metric admits only a finite number of closed geodesics will be called a* regular static spacetime.

Proposition 172 *Let (M, g) be a regular static (2+1)-dimensional spacetime, and $x, y \in M$ such that $y \in J^+(x)$ and $X \cap Y = \emptyset$. Then if the canonical causality surface $\mathcal{C}(x)$ has self-intersections there exists a point $z \in M$ such that $y \in J^+(z)$, $Z \cap Y = \emptyset$, $Z \amalg Y$ is equivalent to $X \amalg Y$ and $\mathcal{C}(z)$ has no self-intersections.*

Proof: Notice that any closed geodesic of the light metric on Σ is necessarily compact. Since only a finite number of such geodesics exist, it is clear that for any $\varepsilon > 0$ there will exist points on $B_\varepsilon(p(x))$ which do not belong to any such geodesic. Thus we can always choose a point z with (say) $t(z) = t(x)$ and such that $p(z) \in B_\varepsilon(p(x))$ does not belong to any closed geodesic of the light metric. Clearly for $\varepsilon > 0$ sufficiently small z satisfies the requirements listed above, and since $p(z)$ does not belong to any closed geodesic of the light metric $\mathcal{C}(z)$ has no self-intersections.□

Thus we shall assume from this point on that canonical causality surfaces do not self-intersect. In other words we are restricting ourselves to regular static (2+1)-dimensional spacetimes. Specifically, for the remainder of this section, (M, g) will be a regular static (2+1)-dimensional spacetime, $x, y \in M$ will be such that such that $y \in J^+(x)$ and $X \cap Y = \emptyset$ and the canonical causality surface $\mathcal{C}(x)$ has no self-intersections.

Recall that

$$\mathcal{C}(x) = \bigcup_{0 \le \tau \le 1} \Psi_\tau$$

where the curves Ψ_τ are lifts to N of curves on Σ which in general self-intersect and contain cusps. It is perhaps puzzling then that a canonical causality surface manages not to develop self-intersections. To see how this happens in a simple case consider an ellipse with the coorientation pointing inwards and evolve it in Minkowski spacetime. As is well known, the resulting curves at a given evolution time will develop self-intersections and cusps; however, because each light ray corresponds to a single direction (which remains constant throughout evolution), these will never meet on N and the union of the lifts to N of all these curves will never intersect.

Because $\mathcal{C}(x)$ is ruled by the lifts of geodesics of the light metric, we can parametrize it through $g : [0, 2\pi) \times [0, 1] \to N$, $g(\varphi_0, t)$ meaning the endpoint of the lift of the geodesic of the light metric with initial conditions $(p(x), \varphi_0)$ and evolved for a time t. Indeed, if $\pi : N \to \Sigma$ is the canonical projection, then $\pi_* g_* \frac{\partial}{\partial t}$ is the (unit) tangent vector to the corresponding geodesic of the light metric, and hence $g_* \frac{\partial}{\partial t}$ cannot vanish; on the other hand, $\pi_* g_* \frac{\partial}{\partial \varphi_0}$ is clearly a Jacobi field and although it *can* vanish at conjugate points $g_* \frac{\partial}{\partial \varphi_0}$ certainly cannot, as this would correspond to infinitesimally close geodesics of the light metric becoming tangent (i.e., to the *derivative* of the Jacobi field vanishing at a zero of the Jacobi field). Since $g_* \frac{\partial}{\partial t}$ and $g_* \frac{\partial}{\partial \varphi_0}$ are clearly globally defined, one has

Proposition 173 *$\mathcal{C}(x)$ is an embedded oriented submanifold of N with boundary.*

Consequently $\mathcal{C}(x)$ is two-sided. Notice that there exists a neighbourhood of Ψ_0 on $\mathcal{C}(x)$ such that $\frac{\partial}{\partial \varphi}$ is only tangent to $\mathcal{C}(x)$ on that neighborhood at Ψ_0.

Definition 174 *The side of $\mathcal{C}(x)$ through which the flow of $\frac{\partial}{\partial \varphi}$ is positive in that neighbourhood is said to be the* positive side *of $\mathcal{C}(x)$. The other side of $\mathcal{C}(x)$ said* *to be the* negative side *of* $\mathcal{C}(x)$.

Remember that because $y \in J^+(x)$, Y intersects $\mathcal{C}(x)$. It is easy to see that Y can only be tangent to $\mathcal{C}(x)$ at a conjugate point of the geodesic of the light metric through that intersection, and one can easily exclude that case by moving y slightly on Σ. Because Y has an orientation, one can then decide at each intersection whether Y crosses $\mathcal{C}(x)$ from the negative to the positive side or vice-versa.

Definition 175 *An intersection of* Y *and* $\mathcal{C}(x)$ *is said to be* positive *if* Y *crosses* $\mathcal{C}(x)$ *from the negative to the positive side, and* negative *if* Y *crosses* $\mathcal{C}(x)$ *from the positive to the negative side.*

Recall that we can always assume that $N \subseteq \mathbb{R}^3$ by using the standard diffeomorphism.

Proposition 176 *The sum of the signs of the crossings of* $\mathcal{C}(x)$ *by* Y *equals* $link(X,Y)$.

Proof: This is obvious from the fact that $\mathcal{C}(x)$ can easily be turned into a (self-intersecting) surface with boundary X by adding disk with boundary Ψ_0 which does not intersect Y.□

We want to prove that X and Y are linked. If $link(X,Y) \neq 0$ we know this to be true, so we assume from this point on that $link(X,Y) = 0$. It follows from proposition 176 that Y intersects $\mathcal{C}(x)$ in $2n$ points ($n \in \mathbb{N}$), n of the crossings being positive and n negative. Each of these crossings is connected to Ψ_0 by a (lift of a) geodesic.

Definition 177 *The* positive/negative push-off *of a (non-self-intersecting) curve on $\mathcal{C}(x)$ is the curve obtained moving each point of the curve a small distance transversely to $\mathcal{C}(x)$ in the positive/negative side.*

We now deform $X \amalg Y$ as follows: first we deform Y into a curve $\widetilde{Y}$ obtained deleting the intersections of Y with a neighbourhood of the crossings, joining it with the positive and negative push-offs of the corresponding geodesics and joining these in a neighbourhood of Ψ_0 in such a way that $\widetilde{Y}$ does not intersect $\mathcal{C}(x)$. Then one reverses the evolution to deform X back into $\Psi_0 \equiv \widetilde{X}$. (See figure 9.9).

This has the advantage that to understand our link we basically only have to consider two integral lines of $\frac{\partial}{\partial\varphi}$ and $2n$ geodesics. Let us then consider the behaviour of the geodesics on $\mathcal{C}(x)$ in more detail.

The natural projection $\pi : N \to \Sigma$ obviously restricts to a projection $\pi : \mathcal{C}(x) \to \Sigma$. If one uses the coordinates (φ_0, t) on $\mathcal{C}(x)$, then as we have discussed $\pi_* \frac{\partial}{\partial t}$ is the (unit) tangent vector to the corresponding geodesic, and $\pi_* \frac{\partial}{\partial\varphi_0}$ is clearly the Jacobi field associated with the congruence of geodesics through x. Hence the singular points of the projection $\pi : \mathcal{C}(x) \to \Sigma$ are the conjugate points of all these geodesics. Since $\left(\pi_* \frac{\partial}{\partial t}, \pi_* \frac{\partial}{\partial\varphi_0}\right)$ clearly reverses orientation at a conjugate point, it is easy to see that the curves formed by conjugate points on $\mathcal{C}(x)$ correspond to folds of this surface with respect to the projection. In other words, these are the points at which $\frac{\partial}{\partial\varphi}$ is tangent to $\mathcal{C}(x)$, and therefore the positive and negative sides of $\mathcal{C}(x)$ reverse orientations with respect to $\frac{\partial}{\partial\varphi}$.

Definition 178 *The set of all conjugate points of the geodesics in Σ are said to*

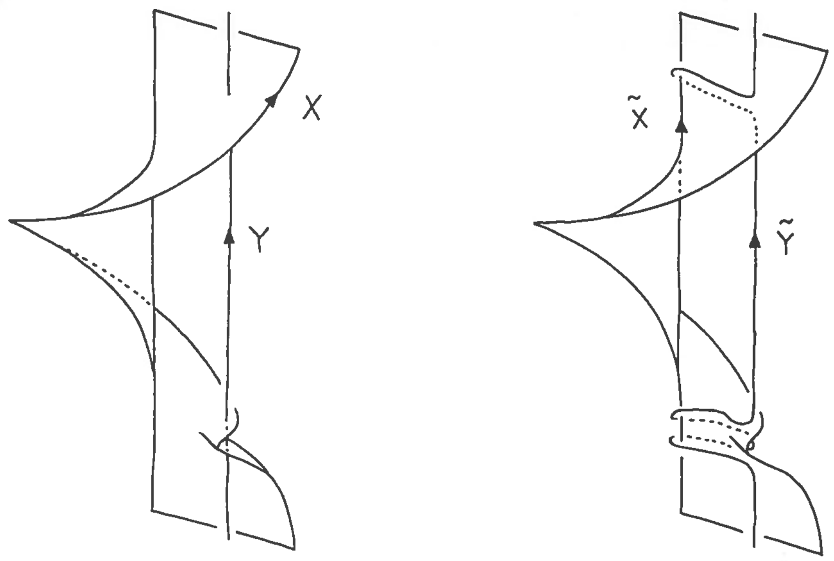

Figure 9.9: Deformation of $X \amalg Y$ into $\widetilde{X} \amalg \widetilde{Y}$.

be the caustics *of the congruence.*

Hence the folds of $\mathcal{C}(x)$ project down to caustics on Σ. Consequently, when a geodesic in $\mathcal{C}(x)$ goes through a conjugate point its projection on Σ hits a caustic but does not cross it. Because the geodesic has an orientation, we can decide if the caustic is to its right or to its left at the conjugate point. If it is to its right, then the geodesic must curve to the left with respect to the caustic, and therefore its lift goes from the under to the upper part of the fold (as its φ coordinate increases more rapidly than that of the fold); if it is to its left, then the geodesic must curve to the right with respect to the caustic, and therefore its lift goes from the upper to the lower part of the fold. (Here upper and lower are taken with respect to the "vertical" defined by $\frac{\partial}{\partial\varphi}$). Notice that in both cases the push-offs of the geodesic

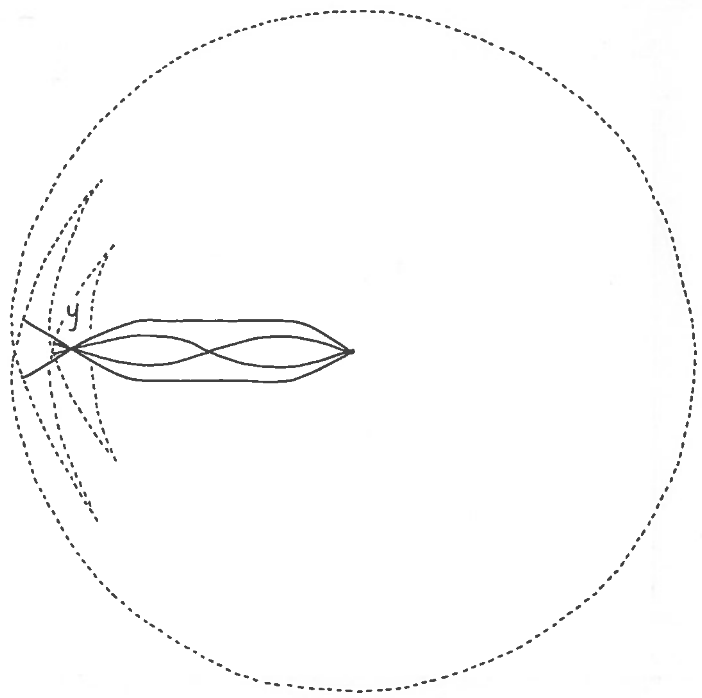

Figure 9.10: Geodesics through y in the 2 thickening spheres optical (2+1)-spacetime

twist through π in the negative direction with respect to the orientation of the geodesic.

Thus we have proved

Proposition 179 *The link* $X \amalg Y \approx \widetilde{X} \amalg \widetilde{Y}$ *can be obtained as follows: thicken each geodesic going from* x *to* y *slightly along the* $\frac{\partial}{\partial\varphi}$ *direction and extend it slightly past* x*; add as many half-twists to the ribbon thus obtained as there are conjugate points on the geodesic, in the negative direction with respect to the orientation of the geodesic; twist the ribbon slightly in the same direction at* $\widetilde{X}$ *so that they become transverse; join the boundary of the ribbon to* Y *and delete their intersection.*

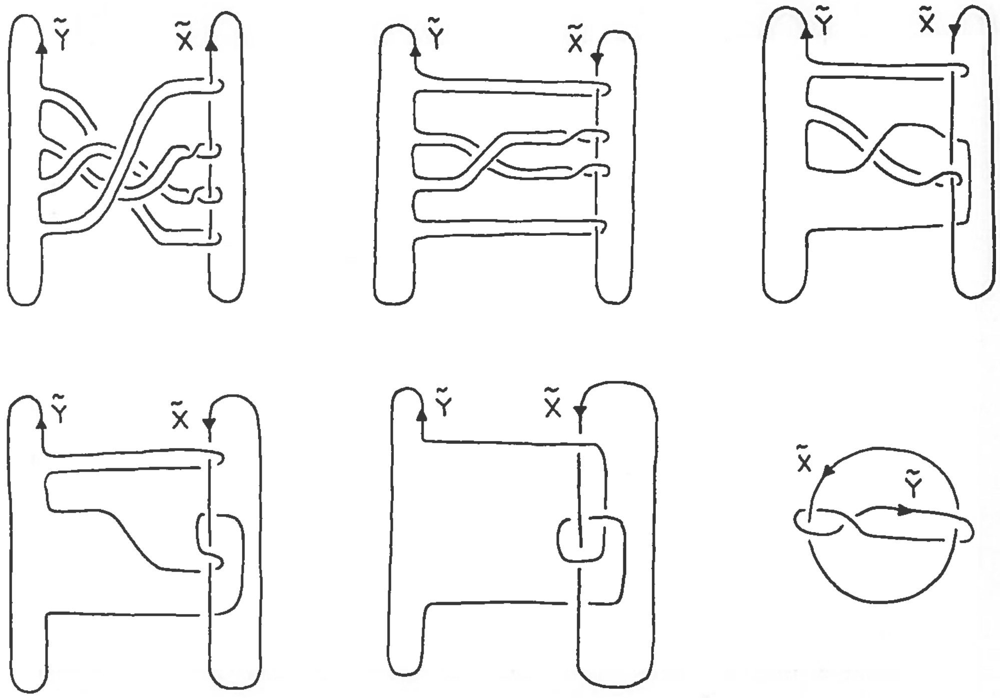

Figure 9.11: Link diagram for $\widetilde{X} \amalg \widetilde{Y}$.

As an example, we draw the link diagram for the skies with zero linking number in the (2+1)-dimensional optical spacetime for two thickening spheres. To do so, we must first obtain the set of geodesics going through y; we have done so numerically, and show a sketch of them in figure 9.10 (it turns out that there are four geodesics through y, two going through one conjugate point and two with no conjugate points). Next we simply follow the procedure of theorem. We do so in figure 9.11 and show it to be in fact a link diagram for the Whitehead link.

We can use these ideas to prove the following

Proposition 180 *Suppose that the geodesics going from x to y are such that their lifts are contained in a set of the form $\{\varphi_0 < \varphi < \varphi_0 + 2\pi\} \subseteq N$. Then the link*

$X \amalg Y \approx \widetilde{X} \amalg \widetilde{Y}$ is not the unlink.

Proof: Notice that $\widetilde{Y}$ spans a fairly obvious disk D, comprising the ribbons and a disk spanned by Y. If $\widetilde{X} \amalg \widetilde{Y}$ were the unlink, then $\widetilde{X}$ would span a disk C which $\widetilde{Y}$ would not intersect. However, since $\widetilde{X}$ does intersect D, the two disks would intersect.

Clearly the linking number of $\widetilde{X} \amalg \widetilde{Y}$ is the sum of the signs of the intersections of $\widetilde{X}$ with D, where we give D an orientation compatible with the orientation of $\widetilde{Y}$. Thus if $\widetilde{X} \amalg \widetilde{Y}$ were the unlink there would be an even number of geodesics connecting x and y, half with an even number of conjugate points and half with an odd number of conjugate points. On the other hand, it is clear $C \cap D$ would be formed by disjoint curves connecting one positive and one negative intersections of $\widetilde{X}$ with D (because of the relative orientations).

Now consider the push-off of one of these intersections curves α from D along, say, C (which is transverse to D along the intersection), in the direction of the side of D connected by $\widetilde{X}$ in the region $\{\varphi_0 < \varphi < \varphi_0 + 2\pi\}$ (see figure 9.12). Now α plus the segment of $\widetilde{X}$ joining its ends is a closed curve β; so is the push-off of α plus the same segment of $\widetilde{X}$ slightly displaced along C, γ. Because γ spans a surface which is not intersected by β (part of C), their linking number should be zero. However, since β must follow the two ribbons, it is easy to see that $link(\beta, \gamma) = -\frac{1}{2}(n+1) \leq -1$, where n is the total number of conjugate points along the two geodesics generating the ribbons. Thus the assumption that $\widetilde{X} \amalg \widetilde{Y}$ is the unlink leads to clear contradiction.□

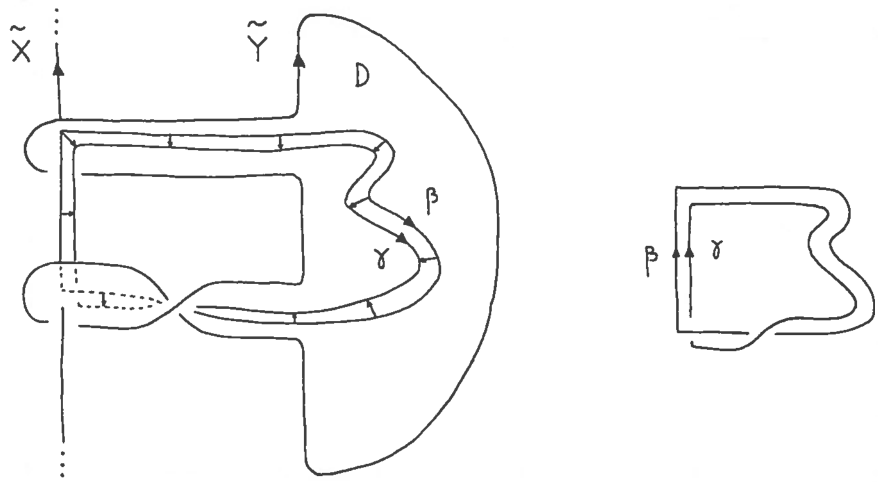

Figure 9.12: Definitions of β and γ

The condition in proposition 180 is useful in the proof, but looks rather unnatural. It is therefore not surprising that it can be removed. The reason is as follows: the argument in the proof of proposition 180 depends basically on the fact that the twisting of the ribbons arises solely due to the conjugate points of the corresponding geodesics. One might think that the ribbon of a geodesic not contained in a region $\{\varphi_0 < \varphi < \varphi_0 + 2\pi\} \subseteq N$ would pick up positive twisting as the geodesic's tangent vector rotates through 2π. In fact it does, but this is exactly canceled by the negative twisting arising from the "belt trick" (see figure 9.13). Consequently the net twisting of the ribbon is the twisting arising from the conjugate points, and the argument in the proof of proposition 180 applies to *all* geodesics. Thus we have proved

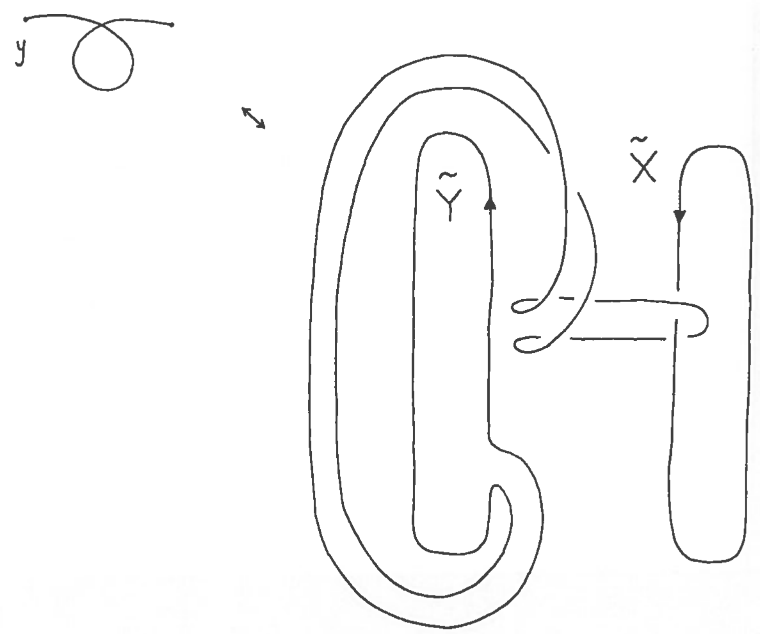

Figure 9.13: Cancelling of twisting

Theorem 181 *Let (M, g) be a regular static (2+1)-dimensional spacetime with Cauchy surface diffeomorphic to $\mathbb{R}^2$, and let N be its manifold of light rays. Then two spacetime points are causally related in M* iff *their skies either intersect or are linked in N.*

In fact we have proven the slightly stronger assertion that we got linking in $\mathbb{R}^3$ under the standard diffeomorphism.

9.4 Legendrian link invariants

Obviously, Low's conjecture implies in particular that two spacetime points x and y are causally related in M *iff* their skies X and Y either intersect or are Legendrian linked in N (because we know that the skies of two non-causally related spacetime points are always Legendrian unlinked). One could think of trying to prove this result directly using Legendrian link invariants.

Definition 182 *Arnold's* J^+ *invariant can be extended to include links by setting* $J^+(L \amalg K) = J^+(L) + J^+(K)$ *for any two unlinked Legendrian links* L *and* K.

Proposition 183 *If* X *and* Y *are nonintersecting skies,* $J^+(X \amalg Y) = 2link(X, Y)$.

Proof: By a judicious choice of the Cauchy surface Σ one can always assume that Y is a CC, which can be deformed either into a small FCC or a small PCC as required. Notice that both the FCC and the PCC have the same (counter-clockwise) orientation. One can imagine unlinking X and Y by taking the CC along a path out to infinity. Each time the path crosses the wavefront $\Phi = \pi(X)$ a dangerous tangency will occur. At each such tangency we deform the CC either into a small FCC or a small PCC in such a way that the dangerous tangency occurs as the circle first touches the wavefront. Then the crossing will be positive if the orientation of the wavefront is to the left of the path, and negative if it is to the right. If n_+ is the number of positive crossings and n_- the number of negative crossings, we'll have $J^+(X \amalg Y) = 2n_+ - 2n_-$; but on the other hand clearly $n_+ - n_-$ is the winding number of Φ about the path's endpoint.□

Thus Arnold's J^+ invariant is not enough to conclude that causally related points yield Legendrian linked skies. The Kauffman and HOMFLY polynomial invariants do distinguish between the Whitehead which arises in the 2 thickening sphere (2+1)-spacetime from the trivial link, but only after a very long calculation, and do not appear to be usable in the general case. They certainly do not distinguish all links, as they are symmetric in their components, whereas we know from theorem 105 that this is not always the case.

One might wonder why is it that for (2+1)-dimensional spacetimes causality appears to correspond to topological linking whereas in the (3+1)-dimensional case it appears to correspond to Legendrian linking. In this regard, it is interesting to note that it is possible to prove that two Legendrian knots in $ST(\mathbb{R}^2)$ are Legendrian linked *iff* they are topologically linked as long as one of them as index $i = 0$ (E. Ferrand, private communication). It is not known if this extends to the case when both the knots are skies and hence both have index $i = 1$. Notice that we have reasons to believe that in the (3+1)-dimensional case this is not so, as we expect Legendrian linking to correspond to causality and we can produce skies of causally related points which are nonetheless unlinked.

An immediate consequence of either Low's conjecture or its (possibly weaker) Legendrian version would be the following

Corollary 184 *Let Φ be a future sky corresponding to the spacetime point $x \in M$; then Φ uniquely determines the points in the Cauchy surface $\Sigma \supseteq \Phi$ which are in the causal future of x.*

CHAPTER 10

Results for (3+1)-spacetimes

10.1 An example

Definition 185 *The Legendrian isotopy* $\Phi_t : [0,1] \times N \to N$ *is said to* undo *the Legendrian link* $X \amalg Y$ *if* $\pi(\Phi_1(X))$ *and* $\pi(\Phi_1(Y))$ *are separated by a plane on* $\Sigma \approx \mathbb{R}^3$.

Proposition 186 *The Legendrian link* $X \amalg Y$ *can be undone* iff *it is the unlink.*

Proof: If $X \amalg Y$ is the unlink then by definition there exists a Legendrian isotopy $\Phi_t : [0,1] \times N \to N$ such that $\pi(\Phi_1(X))$ and $\pi(\Phi_1(Y))$ are two distinct points of Σ, and hence it can be undone; on the other hand, if $X \amalg Y$ can be undone then because both X and Y can be separately deformed through a Legendrian isotopy into fibres of $\Sigma \times S^2 \approx N$ it is not hard to construct a Legendrian isotopy $\Phi_t : [0,1] \times N \to N$ such that $\pi(\Phi_1(X))$ and $\pi(\Phi_1(Y))$ are two distinct points of Σ.□

Proposition 187 *If the Legendrian link* $X \amalg Y$ *is the unlink then it can be undone by a Legendrian isotopy* $\Phi_t : [0,1] \times N \to N$ *such that* $\pi(\Phi_t(Y)) = \pi(Y)$ *(i.e., which fixes* Y*).*□

Proof: Take any Legendrian isotopy $\Phi_t : [0,1] \times N \to N$ which undoes $X \amalg Y$, and consider the set

$$A = \bigcup_{t \in [0,1]} \Phi_t(Y).$$

Clearly A (and hence $\pi(A)$) is compact. Now extend Φ simply by moving $\pi(\Phi_1(X))$ away from $\pi(A)$ in such away that $\pi(\Phi_1(X))$ and $\pi(A)$ are separated by a plane. Call this new isotopy (suitably reparametrized) Ψ. Then Ψ undoes $X \amalg Y$ and

$$\Psi_1(X) \cap \Psi_t(Y) = \emptyset$$

for all $t \in [0,1]$. Finally, use the Legendrian isotopy extension theorem to construct a Legendrian isotopy $\Xi_t : [0,1] \times N \to N$ satisfying

$$\begin{aligned} \Xi_t(\Psi_1(Y)) &= \Psi_{1-t}(Y); \\ \Xi_t(\Psi_1(X)) &= \Psi_1(X). \end{aligned}$$

Then $\Xi_{1-t} \circ \Psi_t$ undoes $X \amalg Y$ and fixes Y.□

We now return to the skies of the points x and y of the optical (3+1)-dimensional spacetime containing one thickening sphere. Recall that these satisfy $y \in J^+(x)$ and yet X and Y do not intersect nor are linked. Recall also that there exists a Cauchy surface Σ such that the projections $\pi(X)$ and $\pi(Y)$ are the revolution surfaces about the z-axis generated by the curves shown on figure 5.1 ($\pi : N \to \Sigma$ is the natural projection). Our objective is to prove that X and Y are Legendrian linked.

The two following propositions are almost immediate:

Proposition 188 *The Legendrian link $X \amalg Y$ cannot be undone in such a way that $\pi(X)$ and $\pi(Y)$ remain revolution surfaces about the z-axis.*

Proof: This is immediate from the observation that any Legendrian deformation such that $\pi(X)$ and $\pi(Y)$ remain revolution surfaces about the z-axis can be thought of as a Legendrian deformation of the lifts of $\pi(X) \cap \{x = 0\}$ and $\pi(Y) \cap \{x = 0\}$ to $ST^*\mathbb{R}^2$. But we know that this Legendrian link in $ST^*\mathbb{R}^2$ is not the trivial link: in fact, the linking number is not even zero.□

Proposition 189 *The Legendrian link $X \amalg Y$ cannot be undone in such a way that $\pi(X)$ and $\pi(Y)$ remain smooth surfaces.*

Proof: Recall that any smooth isotopy of Σ yields a Legendrian isotopy on N. In particular if the Legendrian link $X \amalg Y$ could be undone in such a way that $\pi(X)$ and $\pi(Y)$ remained smooth surfaces, one could, through composition with a suitable smooth isotopy, find a Legendrian isotopy such that $\pi(X)$ remained a smooth surface and $\pi(Y)$ remained a spherical surface ($\{r = 1\}$, say). Now let X_t ($t \in [0, 1]$) be the Legendrian deformation of X corresponding to such an undoing. Because $\pi(X_t)$ is compact we can define

$$\rho(t) = \min\{r(\xi) : \xi \in \pi(X_t)\}$$

and it is clear that $\rho(0) < 1$, $\rho(1) > 1$. Since ρ is clearly continuous, there must exist $t^* \in (0, 1)$ such that $\rho(t^*) = 1$. Take the compact manifold $\pi(X_{t^*})$; the restriction to it of the radial distance function $r : \mathbb{R}^3 \to \mathbb{R}$ is clearly C^∞, as $r \geq 1$ on the submanifold. Let ξ be the point of $\pi(X_{t^*})$ at which the minimum is attained; then clearly the outward pointing normal vector $\mathbf{n}$ to $\pi(X_{t^*})$ at ξ satisfies

$$\mathbf{n} = -\nabla r$$

and since $-\nabla r$ is exactly the coorientation of $\pi(Y)$ we see that X_{t^*} and Y intersect. (Notice that because we are assuming that $\pi(X_t)$ remains a smooth manifold its coorientation must always be the outward pointing normal).□

Notice that initially $\pi(X)$ and $\pi(Y)$ are spheres, and their intersection are two circles which divide $\pi(Y)$ in three connected components. Call the component further away from $\pi(X)$ the *far component* of $\pi(Y)$ (and similarly for the far component of $\pi(Y)$). Then a not so obvious result is

Proposition 190 *Let $\Phi_t : [0,1] \times N \to N$ be a Legendrian isotopy which fixes Y and such that (i) there exists a point $\xi \in \pi(Y)$ on the far component which is not on $\pi(\Phi_t(X))$ for all $t \in [0,1]$, and (ii) $\pi(\Phi_t(X)) \cap \pi(Y)$ is either two disjoint circles, one single circle, or the empty set. Then Φ does not undo $X \amalg Y$.*

Proof: Under the conditions above we can think of $\pi(Y)$ as a plane (the point at infinity being ξ). For definitiveness let this plane be the $\{z=0\}$ plane, and let the far component of $\pi(X)$ be contained in $\{z>0\}$. Then the coorientation of $\pi(Y)$ is simply $-\frac{\partial}{\partial z}$. In general, the surface $\pi(\Phi_t(X)) \cap \{z \leq 0\}$ will be homeomorphic to an immersed annulus. Thus at each $t \in [0,1]$ the coorientation defines a map from the annulus to $S^2 \subseteq \mathbb{R}^3$, to which we shall refer to as the *Gauss map*. Notice that because $\Phi_t(X)$ is a Legendrian manifold all the Gauss maps must be continuous. The condition that X and Y should not intersect is then the condition that the Gauss map should not map any point on the boundary to $-\frac{\partial}{\partial z}$ (points on the interior can obviously be mapped to any point of S^2). For instance, initially the Gauss map maps the boundary of the annulus to lines of constant latitude on S^2, and each radial line of the annulus to a segment of a meridian through $-\frac{\partial}{\partial z}$ joining these two lines. Hence an infinite number of points in the annulus is mapped to $-\frac{\partial}{\partial z}$ by the Gauss map, and it is easy to see that these points form a circle parallel to the boundary (it is the circle where the function $z : \pi(X) \to \mathbb{R}$ attains its minimum).

It is easily seen that we can assume that the two circle forming the boundary of the annulus are mapped to the equator, say. Now under our assumptions the only way of undoing the link would be to somehow $\pi(\Phi_t(X)) \cap \pi(Y)$ collapse to a circle on $\{z \leq 0\}$. This would mean that the original Gauss map would have to be homotopic to a map mapping the annulus to the equator (in the class of maps from the annulus to S^2 mapping the boundary to the equator). But these maps are clearly non-homotopic (for instance, if they were one could easily construct a homotopy showing that $2 = 0$ on $\pi_2(S^2) = \mathbb{Z}$).□

Thus we have some reasons to believe that $X \amalg Y$ is indeed Legendrian linked. Unfortunately, a complete proof could not been obtained. However, the fact that the initial Gauss map above is not homotopic to the equator seems to indicate that a fundamental obstruction to the undoing of the link does indeed exist.

10.2 A conjecture

The example of the previous section naturally leads us to the following

Conjecture 191 *Let (M, g) be a globally hyperbolic (3+1)-spacetime with Cauchy surface diffeomorphic to a subset of $\mathbb{R}^3$, and let N be its manifold of light rays.*

Then two spacetime points are causally related in M iff *their skies either intersect or are Legendrian linked in N.*

This is a natural extension of Low's conjecture for (2+1)-dimensional spacetimes, and appears to be true for at least some cases. However, a proof appears to require new methods of contact topology.

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