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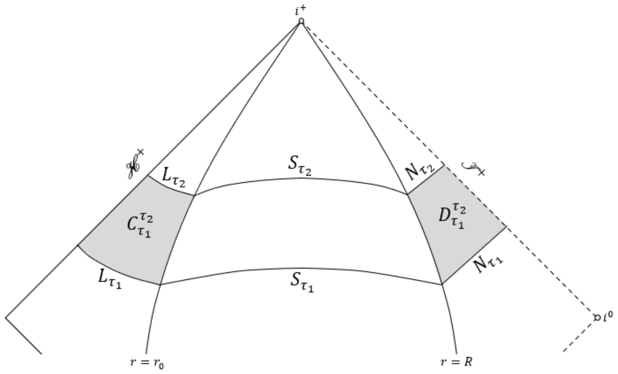

Boundedness and decay of solutions of the wave equation in Minkowski's and Schwarzschild's spacetimes

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Mathematics and Applications

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To my parents,

for all their love and support.

Declaration

I declare that this document is an original work of my own authorship and that it fulfills all the requirements of the Code of Conduct and Good Practices of the Universidade de Lisboa.

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Resumo

Neste trabalho, apresentamos uma prova detalhada da limitação e decaimento das soluções da equação de onda nos espaços-tempo de Minkowski e Schwarzschild, considerando dados iniciais regulares e de suporte compacto. Para tal, recorremos a métodos de energia, devido à sua robustez e aplicabilidade a vários espaços-tempos. Em particular, provamos uma estimativa integral de decaimento da energia local, que permite controlar a energia em regiões espacialmente compactas e, em seguida, deduzimos a chamada hierarquia de Dafermos-Rodnianski. No caso do espaço-tempo de Schwarzschild, ocorre uma degenerescência da energia no horizonte de eventos e na esfera de fotões, estando esta última intimamente relacionada com o aprisionamento dos raios de luz nesta superfície. Para resolver o problema no horizonte do buraco negro, construímos um fluxo de energia não degenerado apropriado, explorando a positividade da gravidade de superfície do horizonte de eventos, o que indica que o fenómeno do desvio para o vermelho acaba por ser crucial para obter estes resultados. Finalmente, utilizamos uma sequência diádica para deduzir o decaimento da energia não degenerada, o que, por sua vez, usando a simetria esférica destes espaços-tempos, leva a estimativas pontuais das soluções da equação de onda.

Palavras-chave: Equação de onda, Método do campo vetorial, Desvio para o vermelho, Esfera de fotões, Limitação, Decaimento

Abstract

In this work, we present a detailed proof of the boundedness and decay of the solutions to the wave equation on the Minkowski and Schwarzschild spacetimes, given smooth and compactly supported initial data. For this purpose, we make use of energy methods, due to their robustness and applicability to various spacetimes. In particular, we prove an Integrated Local Energy Decay estimate, which allows us to control the energy on spatially compact regions, and then we deduce the so-called Dafermos-Rodnianski hierarchy. In the case of the Schwarzschild spacetime, an energy degeneracy occurs at the event horizon and at the photon sphere, the latter being intimately related to the trapping of light rays at this surface. To solve the issue at the black hole's horizon, we construct a well-behaved nondegenerate energy flux by exploiting the positivity of the surface gravity of the event horizon, which shows that the redshift phenomenon turns out to be crucial to obtain these results. Finally, we make use of a dyadic sequence to derive decay of the nondegenerate energy, which then leads, using the spherical symmetry of these spacetimes, to pointwise estimates of the solutions of the wave equation.

Keywords: Wave equation, Vector field method, Redshift, Photon sphere, Boundedness, Decay

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1 Introduction

General Relativity studies the 4-dimensional Lorentzian manifolds $(\mathcal{M}, g)$ which are solutions of the Einstein field equations, first formulated by Albert Einstein in 1915 [14, 15]:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi T_{\mu\nu}\,. \tag{1.1}$$

Here, $R_{\mu\nu}$ and R denote the Ricci tensor and the scalar curvature, respectively, and $T_{\mu\nu}$ is the energy-momentum tensor describing the matter fields modeling the system of interest, so that the equations connect the geometric properties of the spacetime $(\mathcal{M}, g)$, on the left-hand side, with a quantity modeling the physical content of the system, on the right-hand side. In the absence of matter, these equations are called Einstein vacuum equations:

$$R_{\mu\nu} = 0\,. \tag{1.2}$$

The trivial solution to this system of nonlinear partial differential equations is the so-called Minkowski spacetime, corresponding to flat space, i.e. $\mathbb{R}^{1+3}$ with the metric $g = -dt^2 + dx^2 + dy^2 + dz^2$. Arguably, the second simplest solution to the Einstein vacuum equations (1.2) is the so-called Schwarzschild spacetime, first discovered in 1916 [30], which describes a non-rotating black hole with zero electric charge. In the years following the publication of this work, there was some misunderstanding regarding the singularities found in the components of the metric of this solution, and what kind of system it did in fact, describe. Nevertheless, eventually it came to light that the Schwarzschild spacetime corresponded to a black hole, and that some of the singularities found in the metric components of the Schwarzschild solution were merely an artifact of the choice of coordinates, related to the presence of an event horizon. Other exact solutions describing black holes were also eventually discovered, such as the Reissner-Nordström [27, 28] and the Kerr [23] spacetimes. Nonetheless, in this work we will focus only on the Schwarzschild spacetime, since it is the simplest such solution, describing a non-rotating black hole with zero electric charge.

Even though these black hole spacetimes were found to be solutions to the Einstein vacuum equations, one can still pose the question of their stability, as a strong indication that this kind of physical objects could indeed exist in our universe. However, the stability problem turned out to be quite hard to formulate, and even more difficult to tackle. Only after the seminal contribution by Choquet-Bruhat [4] on how to properly define, and solve, the so-called Cauchy problem for the Einstein field equations, was it possible to start studying the stability problem. The first major result was the stability of the Minkowski spacetime, proved by D. Christodoulou and S. Klainerman in 1993 [5]. Further developments have also been made in the past few years regarding the stability problem for black hole spacetimes. An extensive description of the formalism of this problem and the results that have already been proved, as well as open questions, can be found in [18].

Due to the hyperbolic nature of equation (1.2), it turns out that simply understanding the properties of solutions of the homogeneous wave equation on a fixed spacetime background is already a very useful first step before addressing more subtle questions, such as the stability of the spacetime itself. Not only does it require developing suitable techniques, but it also allows dealing with some of the interesting aspects of each particular spacetime and how they might affect its stability properties. In particular, given appropriate initial Cauchy data, one should check whether the solution to the wave equation is bounded and, in that case, if it exhibits any decay properties.

The case of the Minkowski spacetime is already fully understood, given that an explicit formula for the solution of the wave equation is a classical well-known result (see for instance [16]). Nonetheless, one can make use of the simplicity of this spacetime to develop robust tools which might be useful in the context of other spacetimes that share key properties with it, namely staticity and spherical symmetry. One of the tools that has become quite significant in this setting is the so-called Integrated Local Energy Decay estimate, first introduced by Morawetz in 1968 for the Minkowski spacetime [24]. These estimates capture the fact that energy in spatially compact regions decays in time. Another remarkable result is the Dafermos-Rodnianski hierarchy [9], which is a pair of inequalities controlling the behavior of energy in non-compact regions away from the origin.

Unfortunately, no explicit solution is known for the wave equation on the Schwarzschild spacetime, and so it becomes necessary to use other methods to understand its global properties. The proof of boundedness of the solution to the wave equation was given by Kay and Wald in 1987 [22], and the first result concerning decay was proved by Twainy in 1989 [31], without providing an explicit decay rate. A couple of decades later, Dafermos and Rodnianski further improved Twainy's result by determining a decay rate [10], thus describing the main features of the wave equation on the Schwarzschild spacetime. In particular, this work exploited the redshift effect phenomenon near the black hole's event horizon to overcome some additional difficulties arising in this problem when compared to the Minkowski spacetime case. Since then, refinements have been added to this proof, notably the fact that the Dafermos-Rodnianski hierarchy is also valid in this case [9], and that various Integrated Local Energy decay estimates hold [7, 21].

Although it is beyond the scope of this work, we also highlight the main results for the Kerr spacetime. A first decay rate was established in 2010 [11] for the full subextremal range under the assumption that the solution of the wave equation is axisymmetric, and also for the very slowly rotating black holes, without the axisymmetry assumption. The decay result for the full subextremal range with no additional assumptions was later attained in [12, 13]. It is worth mentioning that the wave equation in Kerr's spacetime poses additional difficulties, due to the presence of an ergoregion with the associated superradiance phenomenon, and also due to the complexity of the trapping effect.

1.1 Thesis Overview

In this work, we aim to study the properties of the solutions of the wave equation,

$$\Box_g \phi = 0\,, \tag{1.3}$$

on two specific Lorentzian manifolds $(\mathcal{M}, g)$ corresponding to the two most fundamental spacetimes in General Relativity: the Minkowski spacetime and the exterior region of the black hole in the Schwarzschild spacetime.[1] Our main goal is thus to study the behavior of waves on these fixed spacetime backgrounds. In particular, we intend to prove boundedness and a suitable notion of decay of the solutions of the wave equation, given appropriate initial compactly supported Cauchy data.

In Sections 2.1, 2.2 and 2.3, we introduce some general concepts which will be useful for employing techniques based on energy considerations. Then, in Section 3.1, we prove that the solution to the wave equation in the Minkowski spacetime is bounded. In Section 3.2, we end the discussion of the Minkowski spacetime case by deducing that the solution decays in time at a certain rate. Section 4.1 contains a description of several coordinate systems for the Schwarzschild spacetime, and in Section 4.2 we set up the Cauchy problem for the wave equation on this spacetime. Finally, Sections 4.3 and 4.4 contain the proof of the boundedness and decay results for the solution to the wave equation on the exterior region of the Schwarzschild black hole.

The results in this thesis can be essentially found in several lecture notes [2, 6, 8, 17]. However, there is no one place in the literature where the proofs of these results are spelled out in full detail, as we aim to do here. Our proof also differs slightly from those in the lecture notes above in that we adapt the results of the recent paper [21] to obtain our Integrated Local Energy Decay estimates.

Throughout this work, we use a geometrized system of units, for which $c = G = 1$.

1.2 Outline of the proof

The purpose of this Section is to provide a brief explanation, without excessive detail, of the way in which we will approach the proofs of boundedness and decay of the solutions of the wave equation, on the Minkowski and Schwarzschild spacetimes. Specifically, we will address some of the issues arising throughout the proof of the main results and how they can be overcome.

First, we point out that our proof mostly focuses on energy-type estimates, as they have shown to be quite robust, making them especially useful for spacetimes with less symmetry. We start by constructing a certain notion of energy current, corresponding to the contraction of the energy-momentum tensor of the massless scalar field, associated to the solution of the wave equation, with an arbitrary vector field, usually called a multiplier. Then, we define the associated energy flux of this current across a spacelike or null hypersurface, which should coincide with the boundary terms arising in the application of the divergence theorem to the energy current over some spacetime region. It turns out that the energy current associated to a Killing vector field has a vanishing bulk term in the divergence theorem,

[1] This is usually called the domain of outer communication.

easily yielding energy conservation identities. The energy flux of a global timelike Killing field of a static spacetime is particularly important as it can be used to obtain pointwise bounds of the solution to the wave equation. Therefore, one starts by proving energy boundedness and decay in order to deduce pointwise estimates.

Unfortunately, for the case of the Schwarzschild spacetime, the presence of the event horizon of the black hole comes up as an obstruction to deriving pointwise bounds from energy estimates, due to a degeneracy phenomenon occurring at the horizon. Nevertheless, this challenge can be surmounted by constructing a new multiplier, which coincides with the timelike Killing field away from the horizon, and is equal to the so-called redshift vector field near the event horizon. Such a construction highly depends on the fact that the surface gravity of the Schwarzschild black hole is positive, which is known to be closely related to the gravitational redshift effect taking place in this spacetime. The proof of boundedness then follows from the fact that the energy flux of the redshift vector field is bounded.

In order to derive decay estimates, one has to make a more careful analysis. In this work, we follow the reasoning presented in [9] to achieve such a result. First, one has to choose accurately a collection of hypersurfaces foliating the future domain of dependence of the initial data, such that one expects the energy flux across these hypersurfaces to decay. In both spacetimes, a key characteristic is that these hypersurfaces should be null near null infinity, so that one can capture the fact that energy is radiating away to infinity. For the Schwarzschild spacetime it is also essential to define the hypersurfaces in such a way that they cross the horizon transversally, instead of approaching the bifurcation sphere. The main reason for this choice is that the solution of the wave equation does not necessarily decay towards the event horizon, so one should not expect to recover pointwise estimates from energy decay results involving hypersurfaces that meet at the bifurcation sphere.

After this groundwork, it is crucial to derive an energy-type result designated as an Integrated Local Energy Decay estimate, first discussed by Morawetz in [24]. This is an inequality capturing the fact that energy decays in spatially compact regions. For the Schwarzschild spacetime, we follow the approach in [21] to obtain this kind of estimate. However, this energy-type result also degenerates at the horizon, so one has to make use of the redshift vector field once again to derive a non-degenerate Integrated Local Energy Decay estimate. Another feature of this result is that it also degenerates at the black hole's photon sphere, which has actually been proved to be unavoidable by Sbierski in [29]. Essentially, this is related to a trapping phenomenon near the photon sphere, allowing the construction of light ray trajectories that stay arbitrarily close to the photon sphere for an arbitrarily long time, thus preventing the energy from decaying in a compact neighborhood of this surface. In spite of that, this degeneracy can be sidestepped by making use of the staticty and spherical symmetry of the Schwarzschild spacetime, applying commutations arguments in order to finally get a fully nondegenerate estimate.

Finally, one should deduce the so-called Dafermos-Rodnianski hierarchy, which consists of a pair of inequalities relating the time integral of the energy flux through appropriately defined hypersurfaces with the energy flux itself and also with the flux of the so-called radiation field through null hypersurfaces near null infinity. Having established this hierarchy, one can then use a clever trick to obtain the energy decay result.

All these techniques will be explained and fully developed in the following chapters, which comprise the main body of this thesis.

Wave equation on curved spacetimes 2

The central problem of this work is to study the behavior of waves on a fixed spacetime background $(\mathcal{M}, g)$, and analyze some properties of the solutions of the wave equation,

$$\Box_g \phi = 0 \,. \tag{2.1}$$

Here $\Box_g$ is the Laplace-Beltrami operator for the metric g, given by

$$\Box_g \phi = \nabla^\alpha \nabla_\alpha \phi = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi), \tag{2.2}$$

which, for the Lorentzian metrics of General Relativity, is a hyperbolic operator[1].

As mentioned in the Introduction, we intend to study properties of boundedness and decay for solutions of (2.1) on the Minkowski and Schwarzschild spacetimes, the latter not allowing for explicit solutions, as is generally the case. We then need to rely on more robust methods, that can be applied and exploited independently of whether explicit solutions are available or not. These will mostly be energy methods adapted to the geometry of the spacetime, briefly described in Section 1.2 above, which we now proceed to introduce in detail.

Although explicit solutions for the wave equation on the Minkowski spacetime do exist and are well known, we will nevertheless start by applying the energy method here, to obtain boundedness and decay of the solution to the wave equation, as it is a simpler setting on which to understand the most important features. The Schwarzschild spacetime can then be tackled afterwards, with a better grasp of the method, in order to deal with the added difficulties posed by this black hole spacetime.

We will now present the main mathematical tools that will be used to develop and apply the energy methods required to prove the boundedness and decay of the solutions of the wave equation on Minkowski, and, more particularly, on the Schwarzschild background spacetimes. They are essentially based on a precise use of the divergence theorem on carefully chosen regions in Lorentzian manifolds, for certain energy currents derived from the energy-momentum tensor associated to the wave equation, and appropriately chosen vector fields, namely Killing vector fields, therefore bringing together the analysis structure of the wave equation with the geometric structure of the spacetime. The final mathematical ingredient which we rely on are Sobolev inequalities, which allow us to derive pointwise bounds from the integral energy estimates.

[1] The hyperbolic character of the Laplace-Beltrami operator in the case of Lorentzian metrics leads to the prevailing usage of the d'Alembertian symbol $\Box_g$, instead of the Laplacian symbol Δ_g, usually reserved for Riemannian metrics (where the operator is elliptic).

2.1 Divergence Theorem in Lorentzian manifolds

The energy techniques that will be used require repeated applications of the divergence theorem. In this Section, we describe how we can make use of this theorem in Lorentzian manifolds, since some differences arise when compared to the Riemannian case. Consider a k-dimensional Lorentzian manifold $(\mathcal{M}, g)$ with boundary $\partial\mathcal{M}$. Given a vector field X, the divergence theorem, then, states that

$$\int_{\mathcal{M}} \nabla_\mu X^\mu = \int_{\partial\mathcal{M}} g(X, n)\,, \tag{2.3}$$

where n denotes a unit normal to the boundary $\partial\mathcal{M}$ pointing as depicted in Figure 2.1.

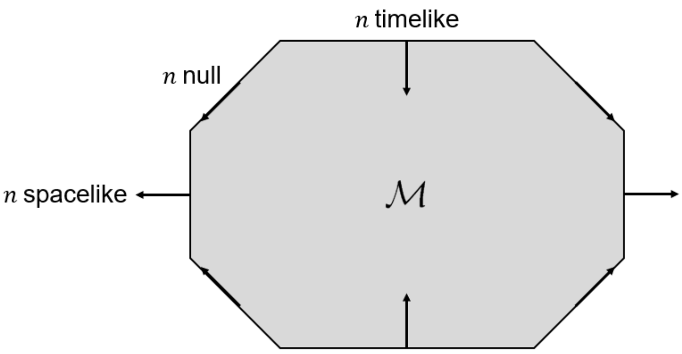

Figure 2.1: Unit normal vector for the divergence theorem in Lorentzian manifolds.

Therefore, we take n to be pointing inwards if it is timelike, and outwards if it is spacelike. Note that there is still the case where n can be a null vector, for which the notion of unitary vector cannot be defined. In this case, we consider a positive orthonormal frame $\{E_1, ..., E_k\}$, where E_1 is timelike and pointing inwards and E_2 is spacelike and pointing outwards. Then, we take n to be

$$n = E_1 + E_2\,, \tag{2.4}$$

and the volume element on the null portion of $\partial\mathcal{M}$ is given by

$$\sigma = E_2^\flat \wedge ... \wedge E_k^\flat \tag{2.5}$$

(where $E_i^\flat$ is the one-form associated by the metric to the vector field E_i). See [26] for a detailed explanation on why we choose the normal vector as described above.

2.2 Energy currents and conservation

Consider a fixed background globally hyperbolic static spacetime $(\mathcal{M}, g)$, and denote by T a future-directed timelike Killing vector field. Let Σ_0 be a Cauchy hypersurface and define $\Sigma_t = \varphi_t(\Sigma_0)$, where φ_t

is the 1-parameter group of diffeomorphims generated by T. Given $t_1 < t_2$, define the spacetime region

$$\mathcal{M}(t_1, t_2) := \bigcup_{t_1 \leq t \leq t_2} \Sigma_t\,, \tag{2.6}$$

delimited in the past and the future by the hypersurfaces Σ_{t_1} and Σ_{t_2}, respectively. Denote by n_{Σ_t} the future-directed unit normal of Σ_t. Given a scalar field ϕ, consider the following covariant two-tensor, written in arbitrary coordinates as

$$T_{\mu\nu}[\phi] = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2}(\partial^\alpha \phi \, \partial_\alpha \phi) g_{\mu\nu}\,. \tag{2.7}$$

From a physical point of view, this tensor represents the energy-momentum tensor corresponding to a massless scalar field ϕ, with Lagrangian $\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi = \frac{1}{2} \partial^\alpha \phi \, \partial_\alpha \phi$, whose Euler-Lagrange equation is precisely the wave equation (2.1). It is, therefore, a particularly well-suited quantity for the study of solutions of the wave equation on Lorentizian manifolds with geometry given by the metric g.

Given a vector field X, define the associated energy current $J^X_\mu[\phi] = T_{\mu\nu}[\phi] X^\nu$ and the associated bulk term $K^X[\phi] = T_{\mu\nu}(\pi^X)^{\mu\nu}$, where $\pi^X = \frac{1}{2}\mathcal{L}_X g$ is the so called deformation tensor of X. An easy computation shows that

$$\nabla^\mu T_{\mu\nu}[\phi] = \partial_\nu \phi \, \Box_g \phi\,, \tag{2.8}$$

implying that the energy current and the bulk term are related by

$$\nabla^\mu J^X_\mu[\phi] = K^X[\phi] + (X\phi) \, \Box_g \phi\,. \tag{2.9}$$

In particular, if X is Killing and ϕ is a solution to the wave equation then $\nabla^\mu J^X_\mu[\phi] = 0$.

By the divergence theorem, assuming that ϕ has compact support on each Σ_t, equation (2.9) implies that

$$\int_{\mathcal{M}(t_1,t_2)} \left(K^X[\phi] + (X\phi)\, \Box_g \phi\right) \, \mathrm{dVol}_{\mathcal{M}} = \int_{\Sigma_{t_1}} J^X_\mu[\phi] n^\mu_{\Sigma_{t_1}} \, \mathrm{dVol}_{\Sigma_{t_1}} - \int_{\Sigma_{t_2}} J^X_\mu[\phi] n^\mu_{\Sigma_{t_2}} \, \mathrm{dVol}_{\Sigma_{t_2}}\,, \tag{2.10}$$

where $\mathrm{dVol}_{\Sigma_{t_i}}$ denotes the volume element of Σ_{t_i}. Defining now the energy as the flux of the current $J^X_\mu[\phi]$ through Σ_t,

$$\mathbb{E}^X[\phi](t) = \int_{\Sigma_t} J^X_\mu[\phi] n^\mu_{\Sigma_t} \, \mathrm{dVol}_{\Sigma_t}\,, \tag{2.11}$$

we finally conclude that, if ϕ satisfies the wave equation, then we have the following energy conservation identity:

$$\mathbb{E}^T[\phi](t) = \mathbb{E}^T[\phi](0)\,, \qquad \forall t > 0\,. \tag{2.12}$$

2.3 Sobolev inequality

Our main goals of this work require deducing pointwise estimates for the solution of the wave equation. However, resorting to energy methods only leads to integral estimates, in particular, to weighted L^2

norms of ϕ and its derivatives. Therefore, we will make use of the so called Sobolev inequality, which gives pointwise bounds by integral quantities (for a proof, see for instance [20]):

Theorem 2.3.1. *Let (M, g) be a compact n-dimensional Riemannian manifold, and let k be an integer. If $k > n/2$ and $f \in H^k(M)$, then f is continuous and there exists a constant $C > 0$ such that*

$$\|f\|_{L^\infty(M)} \leq C\|f\|_{H^k(M)}, \quad \forall f \in H^k(M)\,. \tag{2.13}$$

It might seem that this result cannot be employed in the context of the Minkowski and Schwarzschild spacetimes, since we are working with Lorentzian manifolds whose Cauchy hypersurfaces are not compact Riemannian manifolds. Nonetheless, they are spherically symmetric and therefore can be foliated by spheres. It will be convenient to use coordinates (t, r, ω), where t and r are the usual time and radial coordinates and $\omega \in \mathbb{S}^2$. We can then apply the Sobolev inequality on the unit sphere $\mathbb{S}^2$, which is a compact 2-dimensional Riemannian manifold.

Corollary 2.3.2. *If $k > 1$ and $f \in H^k(\mathbb{S}^2)$, then f is continuous on $\mathbb{S}^2$ and there exists a constant $C > 0$ such that*

$$\|f\|_{L^\infty(\mathbb{S}^2)} \leq C\|f\|_{H^k(\mathbb{S}^2)}, \quad \forall f \in H^k(\mathbb{S}^2)\,. \tag{2.14}$$

Therefore, to obtain global pointwise results, it will be enough to prove that $\|\phi(t, r, \cdot)\|_{H^2(\mathbb{S}^2)}$ is uniformly bounded and decays. Actually, it turns out that it will be sufficient to obtain these results for $\|\phi(t, r, \cdot)\|_{L^2(\mathbb{S}^2)}$, since we can make use of the spherical symmetry of the Minkowski and Schwarzschild spacetimes to apply commutation with angular Killing vector fields and obtain estimates involving higher order derivatives.

Unfortunately, as we will see later, some difficulties will arise when trying to bound $\|\phi(t, r, \cdot)\|_{H^2(\mathbb{S}^2)}$ for $r \to 0$ in the Minkowski spacetime. To solve this problem, we will obtain bounds for $\|\phi(t, \cdot)\|_{H^2(B_R(0))}$, for a fixed $R > 0$, and then apply the following result (see [1]):

Theorem 2.3.3. *Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition: there exists a compact cone C such that each $x \in \Omega$ is the vertex of a cone C_x, contained in Ω and congruent to C. If $k \geq 1$ is an integer such that $k > n/2$ and $f \in H^k(\Omega)$, then f is continuous on Ω and then there exists a constant $C > 0$ such that*

$$\|f\|_{L^\infty(\Omega)} \leq C\|f\|_{H^k(\Omega)}, \quad \forall f \in H^k(\Omega)\,. \tag{2.15}$$

For our purposes, we will take Ω to be the ball $B_R(0) \subset \mathbb{R}^3$, so that it is indeed enough to deduce estimates for $\|\phi(t, \cdot)\|_{H^2(B_R(0))}$ to obtain pointwise results.

3 Minkowski spacetime

As an introduction to the problem of establishing boundedness and decay for solutions of the wave equation on general spacetimes, we will first study it on the Minkowski spacetime. This is the trivial solution to the Einstein Field Equations, where the wave equation has long been fully understood, including explicit formulas for the solutions; therefore, it serves as the simplest framework to develop, test and gauge more general and robust methods, that can then be deployed on different spacetimes. Its metric is given by

$$g = -dt^2 + dr^2 + r^2 d\Omega^2 \,, \tag{3.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$ is the round metric on the unit sphere. In these coordinates, the wave equation is

$$\Box_g \phi = 0 \Leftrightarrow -\partial_t^2 \phi + \partial_r^2 \phi + \frac{2}{r} \partial_r \phi + \frac{1}{r^2} \Delta_{\mathbb{S}^2} \phi = 0 \,, \tag{3.2}$$

where $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator on the unit sphere.

In this case, the timelike Killing field is $T = \partial_t$. The initial data is imposed on the hypersurface $\Sigma_0 = \{t = 0\}$, so we have $\Sigma_\tau = \{t = \tau\}$. As initial conditions, we set $\phi\big|_{\Sigma_0} = \phi_0 \in C_c^\infty(\Sigma_0)$ and $\partial_t \phi\big|_{\Sigma_0} = \phi_1 \in C_c^\infty(\Sigma_0)$. Note that this immediately implies that, due to the geometry of the hypersurfaces Σ_τ and the domain of dependence property of the wave equation, $\phi(t, \cdot)$ has compact support in Σ_t, for all $t > 0$. In what follows, ϕ will always denote the solution to the Cauchy problem described here.

In Section 3.1, we will show that the solution of the wave equation is bounded. In Section 3.2, we will prove a certain decay property for the solution of the wave equation. In both Sections, we will completely avoid any properties relying on the known explicit formulas for the solutions of the wave equation, mostly making use of the fact that the Minkowski spacetime is static and spherically symmetric, so that the same techniques can be employed to prove similar results for the Schwarzschild spacetime.

3.1 Boundedness

To deduce that the solution of the wave equation is bounded, one should consider a non-increasing energy flux from which Sobolev norms of the function ϕ can be controlled, thus yielding pointwise bounds, by the Sobolev inequality. For this purpose, we consider the energy associated to the Killing field $T = \partial_t$, which satisfies energy conservation, as seen in Section 2.2.

Computing the energy current associated to the vector field $T = \partial_t$, we have

$$
\begin{aligned}
J^T_\mu[\phi] &= \Big(\partial_\nu\phi\partial_\mu\phi - \frac{1}{2}(\partial^\alpha\phi\partial_\alpha\phi)g_{\nu\mu}\Big)(\partial_t)^\nu \\
&= \partial_t\phi\partial_\mu\phi - \frac{1}{2}\left(-(\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) g_{t\mu} \\
&= \partial_t\phi\partial_\mu\phi + \frac{1}{2}\left(-(\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) \delta_{t\mu}\,,
\end{aligned} \tag{3.3}
$$

where $\left|\nabla_{\mathbb{S}^2}\phi\right|^2 = (\partial_\theta\phi)^2 + \frac{1}{\sin^2\theta}(\partial_\varphi\phi)^2$. Therefore, given that the unit normal n_{Σ_t} is also ∂_t, this yields the energy density on the hypersurfaces Σ_t:

$$
\begin{aligned}
J^T_\mu[\phi]n^\mu_{\Sigma_t} &= \left(\partial_t\phi\partial_\mu\phi + \frac{1}{2}\left(-(\partial_t\phi)^2 + (\partial_r\phi)^2 + + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right)\delta_{t\mu}\right)(\partial_t)^\mu \\
&= \frac{1}{2}\left((\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right)\,.
\end{aligned} \tag{3.4}
$$

As seen in Section 2.2, the solution of the wave equation then satisfies energy conservation:

$$
\mathbb{E}^T[\phi](0) = \mathbb{E}^T[\phi](t) = \frac{1}{2}\int_{\Sigma_t}\left[(\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right] r^2\,dr\,\mathrm{dVol}_{\mathbb{S}^2}\,, \quad \forall t > 0\,. \tag{3.5}
$$

It turns out that one has to consider two cases and prove that ϕ is bounded for $r \geq R$ and $r \leq R$ separately, for an arbitrary fixed $R > 0$.

Let us then fix $R > 0$ and first prove that the solution to the wave equation is bounded for $r \geq R$. For arbitrary $t_0 > 0$, $r_0 \geq R$ and $\omega \in \mathbb{S}^2$, we have the following pointwise estimate, that follows directly from applying the fundamental theorem of calculus along the radial variable, at fixed $t = t_0$, and Cauchy-Schwarz's inequality:

$$
\begin{aligned}
\phi^2(t_0, r_0, \omega) &= \left(\int_{r_0}^\infty \partial_r\phi(t_0, r, \omega)\,dr\right)^2 \leq \left(\int_{r_0}^\infty (\partial_r\phi(t_0, r, \omega))^2\,r^2\,dr\right)\left(\int_{r_0}^\infty \frac{1}{r^2}\,dr\right) \\
&= \frac{1}{r_0}\int_{r_0}^\infty (\partial_r\phi(t_0, r, \omega))^2\,r^2\,dr \leq \frac{1}{R}\int_0^\infty (\partial_r\phi(t_0, r, \omega))^2\,r^2\,dr\,.
\end{aligned} \tag{3.6}
$$

Integrating the previous equation in $\mathbb{S}^2$, we obtain

$$
\int_{\mathbb{S}^2}\phi^2(t_0, r_0, \omega)\,\mathrm{dVol}_{\mathbb{S}^2}(\omega) \leq \frac{1}{R}\int_{\mathbb{S}^2}\int_0^\infty (\partial_r\phi(t_0, r, \omega))^2\,r^2\,dr\,\mathrm{dVol}_{\mathbb{S}^2}(\omega) \leq C_R\mathbb{E}^T[\phi](t_0)\,, \tag{3.7}
$$

where $C_R > 0$ is a constant that satisfies $C_R \to \infty$ as $R \to 0$.

Denote by Ω_i, $i = 1, ..., 3$, the three rotational Killing vector fields. Since these vector fields are Killing, they commute with the wave equation operator, and so $\Omega_i\phi$ and $\Omega_i\Omega_j\phi$ also satisfy the wave equation. Therefore, equation (3.7) also holds for $\Omega_i\phi$ and $\Omega_i\Omega_j\phi$ in place of ϕ. This implies that

$$
\begin{aligned}
\|\phi(t_0, r_0, \cdot)\|_{H^2(\mathbb{S}^2)} &\lesssim_R \left(\mathbb{E}^T[\phi](t_0) + \sum_i \mathbb{E}^T[\Omega_i\phi](t_0) + \sum_{i,j}\mathbb{E}^T[\Omega_i\Omega_j\phi](t_0)\right)^{1/2} \\
&= \left(\mathbb{E}^T[\phi](0) + \sum_i \mathbb{E}^T[\Omega_i\phi](0) + \sum_{i,j}\mathbb{E}^T[\Omega_i\Omega_j\phi](0)\right)^{1/2}\,,
\end{aligned} \tag{3.8}
$$

where we used the fact that the energies $\mathbb{E}^T[\Omega_i\phi]$ and $\mathbb{E}^T[\Omega_i\Omega_j\phi]$ are also conserved. Using the Sobolev inequality on $\mathbb{S}^2$ (equation (2.14)), we deduce that there exists a constant $E_0 > 0$, depending on $\|\phi_0\|_{H^3(\Sigma_0)}$ and $\|\phi_1\|_{H^2(\Sigma_0)}$, such that

$$\|\phi(t_0, r_0, \cdot)\|_{L^\infty(\mathbb{S}^2)} \lesssim_R E_0\,, \tag{3.9}$$

for all $t_0 > 0$ and $r_0 \geq R$. This yields a uniform pointwise bound for the solution of the wave equation ϕ, on the spheres that foliate the Minkwoski spacetime outside the radius R, as described in Section 2.3.

Because the constant in the previous estimate diverges as $R \to 0$, we need to prove another estimate for $r_0 \leq R$ to get boundedness on the whole spacetime. We first prove a Hardy-type inequality to control the L^2 norm of ϕ for $r \leq R$:

$$\int_0^\infty \phi^2\, dr = \left[r\phi^2\right]_{r=0}^{r=\infty} - 2\int_0^\infty r\phi\,\partial_r\phi\, dr \leq \frac{1}{2}\int_0^\infty \phi^2\, dr + 2\int_0^\infty \left(\partial_r\phi\right)^2 r^2\, dr\,. \tag{3.10}$$

Integrating on the unit sphere, we obtain

$$\int_{\Sigma_t} \frac{\phi^2}{r^2}\,\mathrm{dVol}_{\Sigma_t} \leq 4\int_{\Sigma_t} (\partial_r\phi)^2\,\mathrm{dVol}_{\Sigma_t} \lesssim \mathbb{E}^T[\phi](t)\,, \tag{3.11}$$

which implies that

$$\int_{\Sigma_t\cap\{r\leq R\}} \phi^2\,\mathrm{dVol}_{\Sigma_t} \lesssim_R \mathbb{E}^T[\phi](t) = \mathbb{E}^T[\phi](0)\,, \tag{3.12}$$

where the constant in the previous inequality diverges as $R \to \infty$. Let T_i, $i = 1, ..., 3$, denote three independent translation Killing vector fields. Equation (3.12) also holds for $T_i\phi$ and $T_iT_j\phi$ in place of ϕ, and so we have

$$\|\phi\|_{H^2(\Sigma_t\cap\{r\leq R\})} \lesssim_R \left(\mathbb{E}^T[\phi](0) + \sum_i \mathbb{E}^T[T_i\phi](0) + \sum_{i,j}\mathbb{E}^T[T_iT_j\phi](0)\right)^{1/2}\,, \tag{3.13}$$

where we also used that the energies $\mathbb{E}^T[T_i\phi]$ and $\mathbb{E}^T[T_iT_j\phi]$ are conserved. Since $\Sigma_t\cap\{r \leq R\}$ is a ball of radius R in $\mathbb{R}^3$, it satisfies the interior cone condition, and so we can apply Sobolev's inequality (2.15) to conclude that there exists a constant $\widetilde{E}_0 > 0$ depending, as for the $r > R$ case above, on $\|\phi_0\|_{H^3(\Sigma_0)}$ and $\|\phi_1\|_{H^2(\Sigma_0)}$, such that

$$\|\phi\|_{L^\infty(\Sigma_t\cap\{r\leq R\})} \lesssim_R \widetilde{E}_0\,. \tag{3.14}$$

The constants involved in estimates (3.9) and (3.14) only depend on R and the initial data, so we conclude that there is a constant $C > 0$ such that the following bound holds:

$$|\phi| \leq C\,. \tag{3.15}$$

Remark 3.1.1. *Due to the fact that the Minkowski spacetime is translation invariant, the bound in* (3.14) *is valid for arbitrary balls with fixed radius in Σ_t. Therefore, by applying it to balls of unit radius, for* *example, centered at any point of the spacetime, this estimate is sufficient to prove that the solution to the wave equation is globally bounded. However, we did not use this approach since it would not be useful in the context of the Schwarzschild spacetime, where translation invariance is not available.*

3.2 Decay

To prove pointwise decay of ϕ, we will first show an energy decay result and then apply estimates from Section 3.1 to get the desired result. Once again, we need to separate in the cases $r \leq R$ and $r \geq R$. For the former, we will prove what is usually called an Integrated Local Energy Decay (ILED) estimate, which is a certain energy estimate that captures the fact that energy in a ball with finite radius decays with time. For the latter case, we will establish the so-called Dafermos-Rodnianski hierarchy [9].

3.2.1 Energy estimates and decay

The goal of this work is to obtain pointwise results using energy estimates, since this has shown to be a robust method that can also be applied to more complicated spacetimes. However, as seen in Section 3.1, the energy flux through hypersurfaces of constant time is conserved, suggesting that we should actually consider different hypersurfaces in order to be able to obtain decay properties. The idea will be to pick surfaces that extend to infinity along null directions. Before defining these hypersurfaces, we introduce the so-called null coordinates (u, v):

$$\begin{aligned} u &= t - r\,, \\ v &= t + r\,. \end{aligned} \tag{3.16}$$

In these coordinates, the metric is given by

$$g = -du\, dv + r^2\, d\Omega^2\,. \tag{3.17}$$

In what follows, the derivatives ∂_u and ∂_v are taken with v and u constant, respectively, whereas ∂_t and ∂_r are taken with r and t constant. These vector fields satisfy the relations

$$\partial_u = \frac{1}{2}\left(\partial_t - \partial_r\right)\,, \qquad \partial_v = \frac{1}{2}\left(\partial_t + \partial_r\right)\,. \tag{3.18}$$

The wave equation in null coordinates is as follows:

$$\Box_g \phi = 0 \Leftrightarrow -4\partial_u \partial_v \phi + \frac{2}{r}\left(\partial_v \phi - \partial_u \phi\right) + \frac{1}{r^2}\Delta_{\mathbb{S}^2}\phi = 0\,. \tag{3.19}$$

To prove energy decay estimates, we consider hypersurfaces that capture the fact that energy is radiating away to infinity. We will construct them so that they maintain constant time up to a fixed radius, but from that point continue to infinity along the null direction of constant u. Observe that the solutions ϕ of the wave equation, in general, are no longer compactly supported on such surfaces, as the propagation at the speed of light makes them evolve to infinity precisely along the null directions.

Hence, we fix a value of the radius $R > 0$ large enough such that the support of the initial data is contained in $\{r < R\}$, and we define

$$\begin{aligned} \widetilde{\Sigma}_\tau &= \{t = \tau,\, r \le R\} \cup \{u = \tau - R,\, r \ge R\}\,, \\ N_\tau &= \widetilde{\Sigma}_\tau \cap \{r \ge R\}\,, \\ R_{\tau_1}^{\tau_2} &= \bigcup_{\tau_1 \le \tau \le \tau_2} \widetilde{\Sigma}_\tau\,, \\ D_{\tau_1}^{\tau_2} &= \bigcup_{\tau_1 \le \tau \le \tau_2} N_\tau\,, \end{aligned} \tag{3.20}$$

where (u, v) are the previously defined null coordinates (see Figure 3.1).[1]

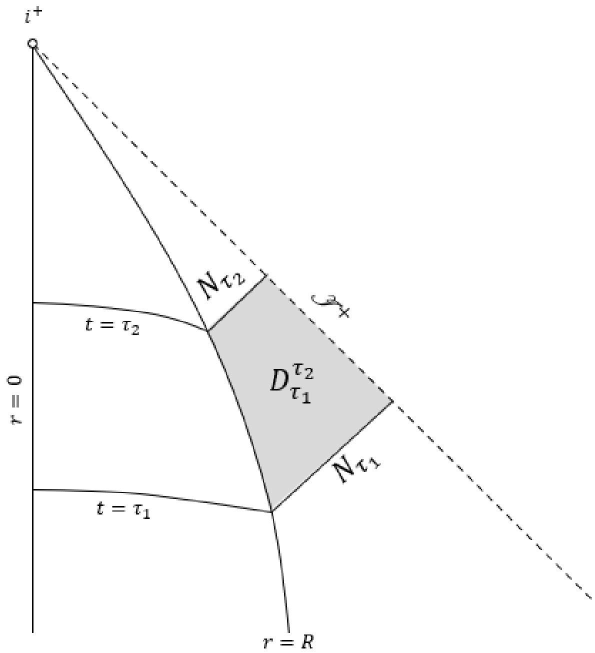

Figure 3.1: Hypersurfaces used in the proof of the energy decay estimates.

Let us now define the energy flux, associated to the vector field X, through $\widetilde{\Sigma}_\tau$ (recall the choice of normal for null surfaces, described in Section 2.1, so that $n_{N_\tau} = \partial_v$ and $\sigma = r^2 dv \wedge \mathrm{dVol}_{\mathbb{S}^2}$):

$$\widetilde{\mathbb{E}}^X[\phi](\tau) = \int_{\{t=\tau,\, r\le R\}} J^X_\mu[\phi]\,(\partial_t)^\mu\, r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2} + \int_{N_\tau} J^X_\mu[\phi]\,(\partial_v)^\mu\, r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2}\,, \tag{3.21}$$

which should coincide with the boundary terms arising in the divergence theorem applied to the energy current $J^X_\mu[\phi]$ on the region $R_{\tau_1}^{\tau_2}$. We also define the energy flux at null infinity as the limit of the flux across hypersurfaces with constant v:

$$\int_{\mathscr{I}^+} J^X_\mu[\phi]\,(\partial_u)^\mu\, r^2\, du\ \mathrm{dVol}_{\mathbb{S}^2} := \lim_{v_0\to+\infty} \int_{\{v=v_0\}} J^X_\mu[\phi]\,(\partial_u)^\mu\, r^2\, du\ \mathrm{dVol}_{\mathbb{S}^2}\ . \tag{3.22}$$

[1] This, and the ensuing figures, display so-called Penrose (or Penrose-Carter) diagrams, corresponding to two-dimensional conformal representations of the quotient $\mathcal{M}/SO(3)$ of spherically symmetric spacetimes as bounded subsets of $\mathbb{R}^{1+1}$, which preserve the causal structure, and whose boundary is a finite representation of conformal infinity. They are frequently used in Mathematical Relativity as convenient depictions of Minkowski and black hole spacetimes (see [19, 26] for detailed expositions on Penrose diagrams).

Given $\tau_2 \geq \tau_1$, an immediate application of the divergence theorem to the region $R_{\tau_1}^{\tau_2}$ with the current $J^T_\mu[\phi]$ implies the following energy identity:

$$\widetilde{\mathbb{E}}^T[\phi](\tau_1) = \widetilde{\mathbb{E}}^T[\phi](\tau_2) + \int_{\mathscr{I}^+ \cap \{\tau_1 - R \leq u \leq \tau_2 - R\}} \left((\partial_u \phi)^2 + \frac{1}{4r^2} |\nabla_{\mathbb{S}^2} \phi| \right) r^2 \, du \ \mathrm{dVol}_{\mathbb{S}^2} \,, \tag{3.23}$$

where, using (3.3) and $n_{N_\tau} = \partial_v = \frac{1}{2}(\partial_t + \partial_r)$ on the null component N_τ of the hypersurface $\widetilde{\Sigma}_\tau$, yields

$$\begin{aligned} J^T_\mu[\phi] \, (\partial_v)^\mu &= \left(\partial_t \phi \partial_\mu \phi + \frac{1}{2} \left(-(\partial_t \phi)^2 + (\partial_r \phi)^2 + + \frac{1}{r^2} |\nabla_{\mathbb{S}^2} \phi|^2 \right) \delta_{t\mu} \right) (\partial_v)^\mu \\ &= \frac{1}{4} (\partial_t \phi)^2 + \frac{1}{2} \partial_t \phi \, \partial_r \phi + \frac{1}{4} (\partial_r \phi)^2 + \frac{1}{4r^2} |\nabla_{\mathbb{S}^2} \phi|^2 \\ &= (\partial_v \phi)^2 + \frac{1}{4r^2} |\nabla_{\mathbb{S}^2} \phi|^2 \,, \end{aligned} \tag{3.24}$$

and the energy now takes the form

$$\begin{aligned} \widetilde{\mathbb{E}}^T[\phi](\tau) = & \frac{1}{2} \int_{\{t=\tau, \, r \leq R\}} \left((\partial_t \phi)^2 + (\partial_r \phi)^2 + \frac{1}{r^2} |\nabla_{\mathbb{S}^2} \phi|^2 \right) r^2 \, dr \ \mathrm{dVol}_{\mathbb{S}^2} + \\ & + \int_{N_\tau} \left((\partial_v \phi)^2 + \frac{1}{4r^2} |\nabla_{\mathbb{S}^2} \phi|^2 \right) r^2 \, dv \ \mathrm{dVol}_{\mathbb{S}^2} \ . \end{aligned} \tag{3.25}$$

Unfortunately, this only implies that

$$\widetilde{\mathbb{E}}^T[\phi](\tau_2) \leq \widetilde{\mathbb{E}}^T[\phi](\tau_1) \,, \quad \tau_2 \geq \tau_1 \geq 0. \tag{3.26}$$

Indeed, we no longer expect energy to be conserved for the hypersurfaces $\widetilde{\Sigma}_\tau$, as ϕ propagates to infinity along null directions and thus the boundary term in (3.23) accounts for the energy radiating to infinity. So we need a more careful analysis to prove that actually the energy flux through $\widetilde{\Sigma}_\tau$ decays with τ.

3.2.2 Integrated Local Energy Decay estimate

To prove an energy decay result, we will obtain estimates involving the integral over τ of the energy flux through $\widetilde{\Sigma}_\tau$. For instance, by using the mean value theorem for integrals, an inequality of the type

$$\int_{\tau_1}^{\tau_2} \widetilde{\mathbb{E}}^T[\phi](\tau) \, d\tau \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) \,, \tag{3.27}$$

would imply the existence of a sequence of times tending to infinity with exponentially decaying energies which, together with equation (3.26), would then lead to an exponential decay of the energy globally in time. Now, we know that such a strong result does not hold, so we need to deduce other integral-type inequalities similar to (3.27). However, it turns out that this inequality does hold if we replace the integrand on the left-hand side by the energy flux through any compact subset of $\widetilde{\Sigma}_\tau$. This kind of result is designated an Integrated Local Energy Decay estimate, and is quite useful when deducing decay results, as it implies that the energy flux through compact subsets of $\widetilde{\Sigma}_\tau$ is integrable as a function of τ, and so it must decay to zero as $\tau \to \infty$.

Before moving on to the Integrated Local Energy Decay estimate, we need the following Hardy-type inequality, which provides an upper bound for an r-weighted L^2 norm of ϕ over $\widetilde{\Sigma}_\tau$. For the proof of this result, we follow the reasoning presented in [17].

Proposition 3.2.1. *If ϕ satisfies the wave equation with compactly supported initial data, then, for $\tau \geq 0$,*

$$\int_{\widetilde{\Sigma}_\tau} \phi^2 \, dr \; \mathrm{dVol}_{\mathbb{S}^2} \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau)\,. \tag{3.28}$$

Proof. To prove this inequality, we perform an integration by parts on both components of $\widetilde{\Sigma}_\tau$, as in (3.10), to obtain:

$$\begin{aligned}
&\int_{\widetilde{\Sigma}_\tau} \phi^2 \, dr \; \mathrm{dVol}_{\mathbb{S}^2} = \int_{\mathbb{S}^2}\int_0^R \phi^2(\tau, r, \omega)\, dr \; \mathrm{dVol}_{\mathbb{S}^2} + \int_{\mathbb{S}^2}\int_R^\infty \phi^2(\tau + r - R, r, \omega)\, dr \; \mathrm{dVol}_{\mathbb{S}^2} \\
&= -2\int_{\{t=\tau,\, r\leq R\}} r\phi\partial_r\phi \, dr \; \mathrm{dVol}_{\mathbb{S}^2} - 4\int_{N_\tau} r\phi\partial_v\phi \, dr \; \mathrm{dVol}_{\mathbb{S}^2} + \\
&+ \lim_{r\to\infty}\int_{\mathbb{S}^2} r\phi^2(\tau + r - R, r, \omega) \; \mathrm{dVol}_{\mathbb{S}^2} \\
&\leq \varepsilon^2 \int_{\{t=\tau,\, r\leq R\}} \phi^2 \, dr \; \mathrm{dVol}_{\mathbb{S}^2} + \frac{1}{\varepsilon^2}\int_{\{t=\tau,\, r\leq R\}} (\partial_r\phi)^2 r^2 \, dr \; \mathrm{dVol}_{\mathbb{S}^2} + 2\varepsilon^2\int_{N_\tau} \phi^2 \, dr \; \mathrm{dVol}_{\mathbb{S}^2} + \\
&+ \frac{2}{\varepsilon^2}\int_{N_\tau} (\partial_v\phi)^2 r^2 \, dr \; \mathrm{dVol}_{\mathbb{S}^2} + \lim_{r\to\infty}\int_{\mathbb{S}^2} r\phi^2(\tau + r - R, r, \omega) \; \mathrm{dVol}_{\mathbb{S}^2} \\
&\leq 2\varepsilon^2 \int_{\widetilde{\Sigma}_\tau} \phi^2 \, dr \; \mathrm{dVol}_{\mathbb{S}^2} + \frac{2}{\varepsilon^2}\widetilde{\mathbb{E}}^T[\phi](\tau) + \lim_{r\to\infty}\int_{\mathbb{S}^2} r\phi^2(\tau + r - R, r, \omega) \; \mathrm{dVol}_{\mathbb{S}^2} \; .
\end{aligned} \tag{3.29}$$

Taking $\varepsilon > 0$ sufficiently small, we have

$$\int_{\widetilde{\Sigma}_\tau} \phi^2 \, dr \; \mathrm{dVol}_{\mathbb{S}^2} \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau) + \lim_{r\to\infty}\int_{\mathbb{S}^2} r\phi^2(\tau + r - R, r, \omega) \; \mathrm{dVol}_{\mathbb{S}^2} \; . \tag{3.30}$$

We will now show that the second term on the right hand side of the previous inequality can be bounded by the energy flux through $\widetilde{\Sigma}_\tau$, and the conclusion will follow. First, we apply the fundamental theorem of calculus along a curve with constant $v = v_0$ (we now consider ϕ to be a function of the null coordinates (u, v)):

$$\begin{aligned}
&\int_{\mathbb{S}^2} \left(\phi(u_0, v_0, \omega) - \phi(-v_0, v_0, \omega)\right)^2 \; \mathrm{dVol}_{\mathbb{S}^2} \\
&= \int_{\mathbb{S}^2} \left(\int_{-v_0}^{u_0} \partial_u\phi(u, v_0, \omega)\, du\right)^2 \; \mathrm{dVol}_{\mathbb{S}^2} \\
&\leq \int_{\mathbb{S}^2} \left(\int_{-v_0}^{u_0} \frac{1}{r^2(u, v_0)}\, du\right)\left(\int_{-v_0}^{u_0} \left(\partial_u\phi(u, v_0, \omega)\right)^2 \, r^2(u, v_0)\, du\right) \mathrm{dVol}_{\mathbb{S}^2} \\
&\lesssim \left(\frac{1}{v_0 - u_0} - \frac{1}{2v_0}\right)\mathbb{E}^T[\phi](0)\,,
\end{aligned} \tag{3.31}$$

where we used the fact that

$$\int_{\mathbb{S}^2}\int_{-v_0}^{u_0} \left(\partial_u\phi(u, v_0, \omega)\right)^2 r^2(u, v_0)\, du \; \mathrm{dVol}_{\mathbb{S}^2} \lesssim \mathbb{E}^T[\phi](0)\,. \tag{3.32}$$

This inequality can be easily obtained by applying the divergence theorem on the spacetime region delimited by the hypersurfaces Σ_0, $\{v = v_0, -v_0 \leq u \leq u_0\}$ and $\Sigma_{\frac{u_0+v_0}{2}}$ (see Figure 3.2).

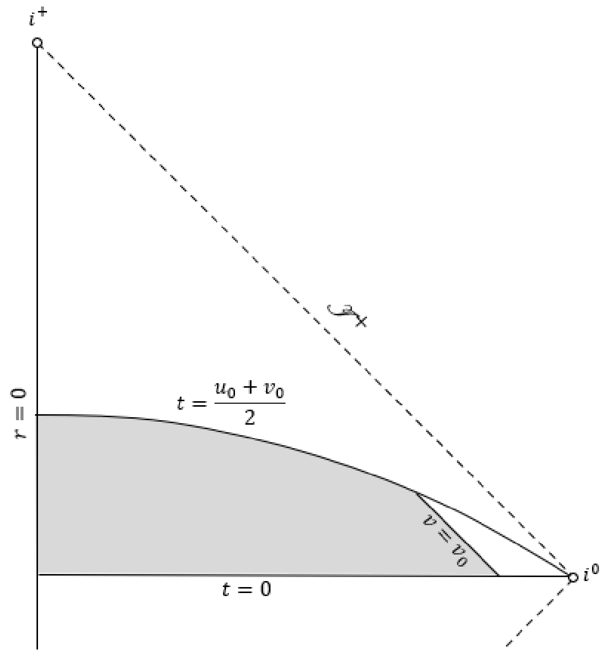

Figure 3.2: Hypersurfaces used in the proof of estimate (3.32).

Since ϕ has compact support on Σ_0, we have $\phi(-v_0, v_0, \omega) = 0$ for v_0 large enough. Hence, equation (3.31) implies that

$$\lim_{v_0\to+\infty} \int_{\mathbb{S}^2} \phi^2(u_0, v_0, \omega)\ \mathrm{dVol}_{\mathbb{S}^2} = 0\,. \tag{3.33}$$

Commuting with the angular Killing vector fields and applying the Sobolev inequality on the unit sphere, we can actually conclude that

$$\lim_{v_0\to+\infty} \|\phi(u_0, v_0, \cdot)\|_{L^\infty(\mathbb{S}^2)} = 0\,. \tag{3.34}$$

All the ingredients can therefore be gathered to finally yield the bound:

$$\begin{aligned}
&\int_{\mathbb{S}^2} r(u_0,v_0)\phi^2(u_0,v_0,\omega)\ \mathrm{dVol}_{\mathbb{S}^2} \\
&= \int_{\mathbb{S}^2} \frac{v_0-u_0}{2}\left(\int_{v_0}^{\infty} \partial_v\phi(u_0,v,\omega)\,dv\right)^2 \mathrm{dVol}_{\mathbb{S}^2} \\
&\leq \int_{\mathbb{S}^2} \frac{v_0-u_0}{2}\left(\int_{v_0}^{\infty} \frac{1}{r^2(u_0,v)}\,dv\right)\left(\int_{v_0}^{\infty} (\partial_v\phi(u_0,v,\omega))^2\, r^2(u_0,v)\,dv\right) \mathrm{dVol}_{\mathbb{S}^2} \\
&= 2\int_{\mathbb{S}^2}\int_{v_0}^{\infty} (\partial_v\phi(u_0,v,\omega))^2\, r^2(u_0,v)\,dv\ \mathrm{dVol}_{\mathbb{S}^2} \lesssim \widetilde{\mathbb{E}}^T[\phi](u_0+R)\,.
\end{aligned} \tag{3.35}$$

□

We now have the tools to prove an inequality which is weaker than the one in equation (3.27), but is sufficient to deduce an Integrated Local Energy Decay estimate. The main idea behind the proof of the following result will be to apply the divergence theorem on the region $R_{\tau_1}^{\tau_2}$ with a modified energy current, for which the energy bulk term is a semi-definite quadratic form of $(\phi, \partial_\mu\phi)$ and the energy flux through $\widetilde{\Sigma}_\tau$ can be bounded by $\widetilde{\mathbb{E}}^T[\phi](\tau)$. The coefficients of the aforementioned quadratic form will be positive functions of the radius that tend to zero as $r \to \infty$, thus only allowing to control the energy on compact subsets of $\widetilde{\Sigma}_\tau$. In what follows, we take the same approach as in [2].

Proposition 3.2.2. *If ϕ satisfies the wave equation with compactly supported initial data then, given arbitrary $\tau_2 \geq \tau_1 \geq 0$,*

$$\int_{R_{\tau_1}^{\tau_2}} \left(\frac{1}{(r+1)^2} (\partial_t \phi)^2 + \frac{1}{(r+1)^2} (\partial_r \phi)^2 + \frac{1}{r^3} |\nabla_{\mathbb{S}^2} \phi|^2 + \frac{1}{r(r+1)^3} \phi^2 \right) r^2 \, dt \, dr \, \mathrm{dVol}_{\mathbb{S}^2} \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) \,. \tag{3.36}$$

Proof. Consider the vector field

$$V = f(r) \partial_r \,, \tag{3.37}$$

where f is a function depending only on the radius. The energy current for this vector field is given by

$$\begin{aligned} J^V_r[\phi] &= \frac{f}{2} (\partial_t \phi)^2 + \frac{f}{2} (\partial_r \phi)^2 - \frac{f}{2} \frac{1}{r^2} |\nabla_{\mathbb{S}^2} \phi|^2 \ , \\ J^V_\mu[\phi] &= f \partial_\mu \phi \, \partial_r \phi \,, \quad \mu = t, \theta, \varphi \,. \end{aligned} \tag{3.38}$$

The deformation tensor is

$$\pi^V = \frac{1}{2} \mathcal{L}_{f(r)\partial_r} g = f' \, dr^2 + f \, r \, d\Omega^2 \,, \tag{3.39}$$

so that we have

$$K^V[\phi] = \left(\frac{f'}{2} + \frac{f}{r} \right) (\partial_t \phi)^2 + \left(\frac{f'}{2} - \frac{f}{r} \right) (\partial_r \phi)^2 - \frac{f'}{2} \frac{1}{r^2} |\nabla_{\mathbb{S}^2} \phi|^2 \ . \tag{3.40}$$

The bulk term K^V is not a positive semidefinite quadratic form of the derivatives of ϕ, so let us define the following modified energy current:

$$\tilde{J}^V_\mu[\phi] = J^V_\mu[\phi] + \left(\frac{f'}{4} + \frac{f}{2r} \right) \nabla_\mu \left(\phi^2 \right) - \nabla_\mu \left(\frac{f'}{4} + \frac{f}{2r} \right) \phi^2 \,. \tag{3.41}$$

Considering that, for a function F depending only on the radius, $\Box_g F = F'' + \frac{2F'}{r}$, we have

$$\begin{aligned} \tilde{K}^V[\phi] := \nabla^\mu \tilde{J}^V_\mu[\phi] &= K^V[\phi] + 2 \left(\frac{f'}{4} + \frac{f}{2r} \right) \nabla^\mu \phi \nabla_\mu \phi - \left(\frac{f'''}{4} + \frac{f''}{r} \right) \phi^2 \\ &= f' \left(\partial_r \phi \right)^2 + \frac{f}{r} \frac{1}{r^2} |\nabla_{\mathbb{S}^2} \phi|^2 - \left(\frac{f'''}{4} + \frac{f''}{r} \right) \phi^2 \,. \end{aligned} \tag{3.42}$$

We now apply the divergence theorem to the region $R_{\tau_1}^{\tau_2} \cap \{r \geq \varepsilon\}$ and then let $\varepsilon \to 0$. For $\varepsilon < R$, we have

$$
\begin{aligned}
&\int_{R_{\tau_1}^{\tau_2}\cap\{r\geq\varepsilon\}} \tilde{K}^V[\phi]\; r^2\,dt\,dr\;\mathrm{dVol}_{\mathbb{S}^2}\\
&=\int_{\{t=\tau_1,\,\varepsilon\leq r\leq R\}} \tilde{J}^V_\mu[\phi]\,(\partial_t)^\mu\, r^2\,dr\;\mathrm{dVol}_{\mathbb{S}^2}+\int_{N_{\tau_1}} \tilde{J}^V_\mu[\phi]\,(\partial_v)^\mu\, r^2\,dv\;\mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{\{t=\tau_2,\,\varepsilon\leq r\leq R\}} \tilde{J}^V_\mu[\phi]\,(\partial_t)^\mu\, r^2\,dr\;\mathrm{dVol}_{\mathbb{S}^2}-\int_{N_{\tau_2}} \tilde{J}^V_\mu[\phi]\,(\partial_v)^\mu\, r^2\,dv\;\mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{R_{\tau_1}^{\tau_2}\cap\{r=\varepsilon\}} \tilde{J}^V_\mu[\phi]\,(\partial_r)^\mu\;\mathrm{dVol}_{\{r=\varepsilon\}}-\int_{\mathscr{I}^+\cap\{\tau_1-R\leq u\leq\tau_2-R\}} \tilde{J}^V_\mu[\phi]\,(\partial_u)^\mu\; r^2\,du\;\mathrm{dVol}_{\mathbb{S}^2}\;.
\end{aligned}
\tag{3.43}
$$

Setting $f=1$, we obtain

$$
\begin{aligned}
&\tilde{K}^V[\phi]=\frac{1}{r^3}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\;,\\
&\tilde{J}^V_t[\phi]=\partial_t\phi\,\partial_r\phi+\frac{1}{r}\phi\,\partial_t\phi\,,\\
&\tilde{J}^V_r[\phi]=\frac{1}{2}\left(\partial_t\phi\right)^2+\frac{1}{2}\left(\partial_r\phi\right)^2-\frac{1}{2r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2+\frac{1}{r}\phi\,\partial_r\phi+\frac{1}{2r^2}\phi^2\,.
\end{aligned}
\tag{3.44}
$$

Using Young's inequality, inequality (3.26) and Proposition 3.2.1, we can control the integrals over $\widetilde{\Sigma}_{\tau_i}$, for $i\in\{1,2\}$, as follows:

$$
\begin{aligned}
&\left|\int_{\{t=\tau_i,\,r\leq R\}} \tilde{J}^V_t[\phi]\,r^2\,dr\;\mathrm{dVol}_{\mathbb{S}^2}+\int_{N_{\tau_i}}\left(\tilde{J}^V_t[\phi]+\tilde{J}^V_r[\phi]\right)r^2\,dr\;\mathrm{dVol}_{\mathbb{S}^2}\right|\\
&\lesssim\int_{\{t=\tau_i,\,r\leq R\}}\left((\partial_t\phi)^2+(\partial_r\phi)^2+\frac{\phi^2}{r^2}\right)r^2\,dr\;\mathrm{dVol}_{\mathbb{S}^2}+\int_{N_{\tau_i}}\left((\partial_v\phi)^2+\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2+\frac{\phi^2}{r^2}\right)r^2\,dr\;\mathrm{dVol}_{\mathbb{S}^2}\\
&\lesssim\int_{\{t=\tau_i,\,r\leq R\}}\left((\partial_t\phi)^2+(\partial_r\phi)^2\right)r^2\,dr\;\mathrm{dVol}_{\mathbb{S}^2}+\int_{N_{\tau_i}}\left((\partial_v\phi)^2+\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right)r^2\,dr\;\mathrm{dVol}_{\mathbb{S}^2}+\widetilde{\mathbb{E}}^T[\phi](\tau_i)\\
&\lesssim\widetilde{\mathbb{E}}^T[\phi](\tau_i)\leq\widetilde{\mathbb{E}}^T[\phi](\tau_1)\,.
\end{aligned}
\tag{3.45}
$$

Since $\mathrm{dVol}_{\{r=\varepsilon\}}=\varepsilon^2\,dt\;\mathrm{dVol}_{\mathbb{S}^2}$, we have

$$
\begin{aligned}
&\lim_{\varepsilon\to0}\int_{R_{\tau_1}^{\tau_2}\cap\{r=\varepsilon\}}\tilde{J}^V_r[\phi]\;\mathrm{dVol}_{\{r=\varepsilon\}}\\
&=\lim_{\varepsilon\to0}\int_{R_{\tau_1}^{\tau_2}\cap\{r=\varepsilon\}}\left(-\frac{1}{2\varepsilon^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2+\frac{1}{2\varepsilon^2}\phi^2\right)\mathrm{dVol}_{\{r=\varepsilon\}}\\
&=\int_{\tau_1}^{\tau_2}\int_{\mathbb{S}^2}\left(-\frac{1}{2}\left|\nabla_{\mathbb{S}^2}\phi(r=0)\right|^2+\frac{1}{2}\phi^2(r=0)\right)\,dt\;\mathrm{dVol}_{\mathbb{S}^2}\\
&=\;2\pi\int_{\tau_1}^{\tau_2}\phi^2(r=0)\,dt\,.
\end{aligned}
\tag{3.46}
$$

Regarding the integral at null infinity, one can show that it is also bounded by the initial energy:

$$
\begin{aligned}
&\left|\int_{\mathscr{I}^+\cap\{\tau_1-R\leq u\leq\tau_2-R\}}\frac{1}{2}\left(\tilde{J}^V_t[\phi]-\tilde{J}^V_r[\phi]\right)r^2\,du\;\mathrm{dVol}_{\mathbb{S}^2}\right|\\
&\lesssim\int_{\mathscr{I}^+\cap\{\tau_1-R\leq u\leq\tau_2-R\}}\left((\partial_u\phi)^2+\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2+\frac{\phi^2}{r^2}\right)r^2\,du\;\mathrm{dVol}_{\mathbb{S}^2}\\
&\lesssim\widetilde{\mathbb{E}}^T[\phi](\tau_1)+\int_{\mathscr{I}^+\cap\{\tau_1-R\leq u\leq\tau_2-R\}}\phi^2\,du\;\mathrm{dVol}_{\mathbb{S}^2}\;.
\end{aligned}
\tag{3.47}
$$

Recalling the definition of integral at null infinity, we obtain for the last integral

$$\begin{aligned}
&\int_{\mathscr{I}^+\cap\{\tau_1-R\le u\le\tau_2-R\}}\phi^2\,du\ \mathrm{dVol}_{\mathbb{S}^2}\\
&=\lim_{v_0\to\infty}\int_{\{\tau_1-R\le u\le\tau_2-R,\ v=v_0\}}\phi^2\,du\ \mathrm{dVol}_{\mathbb{S}^2}\\
&=\int_{\tau_1-R}^{\tau_2-R}\left(\lim_{v_0\to\infty}\int_{\mathbb{S}^2}\phi^2\big|_{v=v_0}\ \mathrm{dVol}_{\mathbb{S}^2}\right)du=0\,,
\end{aligned}\tag{3.48}$$

where we used equation (3.33) to both apply the Dominated Convergence Theorem and conclude that the integral is zero.

Letting $\varepsilon\to 0$ in equation (3.43), the previous estimates imply that

$$\int_{R_{\tau_1}^{\tau_2}}\frac{1}{r^3}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 r^2\,dt\,dr\ \mathrm{dVol}_{\mathbb{S}^2}+\int_{\tau_1}^{\tau_2}\phi^2(r=0)\,dt\lesssim\widetilde{\mathbb{E}}^T[\phi](\tau_1)\,,\tag{3.49}$$

which controls the angular derivatives. Note that we used

$$\lim_{\varepsilon\to0}\int_{\{t=\tau,\,r\le\varepsilon\}}\tilde{J}_t^V[\phi]\,r^2\,dr\ \mathrm{dVol}_{\mathbb{S}^2}=0\,,\tag{3.50}$$

due to the fact that $\tilde{J}_t^V[\phi]\sim\frac{1}{r}$ when $r\to0$.

We will now make a similar reasoning using different functions f to control the radial and time derivatives. First, we set $f(r)=-\frac{1}{r+1}$ to control the radial derivative and ϕ^2. In this case, we have

$$\begin{aligned}
\tilde{K}^V[\phi]&=\frac{1}{(r+1)^2}\left(\partial_r\phi\right)^2-\frac{1}{r^3(r+1)}\left|\nabla_{\mathbb{S}^2}\phi\right|^2+\frac{r+4}{2r(r+1)^4}\phi^2\,,\\
\tilde{J}_t^V[\phi]&=-\frac{1}{r+1}\partial_t\phi\,\partial_r\phi-\frac{r+2}{2r(r+1)^2}\phi\,\partial_t\phi\,,\\
\tilde{J}_r^V[\phi]&=-\frac{1}{2(r+1)}\left(\left(\partial_t\phi\right)^2+\left(\partial_r\phi\right)^2-\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right)-\frac{r+2}{2r(r+1)^2}\phi\,\partial_r\phi-\frac{r^2+3r+1}{2r^2(r+1)^3}\phi^2\,.
\end{aligned}\tag{3.51}$$

The energy current satisfies

$$\begin{aligned}
\left|\tilde{J}_t^V[\phi]\right|&\lesssim\left(\partial_t\phi\right)^2+\left(\partial_r\phi\right)^2+\frac{\phi^2}{r^2}\,,\\
\left|\tilde{J}_t^V[\phi]+\tilde{J}_r^V[\phi]\right|&\lesssim\left(\partial_v\phi\right)^2+\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2+\frac{\phi^2}{r^2}\,,\\
\left|\tilde{J}_t^V[\phi]-\tilde{J}_r^V[\phi]\right|&\lesssim\left(\partial_u\phi\right)^2+\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2+\frac{\phi^2}{r^2}\,,
\end{aligned}\tag{3.52}$$

implying that, as before, the boundary integrals on $\widetilde{\Sigma}_t$ and at null infinity can be controlled by the initial energy. Regarding the integral on the hypersurface $\{r=\varepsilon\}$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_{R_{\tau_1}^{\tau_2}\cap\{r=\varepsilon\}}\tilde{J}_r^V[\phi]\ \mathrm{dVol}_{\{r=\varepsilon\}}\\
&=\int_{\tau_1}^{\tau_2}\int_{\mathbb{S}^2}\left(\frac{1}{2}\left|\nabla_{\mathbb{S}^2}\phi(r=0)\right|^2-\frac{1}{2}\phi^2(r=0)\right)dt\ \mathrm{dVol}_{\mathbb{S}^2}\\
&=-2\pi\int_{\tau_1}^{\tau_2}\phi^2(r=0)\,dt\,.
\end{aligned}\tag{3.53}$$

In this case, we also have $\tilde{J}_t^V[\phi] \sim \frac{1}{r}$ when $r \to 0$, so equation (3.50) also holds. Therefore, the divergence theorem, together with equation (3.49), allows us to deduce the following estimate:

$$\begin{aligned}
&\int_{R_{\tau_1}^{\tau_2}} \left(\frac{1}{(r+1)^2} (\partial_r \phi)^2 + \frac{1}{r(r+1)^3} \phi^2 \right) r^2 \, dt \, dr \, \mathrm{dVol}_{\mathbb{S}^2} \\
&\lesssim \int_{R_{\tau_1}^{\tau_2}} \left(\frac{1}{(r+1)^2} (\partial_r \phi)^2 + \frac{r+4}{2r(r+1)^4} \phi^2 \right) r^2 \, dt \, dr \, \mathrm{dVol}_{\mathbb{S}^2} \\
&\lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{R_{\tau_1}^{\tau_2}} \frac{1}{r^3(r+1)} |\nabla_{\mathbb{S}^2} \phi|^2 \, r^2 \, dt \, dr \, \mathrm{dVol}_{\mathbb{S}^2} + 2\pi \int_{\tau_1}^{\tau_2} \phi^2(r=0) \, dt \\
&\lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{R_{\tau_1}^{\tau_2}} \frac{1}{r^3} |\nabla_{\mathbb{S}^2} \phi|^2 \, r^2 \, dt \, dr \, \mathrm{dVol}_{\mathbb{S}^2} + \int_{\tau_1}^{\tau_2} \phi^2(r=0) \, dt \\
&\lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) \, .
\end{aligned} \tag{3.54}$$

Finally, we will control the time derivative by setting $f(r) = \frac{r}{(r+1)^2}$. For this choice of f, we will use the non-modified currents, so that we have

$$\begin{aligned}
K^V[\phi] &= \frac{r+3}{2(r+1)^3} (\partial_t \phi)^2 - \frac{3r+1}{2(r+1)^3} (\partial_r \phi)^2 + \frac{r-1}{2r^2(r+1)^3} |\nabla_{\mathbb{S}^2} \phi|^2 \, , \\
J_t^V[\phi] &= \frac{r}{(r+1)^2} \partial_t \phi \, \partial_r \phi \, , \\
J_r^V[\phi] &= \frac{r}{2(r+1)^2} (\partial_t \phi)^2 + \frac{r}{2(r+1)^2} (\partial_r \phi)^2 - \frac{1}{2r(r+1)^2} |\nabla_{\mathbb{S}^2} \phi|^2 \, .
\end{aligned} \tag{3.55}$$

Applying the divergence theorem as before, there is no contribution from the boundary integral at $\{r = \varepsilon\}$, since $J_r^V[\phi] \sim \frac{1}{r}$ as $r \to 0$, implying that the corresponding limit is zero. The boundary integrals can be bounded by the initial energy once again, since the energy current satisfies

$$\begin{aligned}
\left| J_t^V[\phi] \right| &\lesssim (\partial_t \phi)^2 + (\partial_t \phi)^2 \, , \\
\left| J_t^V[\phi] + J_r^V[\phi] \right| &\lesssim (\partial_v \phi)^2 + \frac{1}{r^2} |\nabla_{\mathbb{S}^2} \phi|^2 \, , \\
\left| J_t^V[\phi] - J_r^V[\phi] \right| &\lesssim (\partial_u \phi)^2 + \frac{1}{r^2} |\nabla_{\mathbb{S}^2} \phi|^2 \, .
\end{aligned} \tag{3.56}$$

Therefore, we can control the time derivative as follows:

$$\begin{aligned}
&\int_{R_{\tau_1}^{\tau_2}} \frac{1}{(r+1)^2} (\partial_t \phi)^2 \, r^2 \, dt \, dr \, \mathrm{dVol}_{\mathbb{S}^2} \\
&\lesssim \int_{R_{\tau_1}^{\tau_2}} \frac{r+3}{2(r+1)^3} (\partial_t \phi)^2 \, r^2 \, dt \, dr \, \mathrm{dVol}_{\mathbb{S}^2} \\
&\lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{R_{\tau_1}^{\tau_2}} \left(\frac{3r+1}{2(r+1)^3} (\partial_r \phi)^2 + \frac{1-r}{2r^2(r+1)^3} |\nabla_{\mathbb{S}^2} \phi|^2 \right) r^2 \, dt \, dr \, \mathrm{dVol}_{\mathbb{S}^2} \\
&\lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{R_{\tau_1}^{\tau_2}} \left(\frac{1}{(r+1)^2} (\partial_r \phi)^2 + \frac{1}{r^3} |\nabla_{\mathbb{S}^2} \phi|^2 \right) r^2 \, dt \, dr \, \mathrm{dVol}_{\mathbb{S}^2} \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) \, .
\end{aligned} \tag{3.57}$$

Putting equations (3.49), (3.54) and (3.57) together we obtain the desired estimate.

□

Proposition 3.2.2 immediately implies the Integrated Local Energy Decay estimate:

Corollary 3.2.3. ***(Integrated Local Energy Decay)*** *Let $R_0 > 0$ and $\tau_2 \geq \tau_1 \geq 0$. If ϕ satisfies the wave* *equation with compactly supported initial data, then*

$$\int_{R_{\tau_1}^{\tau_2}\cap\{r\leq R_0\}} \left((\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 + \phi^2\right) r^2\, dt\, dr\, \mathrm{dVol}_{\mathbb{S}^2} \lesssim_{R_0} \widetilde{\mathbb{E}}^T[\phi](\tau_1)\,. \tag{3.58}$$

3.2.3 Dafermos-Rodnianski hierarchy

In this Section we will follow [9] to obtain a pair of inequalities, called the Dafermos-Rodnianski hierarchy, by making use of the Integrated Local Energy Decay estimate proved in Section 3.2.2. This result relates the energy fluxes through $\widetilde{\Sigma}_\tau$ and N_τ in such a way that allows us to prove an energy decay estimate.

Since the estimate in Corollary 3.2.3 already controls the energy flux for $r \leq R$, we now focus on the spacetime region $D_{\tau_1}^{\tau_2}$. Therefore, we start by proving an r-weighted energy estimate satisfied by the solution of the wave equation, by applying the vector field method on $D_{\tau_1}^{\tau_2}$ with an appropriate multiplier.

Proposition 3.2.4. *Let $p \in \mathbb{R}$ and define the radiation field $\psi = r\phi$, where ϕ satisfies the wave equation with compactly supported initial data. Then, given $\tau_2 \geq \tau_1 \geq 0$, there exists a constant $C > 0$, independent of τ_1 and τ_2, such that*

$$\begin{aligned}\int_{N_{\tau_2}} r^{p-2}\,(\partial_v\psi)^2\, r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2} + \int_{D_{\tau_1}^{\tau_2}} \left(p\, r^{p-3}\,(\partial_v\psi)^2 + \frac{2-p}{4} r^{p-1}\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dt\, dr\ \mathrm{dVol}_{\mathbb{S}^2} \\ \leq C\widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{N_{\tau_1}} r^{p-2}\,(\partial_v\psi)^2\, r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2}\ .\end{aligned} \tag{3.59}$$

Proof. Consider a smooth cut-off function ζ depending only on the radius function r, satisfying $\zeta(r) = 0$ for $r \leq R + 1/2$ and $\zeta(r) = 1$ for $r \geq R + 1$. Let V be the vector field defined as

$$V = r^q\partial_v\,, \tag{3.60}$$

where $q = p - 2$. Applying the divergence theorem to the current $J_\mu^V[\zeta\psi]$ in the region $D_{\tau_1}^{\tau_2}$, we have

$$\begin{aligned}&\int_{D_{\tau_1}^{\tau_2}} \left(K^V[\zeta\psi] + \Box_g(\zeta\psi)(V(\zeta\psi))\right) r^2\, dt\, dr\ \mathrm{dVol}_{\mathbb{S}^2} \\ &= \int_{N_{\tau_1}} J_\mu^V[\zeta\psi]\,(\partial_v)^\mu\ dv\ \mathrm{dVol}_{\mathbb{S}^2} - \int_{N_{\tau_2}} J_\mu^V[\zeta\psi]\,(\partial_v)^\mu\ dv\ \mathrm{dVol}_{\mathbb{S}^2} - \\ &\quad - \int_{\mathscr{I}^+\cap\{\tau_1 - R\leq u\leq\tau_2 - R\}} J_\mu^V[\zeta\psi]\,(\partial_u)^\mu\ du\ \mathrm{dVol}_{\mathbb{S}^2}\ .\end{aligned} \tag{3.61}$$

Note that there is no boundary integral term in the hypersurface $\{r = R\}$, since we have $J_\mu^V[\zeta\psi] = 0$ for $r \leq R + 1/2$. Now, let us first compute the bulk term. Considering the metric in null coordinates, the deformation tensor of V is given by

$$\pi^V = \frac{1}{4}qr^{q-1}\, du^2 - \frac{1}{4}qr^{q-1}\, du\, dv + \frac{r^{q+1}}{2}\, d\Omega^2\,. \tag{3.62}$$

Using this formula, one can compute the first part of the bulk term

$$K^V[\zeta\psi] = 2r^{q-1}\partial_u(\zeta\psi)\partial_v(\zeta\psi) + qr^{q-1}\left(\partial_v(\zeta\psi)\right)^2 - q\frac{r^{q-1}}{4}\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\zeta\psi\right|^2 , \tag{3.63}$$

which can be written as the similar bulk term for ψ plus error terms supported on $\{r \leq R+1\}$:

$$\begin{aligned} K^V[\zeta\psi] =& K^V[\psi] - (1-\zeta^2)K^V[\psi] + 2r^{q-1}(\partial_u\zeta)(\partial_v\zeta)\psi^2 + 2r^{q-1}(\partial_u\zeta)\zeta(\partial_v\psi)\psi + \\ &+ 2r^{q-1}(\partial_v\zeta)\zeta(\partial_u\psi)\psi + qr^{q-1}(\partial_v\zeta)^2\psi^2 + 2qr^{q-1}(\partial_v\zeta)\zeta(\partial_v\psi)\psi =: K^V[\psi] + Z_1[\zeta,\psi] . \end{aligned} \tag{3.64}$$

Since ζ is smooth, all coefficients in the definition of $Z_1[\zeta,\psi]$ are bounded; by applying Young's inequality, we then have, for $R \leq r \leq R+1$,

$$\begin{aligned} |Z_1[\zeta,\psi]| &\lesssim_R (\partial_u\psi)^2 + (\partial_v\psi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\psi\right|^2 + \phi^2 \\ &\lesssim_R (\partial_u\phi)^2 + (\partial_v\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 + \phi^2 \\ &\sim (\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 + \phi^2 . \end{aligned} \tag{3.65}$$

We now follow a similar procedure for the second part of the bulk term. First note that, since ϕ satisfies the wave equation, ψ satisfies

$$-4\partial_u\partial_v\psi + \frac{1}{r^2}\Delta_{\mathbb{S}^2}\psi = 0 . \tag{3.66}$$

Hence, we see that

$$\begin{aligned} \Box_g(\zeta\psi) &= -4\partial_u\partial_v(\zeta\psi) + \frac{2}{r}\left(\partial_v(\zeta\psi) - \partial_u(\zeta\psi)\right) + \frac{1}{r^2}\Delta_{\mathbb{S}^2}(\zeta\psi) \\ &= \zeta\frac{2}{r}\left(\partial_v\psi - \partial_u\psi\right) - 4\psi\partial_u\partial_v\zeta - 4\partial_u\zeta\partial_v\psi - 4\partial_v\zeta\partial_u\psi + \frac{2}{r}\left(\partial_v\zeta - \partial_u\zeta\right)\psi . \end{aligned} \tag{3.67}$$

Therefore, we can write the second part of the bulk term as

$$\Box_g(\zeta\psi)\left(V(\zeta\psi)\right) = 2r^{q-1}(\partial_v\psi)^2 - 2r^{q-1}\partial_u\psi\partial_v\psi + Z_2[\zeta,\psi] , \tag{3.68}$$

where $Z_2[\zeta,\psi]$ is an error term supported on $\{r \leq R+1\}$, given by

$$\begin{aligned} Z_2[\zeta,\psi] = &\left(-4\psi\partial_u\partial_v\zeta - 4\partial_u\zeta\partial_v\psi - 4\partial_v\zeta\partial_u\psi + \frac{2}{r}\left(\partial_v\zeta - \partial_u\zeta\right)\psi\right)\left(r^q\psi\partial_v\zeta + r^q\zeta\partial_v\psi\right) + \\ &+ 2r^{q-1}\zeta\psi\partial_v\zeta\left(\partial_v\psi - \partial_u\psi\right) - (1-\zeta^2)\left(2r^{q-1}\left(\partial_v\psi\right)^2 - 2r^{q-1}\partial_u\psi\partial_v\psi\right) . \end{aligned} \tag{3.69}$$

Just like in the case of $Z_1[\zeta,\psi]$, one easily shows that, for $R \leq r \leq R+1$,

$$|Z_2[\zeta,\psi]| \lesssim_R (\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 + \phi^2 . \tag{3.70}$$

Using the Integrated Local Energy Decay estimate in Corollary 3.2.3, one then proves that the error terms can be bounded by the initial energy:

$$\begin{aligned}&\int_{D^{\tau_2}_{\tau_1}} \left(|Z_1[\zeta,\psi]| + |Z_2[\zeta,\psi]|\right) r^2\, dt\, dr\, \mathrm{dVol}_{\mathbb{S}^2}\\ &\lesssim_R \int_{D^{\tau_2}_{\tau_1}\cap\{R\le r\le R+1\}} \left((\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 + \phi^2\right) r^2\, dt\, dr\, \mathrm{dVol}_{\mathbb{S}^2}\\ &\lesssim_R \widetilde{\mathbb{E}}^T[\phi](\tau_1)\,.\end{aligned} \tag{3.71}$$

The remaining part of the bulk term is given by

$$K^V[\psi] + 2r^{q-1}(\partial_v\psi)^2 - 2r^{q-1}\partial_u\psi\partial_v\psi = (q+2)r^{q-1}(\partial_v\psi)^2 - q\frac{r^{q+1}}{4}\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\,. \tag{3.72}$$

We use a similar procedure to control the error terms arising in the boundary integrals:

$$\begin{aligned}&J^V_\mu[\zeta\psi](\partial_v)^\mu = r^q(\partial_v(\zeta\psi))^2 =\\ &= r^q(\partial_v\psi)^2 - (1-\zeta^2)r^q(\partial_v\psi)^2 + r^q(\partial_v\zeta)^2\psi^2 + 2r^q(\partial_v\zeta)\zeta(\partial_v\psi)\psi =: r^q(\partial_v\psi)^2 + Z_3[\zeta,\psi]\,,\end{aligned} \tag{3.73}$$

where $Z_3[\zeta,\psi]$ is an error term supported on $\{r\le R+1\}$. Applying Young's inequality and the fact that ζ is smooth, we have, for $R\le r\le R+1$,

$$|Z_3[\zeta,\psi]| \lesssim_R (\partial_v\phi)^2 + \frac{\phi^2}{r^2}\,. \tag{3.74}$$

Using equation (3.26) and Proposition 3.2.1, we have

$$\begin{aligned}&\int_{N_{\tau_i}} |Z_3[\zeta,\psi]r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2}\\ &\lesssim_R \int_{N_{\tau_i}\cap\{R\le r\le R+1\}} \left((\partial_v\phi)^2 + \frac{\phi^2}{r^2}\right) r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2} \lesssim_R \widetilde{\mathbb{E}}^T[\phi](\tau_1)\end{aligned} \tag{3.75}$$

for $i=1,2$. Finally, we just have to notice that the integral over $\mathscr{I}^+$ is non-negative:

$$\begin{aligned}J^V_\mu[\zeta\psi](\partial_u)^\mu &= \partial_u(\zeta\psi)\partial_v(\zeta\psi) - \frac{1}{2}\left(-4\partial_u(\zeta\psi)\partial_v(\zeta\psi) + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\zeta\psi\right|^2\right)\times\left(-\frac{1}{2}\right)\\ &= \frac{1}{4r^2}\left|\nabla_{\mathbb{S}^2}\zeta\psi\right|^2 \ge 0\,.\end{aligned} \tag{3.76}$$

Using all these results in (3.61) allows us to conclude the result:

$$\begin{aligned}&\int_{N_{\tau_2}} r^q\,(\partial_v\psi)^2\, r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2} + \int_{D^{\tau_2}_{\tau_1}}\left((q+2)r^{q-1}(\partial_v\psi)^2 - q\frac{r^{q+1}}{4}\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dt\, dr\ \mathrm{dVol}_{\mathbb{S}^2}\\ &\le -\int_{N_{\tau_2}} Z_3[\zeta,\psi]r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2} - \int_{D^{\tau_2}_{\tau_1}}\left(Z_1[\zeta,\psi] + Z_2[\zeta,\psi]\right) r^2\, dt\, dr\ \mathrm{dVol}_{\mathbb{S}^2} +\\ &\qquad + \int_{N_{\tau_1}}\left(r^q\,(\partial_v\psi)^2 + Z_3[\zeta,\psi]\right) r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2}\\ &\le C_R\,\widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{N_{\tau_1}} r^q\,(\partial_v\psi)^2\, r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2}\ .\end{aligned} \tag{3.77}$$

□

As mentioned before, Proposition 3.2.4 is the main tool to deduce a pair of inequalities, known as the Dafermos-Rodnianski hierarchy, which will be crucial to obtain an energy decay estimate.

Proposition 3.2.5. ***(Dafermos-Rodnianski hierarchy)*** *Let ϕ be a solution of the wave equation with compactly supported initial data, and let $\psi = r\phi$. Then, given $\tau_2 \geq \tau_1 \geq 0$, there exists a constant $C > 0$, independent of τ_1 and τ_2, such that*

$$\begin{aligned} &\int_{\tau_1}^{\tau_2} \widetilde{\mathbb{E}}^T[\phi](\tau)\, d\tau \leq C\widetilde{\mathbb{E}}^T[\phi](\tau_1) + C\int_{N_{\tau_1}} \frac{(\partial_v\psi)^2}{r} r^2\, dv \ \mathrm{dVol}_{\mathbb{S}^2}\,, \\ &\int_{\tau_1}^{\tau_2} \left(\int_{N_\tau} \frac{(\partial_v\psi)^2}{r} r^2\, dv \ \mathrm{dVol}_{\mathbb{S}^2} \right) d\tau \leq C\widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{N_{\tau_1}} (\partial_v\psi)^2\, r^2\, dv \ \mathrm{dVol}_{\mathbb{S}^2}\ . \end{aligned} \tag{3.78}$$

Proof. The second inequality in the hierarchy follows trivially from setting $p = 2$ in Proposition 3.2.4. To prove the first inequality, we set $p = 1$ in Proposition 3.2.4 to obtain

$$\int_{D_{\tau_1}^{\tau_2}} \left((\partial_v\psi)^2 + \frac{1}{4} |\nabla_{\mathbb{S}^2}\phi|^2 \right) dt\, dr \ \mathrm{dVol}_{\mathbb{S}^2} \leq C\widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{N_{\tau_1}} \frac{(\partial_v\psi)^2}{r} r^2\, dv \ \mathrm{dVol}_{\mathbb{S}^2}\ . \tag{3.79}$$

Consider now the following computation regarding the first term on the left-hand side of this inequality:

$$\begin{aligned} &\int_{D_{\tau_1}^{\tau_2}} (\partial_v\psi)^2 \ dt\, dr \ \mathrm{dVol}_{\mathbb{S}^2} \\ &= \frac{1}{2} \int_{\tau_1 - R}^{\tau_2 - R} \int_{\mathbb{S}^2} \int_{u+2R}^{\infty} (\partial_v(r\phi))^2 \ dv \ \mathrm{dVol}_{\mathbb{S}^2} \ du \\ &= \frac{1}{2} \int_{\tau_1 - R}^{\tau_2 - R} \int_{\mathbb{S}^2} \int_{u+2R}^{\infty} \left((\partial_v\phi)^2\, r^2 + r\phi\partial_v\phi + \frac{1}{4}\phi^2 \right) dv \ \mathrm{dVol}_{\mathbb{S}^2} \ du \\ &= \frac{1}{2} \int_{\tau_1 - R}^{\tau_2 - R} \int_{\mathbb{S}^2} \int_{u+2R}^{\infty} \left((\partial_v\phi)^2\, r^2 + \frac{1}{2}\partial_v(r\phi^2) \right) dv \ \mathrm{dVol}_{\mathbb{S}^2} \ du\,. \end{aligned} \tag{3.80}$$

Notice that we cannot bound the boundary integral on $R_{\tau_1}^{\tau_2} \cap \{r = R\}$; therefore, instead of applying the Fundamental Theorem of Calculus on the second term of the previous equation, we make use of the cut-off function introduced in the proof of Proposition 3.2.4 as follows:

$$\begin{aligned} &\int_{\tau_1 - R}^{\tau_2 - R} \int_{\mathbb{S}^2} \int_{u+2R}^{\infty} \partial_v(r\phi^2)\, dv \ \mathrm{dVol}_{\mathbb{S}^2} \ du \\ &= \int_{\tau_1 - R}^{\tau_2 - R} \int_{\mathbb{S}^2} \int_{u+2R}^{\infty} \big(\partial_v(r\zeta\phi^2) + \partial_v(r(1-\zeta)\phi^2) \big) \ dv \ \mathrm{dVol}_{\mathbb{S}^2} \ du \\ &\geq \int_{\tau_1 - R}^{\tau_2 - R} \int_{\mathbb{S}^2} \int_{u+2R}^{\infty} \partial_v(r(1-\zeta)\phi^2)\, dv \ \mathrm{dVol}_{\mathbb{S}^2} \ du\,. \end{aligned} \tag{3.81}$$

The last expression is an error term supported on $\{r \leq R + 1\}$, which is bounded by the energy at τ_1 by applying Young's inequality and Corollary 3.2.3. Therefore, we have

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2} \left(\int_{N_\tau} \left((\partial_v \phi)^2 + \frac{1}{4r^2} \left| \nabla_{\mathbb{S}^2} \phi \right|^2 \right) r^2 \, dv \, \mathrm{dVol}_{\mathbb{S}^2} \right) d\tau \\
&\lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{N_{\tau_1}} \frac{(\partial_v \psi)^2}{r} r^2 \, dv \, \mathrm{dVol}_{\mathbb{S}^2} - \int_{\tau_1 - R}^{\tau_2 - R} \int_{\mathbb{S}^2} \int_{u+2R}^{\infty} \partial_v (r(1-\zeta)\phi^2) \, dv \, \mathrm{dVol}_{\mathbb{S}^2} \, du \\
&\lesssim_R \widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{N_{\tau_1}} \frac{(\partial_v \psi)^2}{r} r^2 \, dv \, \mathrm{dVol}_{\mathbb{S}^2} \; .
\end{aligned}
\tag{3.82}
$$

Adding the Integrated Local Energy Decay estimate in Corollary 3.2.3 with $R_0 = R$ to the previous inequality we get the first inequality of the hierarchy. □

Remark 3.2.6. *Note the similarity between the first inequality of the hierarchy and equation* (3.27)*. In the hierarchy, we have an additional term that is not bounded by the energy* $\widetilde{\mathbb{E}}^T[\phi](\tau_1)$*, essentially due to a factor of* r *in the integrand function, thus preventing us from obtaining an exponential decay estimate.*

3.2.4 Energy decay and pointwise estimate

We conclude this chapter by establishing an energy decay result, as a consequence of the Dafermos-Rodnianksi hierarchy, with a proof based on the mean value theorem for integrals, analogous to the reasoning that (3.27) leads to exponential decay. Afterwards, we deduce a pointwise decay estimate, in the same way as we used energy conservation to prove that the solution of the wave equation is bounded.

Theorem 3.2.7. *If* ϕ *satisfies the wave equation with compactly supported initial data then there exists a constant* $C > 0$ *such that*

$$
\widetilde{\mathbb{E}}^T[\phi](\tau) \le \frac{C}{\tau^2}, \qquad \forall \tau > 0 \, . \tag{3.83}
$$

Proof. Denote the integrals over N_τ in the hierarchy by

$$
\begin{aligned}
f_1(\tau) &= \int_{N_\tau} \frac{(\partial_v \psi)^2}{r} r^2 \, dv \, \mathrm{dVol}_{\mathbb{S}^2} \, , \\
f_2(\tau) &= \int_{N_\tau} (\partial_v \psi)^2 r^2 \, dv \, \mathrm{dVol}_{\mathbb{S}^2} \; .
\end{aligned}
\tag{3.84}
$$

Then the Dafermos-Rodnianski hierarchy can be expressed as

$$
\begin{aligned}
\int_{\tau_1}^{\tau_2} \widetilde{\mathbb{E}}^T[\phi] \, d\tau &\le C \widetilde{\mathbb{E}}^T[\phi](\tau_1) + C f_1(\tau_1) \, , \\
\int_{\tau_1}^{\tau_2} f_1(\tau) \, d\tau &\le C \widetilde{\mathbb{E}}^T[\phi](\tau_1) + f_2(\tau_1) \, .
\end{aligned}
\tag{3.85}
$$

First note that the energy decays with $1/\tau$:

$$
\tau \, \widetilde{\mathbb{E}}^T[\phi](\tau) = \int_0^\tau \widetilde{\mathbb{E}}^T[\phi](\tau) \, d\tau' \le \int_0^\tau \widetilde{\mathbb{E}}^T[\phi](\tau') \, d\tau' \le C \widetilde{\mathbb{E}}^T[\phi](0) + C f_1(0) \, . \tag{3.86}
$$

Setting R large enough so that the support of $\phi\big|_{\Sigma_0}$ is contained in $\{r < R\}$, we have $f_1(0) = f_2(0) = 0$, so that the energy satisfies

$$\widetilde{\mathbb{E}}^T[\phi](\tau) \leq \frac{C\widetilde{\mathbb{E}}^T[\phi](0)}{\tau}\,. \tag{3.87}$$

The second inequality of the hierarchy implies that the function f_1 is integrable on $[0, \infty)$:

$$\int_0^\infty f_1(\tau)\,d\tau \leq C\widetilde{\mathbb{E}}^T[\phi](0) < \infty\,. \tag{3.88}$$

Using the mean value theorem for integrals, define a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \in [2^n, 2^{n+1})$ and

$$f_1(\tau_n) = \frac{\int_{2^n}^{2^{n+1}} f_1(\tau)\,d\tau}{2^n}, \qquad \forall n \in \mathbb{N}\,. \tag{3.89}$$

Therefore, f_1 satisfies

$$f_1(\tau_n) \leq \frac{C\widetilde{\mathbb{E}}^T[\phi](0)}{2^n} \lesssim \frac{1}{2^{n+1}} \leq \frac{1}{\tau_n}\,, \tag{3.90}$$

where the constants do not depend on n. This immediately implies that

$$\widetilde{\mathbb{E}}^T[\phi](\tau_{n+2}) \leq \frac{\int_{\tau_n}^{\tau_{n+2}} \widetilde{\mathbb{E}}^T[\phi](\tau)\,d\tau}{\tau_{n+2} - \tau_n} \leq \frac{C\widetilde{\mathbb{E}}^T[\phi](\tau_n) + Cf_1(\tau_n)}{2^{n+2} - 2^{n+1}} \lesssim \frac{1}{\tau_n 2^n} \lesssim \frac{1}{\tau_{n+2}^2}\,. \tag{3.91}$$

Given $\tau > 0$, let $n \in \mathbb{N}$ be such that $\tau \in [2^n, 2^{n+1})$. Then,

$$\widetilde{\mathbb{E}}^T[\phi](\tau) \leq \widetilde{\mathbb{E}}^T[\phi](2^n) \leq \widetilde{\mathbb{E}}^T[\phi](\tau_{n-1}) \lesssim \frac{1}{\tau_{n-1}^2} \lesssim \frac{1}{\tau_{n+1}^2} \leq \frac{1}{\tau^2}\,. \tag{3.92}$$

□

Finally, we can use this energy decay to obtain a pointwise decay result, by proving that the L^∞ norm of ϕ^2 over $\widetilde{\Sigma}_\tau$ can be bounded by a sum of energy fluxes.

Theorem 3.2.8. *Let ϕ be a solution of the wave equation with compactly supported initial data. Then there exists a constant $C > 0$ such that*

$$\sup_{x\in\widetilde{\Sigma}_\tau} |\phi(x)| \leq \frac{C}{\tau}, \qquad \forall \tau > 0\,. \tag{3.93}$$

Proof. First note that Proposition 3.2.1 implies that

$$\int_{\{t=\tau,\, r\leq R\}} \phi^2 r^2\,dr\;\mathrm{dVol}_{\mathbb{S}^2} \lesssim_R \widetilde{\mathbb{E}}^T[\phi](\tau) \lesssim \frac{1}{\tau^2}\,. \tag{3.94}$$

Note that this equation also holds for $T_i\phi$ and $T_iT_j\phi$ in place of ϕ so we have

$$\|\phi\|^2_{H^2(\{t=\tau,\, r\leq R\})} \lesssim_R \frac{1}{\tau^2}\,. \tag{3.95}$$

Then, using the Sobolev inequality on the ball of radius R, we have

$$\|\phi\|_{L^\infty(\{t=\tau,\, r\leq R\})} \lesssim_R \frac{1}{\tau}\,. \tag{3.96}$$

For the points in N_τ, note that equation (3.35) implies that, for $r \geq R$,

$$\int_{\mathbb{S}^2} \phi(\tau + r - R, r, \omega)^2 \ \mathrm{dVol}_{\mathbb{S}^2} \lesssim_R \widetilde{\mathbb{E}}^T[\phi](\tau) \lesssim \frac{1}{\tau^2}\,. \tag{3.97}$$

This equation also holds for $\Omega_i\phi$ and $\Omega_i\Omega_j\phi$ in place of ϕ, so we have that, given $r \geq R$,

$$\|\phi(\tau + r - R, r, \cdot)\|^2_{H^2(\mathbb{S}^2)} \lesssim_R \frac{1}{\tau^2}\,. \tag{3.98}$$

Applying the Sobolev inequality on the unit sphere, we have

$$\|\phi(\tau + r - R, r, \cdot)\|_{L^\infty(\mathbb{S}^2)} \lesssim_R \frac{1}{\tau}\,, \tag{3.99}$$

and the result follows. □

The proof of the decay result presented in Theorem 3.2.8 only makes use of the translation invariance of Minkowski spacetime to obtain decay in a ball of finite radius centered at the origin. Therefore, it is to be expected that most of the techniques used up until now may be adapted to the Schwarzschild spacetime, since we aim to prove boundedness and decay of the solution to the wave equation in the domain of outer communications.

The decay estimate in Theorem 3.2.8 is not quite the classical decay result for the wave equation in flat spacetime, stated in Theorem 3.2.9, because it makes use of the hypersurfaces $\widetilde{\Sigma}_\tau$ instead of hypersurfaces Σ_τ of constant time.

Theorem 3.2.9. *Let ϕ be a solution of the wave equation with compactly supported initial data. Then there exists a constant $C > 0$ such that*

$$\sup_{x\in\Sigma_\tau} |\phi(x)| \leq \frac{C}{\tau}, \qquad \forall \tau > 0\,. \tag{3.100}$$

Remark 3.2.10. *Note that one cannot apply the same reasoning as for the boundedness result to derive the classical decay result for the solutions of the wave equation. This is due to the fact that the hypersurfaces $\widetilde{\Sigma}_\tau$, and consequently the energies $\widetilde{\mathbb{E}}^T[\phi](\tau)$, are not translation invariant, as opposed to the hypersurfaces of constant t used in the proof of boundedness of the solutions of the wave equation.*

Schwarzschild spacetime 4

Apart from the trivial Minkowski spacetime, the Schwarzschild spacetime was the first solution to be found for the Einstein vacuum equations,

$$R_{\mu\nu} = 0\,. \tag{4.1}$$

It describes the physical system composed by a non-rotating black hole without electric charge. It is a static and spherically symmetric spacetime, which is also asympotically flat.

4.1 Coordinates

4.1.1 (t,r) coordinates

To describe the region of this spacetime outside the black hole, one tipically chooses the usual time and radius coordinates, here denoted by (t, r). The metric is written in these coordinates as

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2\,, \tag{4.2}$$

where $m > 0$ is the mass of the black hole. In what follows, we consider the convention that ∂_t and ∂_r correspond to derivatives with r and t constant, unless otherwise indicated (by adding a subscript with the variable which is held constant).

Although these might be the most intuitive coordinates, they are not well defined for $r = 2m$, which corresponds to the black hole's event horizon. For this reason, it is often useful to consider other coordinate systems which can be extended across the horizon.

4.1.2 Regge-Wheeler coordinates

We now define a new set of coordinates by redefining the radial coordinate using the so called tortoise function,

$$r^* = r + 2m \ln(r - 2m)\,, \tag{4.3}$$

which satisfies

$$\frac{dr^*}{dr} = \frac{1}{1 - \frac{2m}{r}}\,. \tag{4.4}$$

In these coordinates, known as Regge-Wheeler coordinates, the metric can be written as

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right) dr^{*2} + r^2\, d\Omega^2 \,. \tag{4.5}$$

The tortoise coordinate transformation (4.3) is only valid for $r > 2m$ and despite, in these coordinates, the metric no longer exhibiting the singularity at the event horizon $r = 2m$, as in (4.2), it still degenerates there.

The coordinate vector fields satisfy the following relations:

$$\begin{aligned} (\partial_t)_{r^*} &= \partial_t \,, \\ (\partial_{r^*})_t &= \left(1 - \frac{2m}{r}\right) \partial_r \,. \end{aligned} \tag{4.6}$$

We always consider that ∂_{r^*} denotes partial derivation with constant t.

4.1.3 Lemaître coordinates

The Lemaître coordinates correspond to the following coordinate transformation:

$$t^* = t + 2m \ln(r - 2m) \,. \tag{4.7}$$

In the coordinates (t^*, r), the metric is written as follows:

$$g = -\left(1 - \frac{2m}{r}\right) dt^{*2} + \frac{4m}{r} dt^*\, dr + \left(1 + \frac{2m}{r}\right) dr^2 + r^2 d\Omega^2 \,. \tag{4.8}$$

Note now that, although the coordinate transformation is only valid for $r > 2m$, as in the Regge-Wheeler case, the metric in (4.8) is well defined for all $r > 0$, which is why it describes both the exterior and the interior of the black hole, as well as across the event horizon. For these coordinates, the following relations are satisfied:

$$\begin{aligned} (\partial_{t^*})_r &= \partial_t \,, \\ Y := (\partial_r)_{t^*} &= -\frac{2m/r}{1 - 2m/r} \partial_t + \partial_r \,. \end{aligned} \tag{4.9}$$

4.1.4 Ingoing Eddington-Finkelstein coordinates

We also define another useful coordinate system by redefining the time variable as

$$v = t + \int \left(1 - \frac{2m}{r}\right)^{-1} dr = t + r + 2m \ln(r - 2m) \,, \tag{4.10}$$

so that the metric can be written in the following form:

$$g = -\left(1 - \frac{2m}{r}\right) dv^2 + 2dv\, dr + r^2 d\Omega^2 \,. \tag{4.11}$$

These coordinates also have the advantage that they can be extended across the horizon to describe the interior of the black hole. The coordinate vector fields satisfy

$$
\begin{aligned}
&(\partial_v)_r = \partial_t \,, \\
&Z := (\partial_r)_v = -\left(1 - \frac{2m}{r}\right)^{-1} \partial_t + \partial_r = Y - \partial_t \,.
\end{aligned}
\tag{4.12}
$$

4.1.5 Null coordinates

Sometimes it is also useful to define a pair of null coordinates, whose level sets correspond to ingoing and outgoing light rays:

$$
\begin{aligned}
u &= t - \int \left(1 - \frac{2m}{r}\right)^{-1} dr = t - r - 2m \ln(r - 2m) \,, \\
v &= t + \int \left(1 - \frac{2m}{r}\right)^{-1} dr = t + r + 2m \ln(r - 2m) \,.
\end{aligned}
\tag{4.13}
$$

In null coordinates (u, v), the metric is given by

$$
g = -\left(1 - \frac{2m}{r}\right) du\, dv + r^2\, d\Omega^2 \,. \tag{4.14}
$$

Just like in the case of (t, r) coordinates, ∂_u and ∂_v denote derivation with constant v and u, respectively, unless otherwise indicated. These vector fields satisfy

$$
\begin{aligned}
\partial_u &= \frac{1}{2} \left(\partial_t - \partial_{r^*}\right) \,, \\
\partial_v &= \frac{1}{2} \left(\partial_t + \partial_{r^*}\right) \,.
\end{aligned}
\tag{4.15}
$$

4.2 Wave Equation on Schwarzschild Spacetime

In this Section, we consider the covariant wave equation on the Schwarzschild spacetime, $(\mathcal{M}, g)$. We start by working in the usual (t, r) coordinates. The wave equation in these coordinates takes the following form:

$$
\Box_g \phi = 0 \Leftrightarrow -\left(1 - \frac{2m}{r}\right)^{-1} \partial_t^2 \phi + \frac{1}{r^2} \partial_r \left(r^2 \left(1 - \frac{2m}{r}\right) \partial_r \phi\right) + \frac{1}{r^2} \Delta_{\mathbb{S}^2} \phi = 0 \tag{4.16}
$$

We focus on the exterior region of the black hole, which we denote by $\mathcal{R}$. The timelike vector field we consider is $T = \partial_t$. Once again, the initial data is imposed on $\Sigma_0 = \{t = 0\}$, so that $\Sigma_\tau = \{t = \tau\}$. As initial conditions, we set $\phi\big|_{\Sigma_0} = \phi_0 \in C_c^\infty(\Sigma_0)$ and $\partial_t \phi\big|_{\Sigma_0} = \phi_1 \in C_c^\infty(\Sigma_0)$. The domain of dependence property also holds for the Schwarzschild spacetime, so we have that $\phi(t, \cdot)$ has compact support in Σ_t. In particular, for fixed t, $\phi(t, r, \omega) \to 0$ as $r \to 2m$ and as $r \to \infty$, and the same holds for all derivatives. In this Chapter, ϕ will always denote the solution to the Cauchy problem described here.

Under these conditions, we aim to prove boundedness and decay of the solution of the wave equation, using a similar approach as to what was done for the Minkowski spacetime. However, an energy degeneracy phenomenon will occur at the horizon, posing additional problems that need to be addressed.

4.3 Boundedness

As was seen in the case of the Minkowski spacetime, we need to be careful when choosing the hypersurfaces we are working with. For the Schwarzschild spacetime, we will define hypersurfaces suited to prove both energy boundedness and decay estimates. For this purpose, just like we did for the Minkowski spacetime, it will be convenient to work with null hypersurfaces near null infinity. Unfortunately, near the horizon, some additional problems (related to a degeneracy phenomenon) arise with the energy flux through the level sets of t. Moreover, the energy on these hypersurfaces is never allowed to enter the black hole, and so we should not expect to find that the energy flux through constant time hypersurfaces (that presumably bounds ϕ pointwise) decays to zero.[1] Consequently, we will make use of spacelike hypersurfaces crossing $\mathscr{H}^+$, whose normal is timelike on the horizon, which will allow to define a non-degenerate energy flux satisfying the necessary properties to obtain decay. Specifically, we make the following definitions (see Figure 4.1):

$$\begin{aligned}
L_\tau &= \{t^* = \tau + 2m\ln(r_0 - 2m),\, 2m \le r \le r_0\}\,, \\
S_\tau &= \{t = \tau,\, r_0 \le r \le R\}\,, \\
N_\tau &= \{u = \tau - R - 2m\ln(R - 2m),\, r \ge R\}\,, \\
\widetilde{\Sigma}_\tau &= L_\tau \cup S_\tau \cup N_\tau\,, \\
R_{\tau_1}^{\tau_2} &= \bigcup_{\tau_1 \le \tau \le \tau_2} \widetilde{\Sigma}_\tau\,, \\
C_{\tau_1}^{\tau_2} &= \bigcup_{\tau_1 \le \tau \le \tau_2} L_\tau\,, \\
D_{\tau_1}^{\tau_2} &= \bigcup_{\tau_1 \le \tau \le \tau_2} N_\tau\,,
\end{aligned} \tag{4.17}$$

where r_0 and R are taken such that the supports of ϕ_0 and ϕ_1 are contained in $\{r_0 < r < R\}$. Note that, for these values of r_0 and R, the hypersurfaces $\widetilde{\Sigma}_\tau$ foliate the causal future of the support of the initial data, which is the region where the solution to the wave equation might be nonzero.

Given a vector field X, let us define the energy flux across these hypersurfaces:

$$\begin{aligned}
\widetilde{\mathbb{E}}^X[\phi](\tau) = &\int_{L_\tau} J^X_\mu[\phi] \left(\left(1 + \frac{2m}{r}\right) T^\mu - \frac{2m}{r} Y^\mu\right) r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2} + \\
&+ \int_{S_\tau} J^X_\mu[\phi]\, T^\mu \left(1 - \frac{2m}{r}\right)^{-1} r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2} + \int_{N_\tau} J^X_\mu[\phi]\, (\partial_v)^\mu\, r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2}\ .
\end{aligned} \tag{4.18}$$

In particular, for the timelike Killing field T, we have

[1] In fact, one cannot expect to prove a pointwise decay result using these hypersurfaces, as they accumulate on the event horizon, where the solution to the wave equation does not have to be zero.

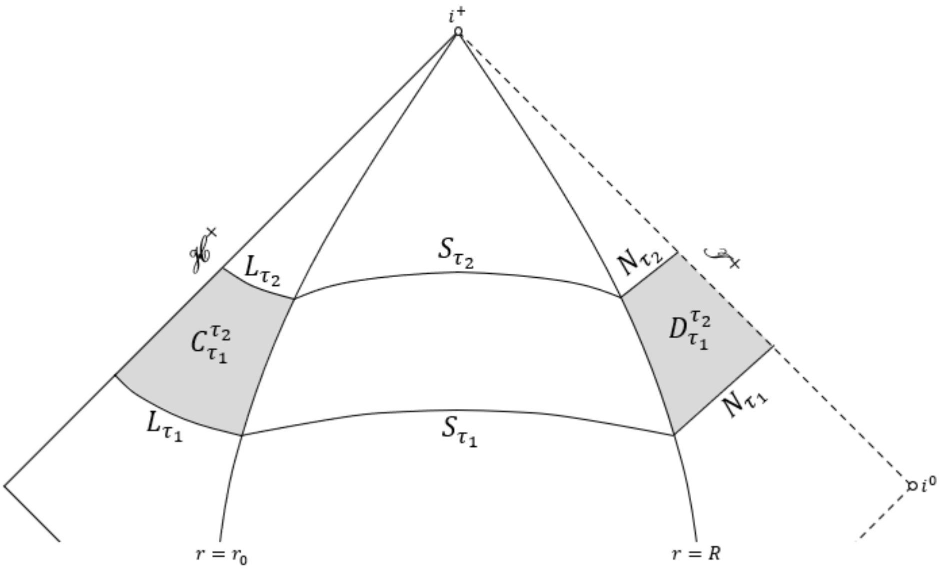

Figure 4.1: Hypersurfaces used in the proof of the energy boundedness and decay estimates.

$$\begin{aligned}\widetilde{\mathbb{E}}^T[\phi](\tau) = &\int_{L_\tau} \left(\frac{1}{2}\left(1+\frac{2m}{r}\right)(T\phi)^2 + \frac{1}{2}\left(1-\frac{2m}{r}\right)(Y\phi)^2 + \frac{1}{2r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2} + \\ &+ \int_{S_\tau} \left(\frac{1}{2}\left(1-\frac{2m}{r}\right)^{-1}(T\phi)^2 + \frac{1}{2}\left(1-\frac{2m}{r}\right)(\partial_r\phi)^2 + \frac{1}{2r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2} \\ &+ \int_{N_\tau} \left(2\left(1-\frac{2m}{r}\right)^{-1}(\partial_v\phi)^2 + \frac{1}{2r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2}\ .\end{aligned} \tag{4.19}$$

Note that the extra factor of 2 in the last integral is due to changing the integration variable from v to r, when comparing with the energy for the Minkowski spacetime in equation (3.25). The main difference one notices immediately is the aforementioned degeneracy phenomenon, since the term incorporating the radial derivative vanishes on the horizon. This makes it impossible to prove uniform bounds over the whole hypersurface $\widetilde{\Sigma}_\tau$, since we lose control of the radial derivative when we are arbitrarily close to the event horizon. The energy flux also satisfies

$$\begin{aligned}\widetilde{\mathbb{E}}^T[\phi](\tau) \sim &\int_{L_\tau} \left((T\phi)^2 + \left(1-\frac{2m}{r}\right)(Y\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2} + \\ &+ \int_{S_\tau} \left((T\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2} \\ &+ \int_{N_\tau} \left((\partial_v\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2}\ ,\end{aligned} \tag{4.20}$$

where the constants involved in the previous relation depend only on the fixed quantities r_0, R and m.

Given $\tau_2 \geq \tau_1 \geq 0$, an immediate consequence of the divergence theorem in the region delimited by $\widetilde{\Sigma}_{\tau_1}$ and $\widetilde{\Sigma}_{\tau_2}$ is that

$$\widetilde{\mathbb{E}}^T[\phi](\tau_2) \leq \widetilde{\mathbb{E}}^T[\phi](\tau_1)\,. \tag{4.21}$$

This inequality is true since the bulk term is zero and the boundary terms arising on the horizon and at null infinity have the correct sign.

Remark 4.3.1. *The flux of a vector field at null infinity is defined just like for the Minkowski spacetime in equation* (3.22)*, but now we consider* (u, v) *to be the null coordinates defined for the Schwarzschild spacetime.*

In what follows, we employ the procedure in [2] to try to find a vector field with an associated non-degenerate energy, whose bulk term should also be a positive definite quadratic form on the derivatives of ϕ near the event horizon. This multiplier will be designated the redshift vector field, since its construction depends crucially on the fact that the surface gravity of the event horizon of the Schwarzschild black hole is positive. For more insight on why the positivity of the surface gravity implies the occurrence of the redshift phenomenon see for instance [26].

Proposition 4.3.2. ***(Redshift vector field)*** *There exists* $r_1 > 2m$ *and a vector field* N *which satisfies the following conditions in the region* $\{2m \leq r \leq \min\{r_0, r_1\}\}$*:*

1. N *is timelike future-pointing;*
2. $K^N[\phi] \sim T_{\mu\nu}[\phi]N^\mu n^\nu_{L_\tau} \sim (T\phi)^2 + (Y\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2$.

Proof. For this proof, we will first work with ingoing Eddington-Finkelstein coordinates, since they are defined across the horizon and make the computations much simpler. Note that

$$(T\phi)^2 + (Z\phi)^2 \sim (T\phi)^2 + (Y\phi)^2 \,. \tag{4.22}$$

We make the following ansatz for N:

$$N = \alpha(r)T + \beta(r)Z \,. \tag{4.23}$$

Given such a vector field, we have

$$\begin{aligned}
&T_{\mu\nu}[\phi]N^\mu n^\nu_{L_\tau} \\
&\sim T_{\mu\nu}[\phi]N^\mu\left(\left(1+\frac{2m}{r}\right)T^\nu - \frac{2m}{r}Y^\nu\right) \\
&= T_{\mu\nu}[\phi](\alpha T + \beta Z)^\mu\left(T^\nu - \frac{2m}{r}Z^\nu\right) \\
&= \alpha(T\phi)^2 + \frac{1}{2}\left(\left(1-\frac{2m}{r}\right)\alpha - \left(1+\frac{2m}{r}\right)\beta\right)(Z\phi)^2 + \\
&\quad + \alpha\left(1-\frac{2m}{r}\right)(T\phi)(Z\phi) + \frac{1}{2}(\alpha-\beta)\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 \,.
\end{aligned} \tag{4.24}$$

The deformation tensor of N is

$$\pi^N = -\frac{m}{r^2}\beta\, dv^2 + \left(\beta' - \left(1-\frac{2m}{r}\right)\alpha'\right) dv\, dr + \alpha'\, dr^2 + r\beta\, d\Omega^2 \,, \tag{4.25}$$

and the energy bulk term is

$$
\begin{aligned}
K^N[\phi] =& \alpha'(T\phi)^2 + \left[\left(1-\frac{2m}{r}\right)\left(\frac{\beta'}{2}-\frac{\beta}{r}\right)-\frac{m}{r^2}\beta\right](Z\phi)^2 + \\
&+\left[\left(1-\frac{2m}{r}\right)\alpha' - \frac{2\beta}{r}\right](T\phi)(Z\phi) - \frac{1}{2}\beta'\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 .
\end{aligned}
\tag{4.26}
$$

At the horizon, we have

$$
\begin{aligned}
&T_{\mu\nu}[\phi]N^\mu\left(\left(1+\frac{2m}{r}\right)T^\nu - \frac{2m}{r}Y^\nu\right)\Big|_{r=2m} = \alpha(2m)(T\phi)^2 - \beta(2m)(Z\phi)^2 + \frac{1}{8m^2}(\alpha(2m)-\beta(2m))\left|\nabla_{\mathbb{S}^2}\phi\right|^2 , \\
&K^N[\phi]\big|_{r=2m} = \alpha'(2m)(T\phi)^2 - \frac{1}{4m}\beta(2m)(Z\phi)^2 - \frac{\beta(2m)}{m}(T\phi)(Z\phi) - \frac{1}{8m^2}\beta'(2m)\left|\nabla_{\mathbb{S}^2}\phi\right|^2 ,
\end{aligned}
\tag{4.27}
$$

which trivially satisfy

$$
\begin{aligned}
&T_{\mu\nu}[\phi]N^\mu\left(\left(1+\frac{2m}{r}\right)T^\nu - \frac{2m}{r}Y^\nu\right)\Big|_{r=2m} \lesssim (T\phi)^2 + (Z\phi)^2 + \frac{1}{4m^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 , \\
&K^N[\phi]\big|_{r=2m} \lesssim (T\phi)^2 + (Z\phi)^2 + \frac{1}{4m^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 .
\end{aligned}
\tag{4.28}
$$

For the second condition in the statement to hold at $r = 2m$, it is necessary that

$$
\alpha(2m) > 0, \qquad \beta(2m) < 0, \qquad \alpha'(2m) > 0, \qquad \beta'(2m) < 0.
\tag{4.29}
$$

Under these conditions, we have

$$
\begin{aligned}
&T_{\mu\nu}[\phi]N^\mu\left(\left(1+\frac{2m}{r}\right)T^\nu - \frac{2m}{r}Y^\nu\right)\Big|_{r=2m} \gtrsim (T\phi)^2 + (Z\phi)^2 + \frac{1}{4m^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 , \\
&K^N[\phi]\Big|_{r=2m} \gtrsim \left(\alpha'(2m) - \frac{\beta^2(2m)}{2m}\right)(T\phi)^2 + \left(-\frac{\beta(2m)}{4m} - \frac{1}{2m}\right)(Z\phi)^2 - \frac{1}{8m^2}\beta'(2m)\left|\nabla_{\mathbb{S}^2}\phi\right|^2 .
\end{aligned}
\tag{4.30}
$$

It is then clear that the functions α and β must also satisfy

$$
\begin{aligned}
&\alpha'(2m) - \frac{\beta^2(2m)}{2m} > 0 , \\
&-\frac{\beta(2m)}{4m} - \frac{1}{2m} > 0 .
\end{aligned}
\tag{4.31}
$$

Regarding the first condition in the statement, we have

$$
\begin{aligned}
&g(N,N)\big|_{r=2m} = 2\alpha(2m)\beta(2m) , \\
&g(N,T)\big|_{r=2m} = \beta(2m) ,
\end{aligned}
\tag{4.32}
$$

and so, once again, we need that $\alpha(2m) > 0$ and $\beta(2m) < 0$. Conditions (4.29) and (4.31) are satisfied, for instance, by the following two functions:

$$
\begin{aligned}
&\alpha(r) = 1 + \frac{5}{m}(r-2m) , \\
&\beta(r) = -3 - \frac{1}{m}(r-2m) .
\end{aligned}
\tag{4.33}
$$

For such functions, both conditions in the statement are satisfied at the horizon, and so, by continuity, there exists an $r_1 > 2m$ such that they are also satisfied in the region $\{2m \leq r \leq \min\{r_0, r_1\}\}$.

□

Remark 4.3.3. *If the surface gravity was negative, the coefficient proportional to $(Z\phi)^2$ in the energy bulk term in equation* (4.27) *would have the opposite sign. Hence, it would be impossible to construct a vector field such that both the energy flux and the bulk term were simultanesouly positive definite quadratic forms of the derivatives of ϕ near the event horizon. This makes it clear how the redshift effect is a key phenomenon to overcome the degeneracy of the energy flux, which is an obstacle to obtaining pointwise estimates of the solution to the wave equation.*

If necessary, we now redefine r_0 so that $r_0 < r_1$ and the vector field N satisfies the conditions above on the whole region $\{2m \leq r \leq r_0\}$. Additionally, we define an extension of the redshift vector field which coincides with the Killing field T away from the horizon. For this purpose, consider a smooth function γ depending only on the radius such that $\gamma(r) = 1$ for $r \leq 0.9\, r_0$ and $\gamma(r) = 0$ for $r \geq r_0$. Using this function, we define the extension vector field as follows (from now on, we will also refer to this vector field as redshift vector field):

$$\tilde{N} = \gamma(r)N + (1 - \gamma(r))T\,. \tag{4.34}$$

Note that $\tilde{N}$ also satisfies the conditions in Proposition 4.3.2 for $2m \leq r \leq 0.9\, r_0$, since it coincides with N in this region. For $r \geq r_0$ the bulk term is zero, since $\tilde{N} = T$. In the transition region $\{0.9\, r_0 \leq r \leq r_0\}$, we know that $K^{\tilde{N}}[\phi]$ is a quadratic form on the derivatives of ϕ. Therefore, using Young's inequality and boundedness of the coefficients of the quadratic form, we have

$$\left|K^{\tilde{N}}[\phi]\right| \lesssim (T\phi)^2 + (Y\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\,, \quad 0.9\, r_0 \leq r \leq r_0\,. \tag{4.35}$$

Moreover, since the energy flux is linear on the vector field, we have

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau) = \gamma(r)\widetilde{\mathbb{E}}^{N}[\phi](\tau) + (1 - \gamma(r))\widetilde{\mathbb{E}}^{T}[\phi](\tau)\,. \tag{4.36}$$

Proposition 4.3.2 then implies that the energy associated to $\tilde{N}$ is indeed non-degenerate:

$$\begin{aligned}\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau) \sim &\int_{L_\tau}\left((T\phi)^2 + (Y\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2} + \\ &+ \int_{S_\tau}\left((T\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2} + \\ &+ \int_{N_\tau}\left((\partial_v\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2}\ .\end{aligned} \tag{4.37}$$

These considerations lead to the analogue of equation (4.21) but for the non-degenerate energy, immediately implying that this energy is also bounded.

Proposition 4.3.4. ***(Non-degenerate energy boundedness)*** *Let $\tilde{N}$ be the redshift vector field, defined in equation* (4.34)*. If ϕ is a solution of the wave equation in the exterior region of the Schwarzschild* *spacetime with compactly supported initial data on Σ_0, then*

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_2) \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1)\,, \quad \forall \tau_2 \geq \tau_1 \geq 0\,. \tag{4.38}$$

Proof. Given $\tau_2 \geq \tau_1 \geq 0$, let us apply the divergence theorem in the region $R_{\tau_1}^{\tau_2}$ to the vector field $J^{\tilde{N}}[\phi]$. Since the energy flux at the horizon and at null infinity are non-negative, we obtain

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_2) + \int_{R_{\tau_1}^{\tau_2}} K^{\tilde{N}}[\phi]\ \mathrm{dVol}_{\mathcal{M}} \leq \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1)\,. \tag{4.39}$$

Taking into account the previous considerations regarding the bulk term, we have

$$\begin{aligned}
&\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_2) + \int_{R_{\tau_1}^{\tau_2}\cap\{r\leq 0.9\,r_0\}} K^{N}[\phi]\ \mathrm{dVol}_{\mathcal{M}} \\
&\leq \int_{R_{\tau_1}^{\tau_2}\cap\{0.9\,r_0\leq r\leq r_0\}} -K^{\tilde{N}}[\phi]\ \mathrm{dVol}_{\mathcal{M}} + \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) \\
&\leq C_1 \int_{R_{\tau_1}^{\tau_2}\cap\{0.9\,r_0\leq r\leq r_0\}} \left((T\phi)^2 + (Y\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) \mathrm{dVol}_{\mathcal{M}} + \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) \\
&\leq C_2 \int_{R_{\tau_1}^{\tau_2}\cap\{0.9\,r_0\leq r\leq r_0\}} J^{N}_{\mu}[\phi] n^{\mu}_{L}\ \mathrm{dVol}_{\mathcal{M}} + \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1)\,,
\end{aligned} \tag{4.40}$$

for some positive constants C_1 and C_2. This implies that

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_2) + B_1 \int_{R_{\tau_1}^{\tau_2}\cap\{r\leq 0.9\,r_0\}} J^{N}_{\mu}[\phi] n^{\mu}_{L}\ \mathrm{dVol}_{\mathcal{M}} \leq C_2 \int_{R_{\tau_1}^{\tau_2}\cap\{0.9\,r_0\leq r\leq r_0\}} J^{N}_{\mu}[\phi] n^{\mu}_{L}\ \mathrm{dVol}_{\mathcal{M}} + \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) \tag{4.41}$$

for some positive constant B_1. Note that, for $0.9\,r_0 \leq r \leq r_0$,

$$J^{N}_{\mu}[\phi] n^{\mu}_{L} \sim J^{\tilde{N}}_{\mu}[\phi] n^{\mu}_{L}\,. \tag{4.42}$$

Adding $B_1 \int_{R_{\tau_1}^{\tau_2}\cap\{0.9\,r_0\leq r\leq r_0\}} J^{\tilde{N}}_{\mu}[\phi] n^{\mu}_{L}\ \mathrm{dVol}_{\mathcal{M}}$ to both sides of the previous inequality we get

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_2) + B_1 \int_{R_{\tau_1}^{\tau_2}\cap\{r\leq r_0\}} J^{\tilde{N}}_{\mu}[\phi] n^{\mu}_{L}\ \mathrm{dVol}_{\mathcal{M}} \leq C_3 \int_{R_{\tau_1}^{\tau_2}\cap\{0.9\,r_0\leq r\leq r_0\}} J^{\tilde{N}}_{\mu}[\phi] n^{\mu}_{L}\ \mathrm{dVol}_{\mathcal{M}} + \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) \tag{4.43}$$

for some constant $C_3 > 0$. Notice that the Lemaître time coordinate satisfies

$$g(\nabla t^*, \nabla t^*) = -\left(1 + \frac{2m}{r}\right) \sim -1\,. \tag{4.44}$$

Therefore, using the coarea formula, there exist constants $B_2, C_4 > 0$ such that

$$\begin{aligned}
&\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_2) + B_2 \int_{\tau_1}^{\tau_2} \left(\int_{L_\tau} J^{\tilde{N}}_{\mu}[\phi] n^{\mu}_{L_\tau}\ \mathrm{dVol}_{L_\tau}\right) d\tau \\
&\leq\ C_4 \int_{\tau_1}^{\tau_2} \left(\int_{L_\tau\cap\{r\geq 0.9\,r_0\}} J^{T}_{\mu}[\phi] n^{\mu}_{L_\tau}\ \mathrm{dVol}_{L_\tau}\right) d\tau + \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1)\,,
\end{aligned} \tag{4.45}$$

where we also used

$$J^{\tilde{N}}_\mu[\phi]n^\mu_{L_\tau} \sim J^T_\mu[\phi]n^\mu_{L_\tau} \quad \text{for} \quad 0.9r_0 \leq r \leq r_0 \,. \tag{4.46}$$

Adding a multiple of the integral between τ_1 and τ_2 of the energy flux of $J^T[\phi]$ through $\widetilde{\Sigma}_\tau \cap \{r \geq r_0\}$, we obtain

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_2) + B\int_{\tau_1}^{\tau_2} \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau)\,d\tau \leq C\int_{\tau_1}^{\tau_2} \widetilde{\mathbb{E}}^T[\phi](\tau)\,d\tau + \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) \tag{4.47}$$

for some constants $B, C > 0$. Hence, given $0 \leq \tau_0 \leq \tau_1$,

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_2) + B\int_{\tau_1}^{\tau_2} \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau)\,d\tau \leq C(\tau_2-\tau_1)\widetilde{\mathbb{E}}^T[\phi](\tau_0) + \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) \,. \tag{4.48}$$

Dividing the previous equation by $\tau_2 - \tau_1$ and taking the limit $\tau_2 \to \tau_1$ we get the following inequality:

$$\left(\frac{d}{d\tau}\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau)\right)\Big|_{\tau=\tau_1} + B\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) \leq C\widetilde{\mathbb{E}}^T[\phi](\tau_0) \leq C'\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_0) \tag{4.49}$$

for some positive constant C'. Hence, setting $\tilde{C} = C'/B$, we have

$$\begin{aligned}
&\frac{d}{d\tau}\left(e^{B\tau}\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau) - \tilde{C}e^{B\tau}\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_0)\right)\Big|_{\tau=\tau_1} \leq 0 \\
&\Rightarrow e^{B\tau_1}\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) - \tilde{C}e^{B\tau_1}\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_0) \leq (1-\tilde{C})e^{B\tau_0}\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_0) \\
&\Rightarrow \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) \leq (\tilde{C} + e^{-B(\tau_1-\tau_0)}(1-\tilde{C}))\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_0) \leq (\tilde{C}+1)\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_0) \,.
\end{aligned} \tag{4.50}$$

□

This last result corresponds to a non-degenerate energy boundedness result. In what follows, we will prove a lemma that will allow to prove a pointwise estimate as a consequence of the energy result.

Lemma 4.3.5. *Consider ϕ as a function of null coordinates (u, v). If ϕ is a solution of the wave equation in the exterior region of the Schwarzschild spacetime with compactly supported initial data on Σ_0, then, given $u_0 \in \mathbb{R}$ and $\omega \in \mathbb{S}^2$,*

$$\lim_{v_0 \to +\infty} \phi(u_0, v_0, \omega) = 0 \,. \tag{4.51}$$

Proof. Let v_0 be large enough so that $\phi(-v_0, v_0, \omega) = 0$. Note that such v_0 exists since initial data has compact support. Then, we apply the fundamental theorem of calculus at constant v, and the Cauchy-Schwarz inequality, as follows:

$$\begin{aligned}
\int_{\mathbb{S}^2} \phi^2(u_0, v_0, \omega)\ \mathrm{dVol}_{\mathbb{S}^2} &= \int_{\mathbb{S}^2}\left(\int_{-v_0}^{u_0} \partial_u\phi(u, v_0, \omega)\,du\right)^2 \mathrm{dVol}_{\mathbb{S}^2} \\
&\leq \left(\int_{-v_0}^{u_0} \frac{1}{r^2(u, v_0)}\,du\right)\left(\int_{\mathbb{S}^2}\int_{-v_0}^{u_0} (\partial_u\phi(u, v_0, \omega))^2\, r^2(u, v_0)\,du\ \mathrm{dVol}_{\mathbb{S}^2}\right) \,.
\end{aligned} \tag{4.52}$$

Applying the divergence theorem to the vector field $J^T[\phi]$ in the region delimited by the hypersurfaces Σ_0, $\Sigma_{\frac{u_0+v_0}{2}}$ and $\{v = v_0\}$, we easily obtain (compare with estimate (3.32) in Minkowski's spacetime)

$$\int_{\mathbb{S}^2} \int_{-v_0}^{u_0} \left(\partial_u \phi(u, v_0, \omega)\right)^2 r^2(u, v_0)\, du \,\mathrm{dVol}_{\mathbb{S}^2} \leq \mathbb{E}^T[\phi](0)\,, \tag{4.53}$$

where $\mathbb{E}^T[\phi](0)$ corresponds to energy flux through Σ_0 (which coincides with $\widetilde{\mathbb{E}}^T[\phi](0)$, due to the definition of r_0 and R). Hence, combining the previous two estimates, we get

$$\int_{\mathbb{S}^2} \phi^2(u_0, v_0, \omega) \,\mathrm{dVol}_{\mathbb{S}^2} \leq \left(\int_{-v_0}^{u_0} \frac{1}{r^2(u, v_0)}\, du\right) \mathbb{E}^T[\phi](0)\,. \tag{4.54}$$

The integral on the right-hand side can be bounded by a simple computation:

$$\begin{aligned} &\int_{-v_0}^{u_0} \frac{1}{r^2(u, v_0)}\, du = 2\int_{r(u_0,v_0)}^{r(-v_0,v_0)} \frac{1}{r^2\left(1 - \frac{2m}{r}\right)}\, dr \\ &\leq 2\int_{r(u_0,v_0)}^{r(-v_0,v_0)} \frac{1}{(r-2m)^2}\, dr = 2\left(\frac{1}{r(u_0, v_0) - 2m} - \frac{1}{r(-v_0, v_0) - 2m}\right)\,. \end{aligned} \tag{4.55}$$

Since the last expression tends to zero when we take the limit $v_0 \to +\infty$, the conclude that

$$\lim_{v_0 \to +\infty} \int_{\mathbb{S}^2} \phi^2(u_0, v_0, \omega) \,\mathrm{dVol}_{\mathbb{S}^2} = 0\,. \tag{4.56}$$

The Schwarzschild spacetime is spherically symmetric, so the previous equation also holds with $\Omega_i\phi$ and $\Omega_i\Omega_j\phi$ in place of ϕ, implying that

$$\lim_{v_0 \to +\infty} \|\phi(u_0, v_0, \cdot)\|_{H^2(\mathbb{S}^2)} = 0\,. \tag{4.57}$$

Applying the Sobolev inequality on the unit sphere, the result follows.

□

We end this section with the uniform boundedness result.

Theorem 4.3.6. *If ϕ is a solution of the wave equation in the exterior region of the Schwarzschild spacetime with compactly supported initial data on Σ_0 then there exists a constant $C > 0$ such that*

$$|\phi| \leq C\,. \tag{4.58}$$

Proof. Denote by ∂_ρ the vector field which is equal to Y for $r < r_0$, ∂_r for $r_0 \leq r \leq R$ and $(\partial_r)_u$ for $r > R$. Consider ϕ to be a function of τ and r, where τ is the coordinate whose level sets are $\widetilde{\Sigma}_\tau$. Then, given $\tau_1 \geq 0$, $r_1 \geq 2m$ and $\omega \in \mathbb{S}^2$, and applying the lemma, we have

$$\begin{aligned} \phi^2(\tau_1, r_1, \omega) = \left(\int_{r_1}^{\infty} \partial_\rho \phi(\tau_1, r, \omega)\, dr\right)^2 &\leq \left(\int_{r_1}^{\infty} \frac{1}{r^2}\, dr\right)\left(\int_{r_1}^{\infty} \left(\partial_\rho \phi(\tau_0, r, \omega)\right)^2 r^2\, dr\right) \\ &\leq \frac{1}{2m}\left(\int_{r_1}^{\infty} \left(\partial_\rho \phi(\tau_1, r, \omega)\right)^2 r^2\, dr\right)\,. \end{aligned} \tag{4.59}$$

Note that, for $r > R$,

$$\left(\partial_\rho \phi\right)^2 \sim \left(\partial_v \phi\right)^2\,. \tag{4.60}$$

Hence, integrating estimate (4.59) on $\mathbb{S}^2$, we have

$$\int_{\mathbb{S}^2} \phi^2(\tau_1, r_1, \omega)\, \mathrm{dVol}_{\mathbb{S}^2} \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](0)\,. \tag{4.61}$$

This equation also holds with $\Omega_i \phi$ and $\Omega_i \Omega_j \phi$ in place of ϕ, so there exists a constant $C > 0$ such that

$$\|\phi(\tau_1, r_1, \cdot)\|_{H^2(\mathbb{S}^2)} \leq C\,. \tag{4.62}$$

Applying the Sobolev inequality on the unit sphere and noting that C does not depend on τ_1 and r_1, we conclude that ϕ is bounded. □

4.4 Decay

In this Section, we prove pointwise decay of the solution of the wave equation using a similar approach to the case of the Minkowski spacetime. We start by proving an Integrated Local Energy Decay estimate which degenerates at the event horizon and at $r = 3m$ (corresponding to the photon sphere). Next, we make use of the redshift vector field introduced in Section 4.3 to obtain an Integrated Local Energy Decay estimate which does non degenerate at $\mathscr{H}^+$. We also briefly explain why we cannot get rid of the degeneracy at the photon sphere. Finally, it turns out that the Dafermos-Rodnianski hierarchy also holds for the Schwarzschild spacetime, allowing us to prove an energy decay result and then obtain the desired pointwise estimate, in the same way as for the Minkowski spacetime.

4.4.1 Integrated Local Energy Decay estimate

Once again, we start by proving a Hardy-type inequality which allows to control a weighted L^2 norm of ϕ. However, for the Schwarzschild spacetime, since the energy flux associated to T degenerates at the horizon, we will have to resort to the nondegenerate energy associated to the redshift vector field $\tilde{N}$. As expected, we also show that the weighted L^2 norm of ϕ on subsets of $\tilde{\Sigma}_\tau$ away from the horizon can be bounded by the standard energy $\widetilde{\mathbb{E}}^T[\phi](\tau)$.

Proposition 4.4.1. *If ϕ satisfies the wave equation with compactly supported initial data on Σ_0 then*

$$\int_{\widetilde{\Sigma}_\tau} \phi^2\, dr\; \mathrm{dVol}_{\mathbb{S}^2} \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau), \quad \forall \tau \geq 0\,. \tag{4.63}$$

Moreover, if we are only interested on a subset of $\widetilde{\Sigma}_\tau$ away from the horizon, we can bound the integral by the degenerate energy: given $R_0 > 2m$,

$$\int_{\widetilde{\Sigma}_\tau \cap \{r \geq R_0\}} \phi^2\, dr\; \mathrm{dVol}_{\mathbb{S}^2} \lesssim_{R_0} \widetilde{\mathbb{E}}^T[\phi](\tau), \quad \forall \tau \geq 0\,, \tag{4.64}$$

where the constant in the previous inequality diverges as $R_0 \to 2m$.

Proof. Consider ϕ to be a function of τ and r as in the proof of Theorem 4.3.6. Then, given $\tau \geq 0$, we perform an integration by parts as follows:

$$
\begin{aligned}
&\int_{\widetilde{\Sigma}_\tau} \phi^2 \, dr \, \mathrm{dVol}_{\mathbb{S}^2} \\
&= -\int_{\widetilde{\Sigma}_\tau} 2r\phi\partial_\rho\phi \, dr \, \mathrm{dVol}_{\mathbb{S}^2} + \lim_{r\to\infty}\int_{\mathbb{S}^2} r\phi^2(\tau,r,\omega) \, \mathrm{dVol}_{\mathbb{S}^2} - \int_{\mathbb{S}^2} 2m\phi^2(\tau,2m,\omega) \, \mathrm{dVol}_{\mathbb{S}^2} \\
&\leq \varepsilon^2 \int_{\widetilde{\Sigma}_\tau} \phi^2 \, dr \, \mathrm{dVol}_{\mathbb{S}^2} + \frac{1}{\varepsilon^2}\int_{\widetilde{\Sigma}_\tau} \left(\partial_\rho\phi\right)^2 r^2 \, dr \, \mathrm{dVol}_{\mathbb{S}^2} + \lim_{r\to\infty}\int_{\mathbb{S}^2} r\phi^2(\tau,r,\omega) \, \mathrm{dVol}_{\mathbb{S}^2} \; .
\end{aligned}
\tag{4.65}
$$

Choosing ε such that $0 < \varepsilon < 1$ and taking equation (4.60) into account, we have

$$
\int_{\widetilde{\Sigma}_\tau} \phi^2 \, dr \, \mathrm{dVol}_{\mathbb{S}^2} \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau) + \lim_{r\to\infty}\int_{\mathbb{S}^2} r\phi^2(\tau,r,\omega) \, \mathrm{dVol}_{\mathbb{S}^2} \; . \tag{4.66}
$$

We now need to prove that the limit in the previous inequality can be bounded by the energy. This can be easily obtained with a computation similar to the one in equation (3.35) and using equations (4.54) and (4.55) together with the fact that $\widetilde{\mathbb{E}}^T[\phi](\tau) \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau)$.

We can also apply the previous reasoning by integrating by parts in the region $\widetilde{\Sigma}_\tau \cap \{r \geq R_0\}$. The second inequality then follows from the fact that

$$
\int_{\widetilde{\Sigma}_\tau \cap \{r \geq R_0\}} \left(\partial_\rho\phi\right)^2 r^2 \, dr \, \mathrm{dVol}_{\mathbb{S}^2} \lesssim_{R_0} \widetilde{\mathbb{E}}^T[\phi](\tau) \, . \tag{4.67}
$$

$\square$

We now proceed to deduce an Integrated Local Energy Decay estimate, since it will also be useful to prove energy decay results, just like in the case of the Minkowski spacetime. However, in the Schwarzschild spacetime, the first Integrated Local Energy Decay estimate we will prove degenerates both at the event horizon and at the photon sphere. The first degeneracy phenomenon can be overcome by making use of the redshift vector field, which was previously constructed to obtain a bounded energy flux that does not degenerate on the horizon. On the other hand, the degeneracy occurring at the photon sphere unavoidable, as proven in [29]. The key feature explored in this work is that one can construct special solutions to the wave equation, designated by Gaussian beams, whose associated energy is localized in an arbitrary spatially compact neighborhood of the photon sphere for finite, but arbitrarily long, times. This then comes up as an obstruction to uniform results about the temporal behavior of the energy of waves, here quantitatively described by the energy flux $\widetilde{\mathbb{E}}^T[\phi](\tau)$. In particular, the main consequence is that there does not exist a constant $C > 0$ such that

$$
\int_0^\infty \widetilde{\mathbb{E}}^T[\phi](\tau) \, d\tau \leq C\widetilde{\mathbb{E}}^T[\phi](0) \, . \tag{4.68}
$$

In fact: we can even say more: given an arbitrary spatially compact neighborhood K of the photon sphere, there does not exist a constant $C > 0$ such that

$$
\int_0^\infty \widetilde{\mathbb{E}}^T_K[\phi](\tau) \leq C\widetilde{\mathbb{E}}^T[\phi](0) \, , \tag{4.69}
$$

where $\widetilde{\mathbb{E}}^T_K[\phi](\tau)$ denotes the energy flux restricted to $\widetilde{\Sigma}_\tau \cap K$. This statement is the main reason behind the fact that the Integrated Local Energy Decay estimate must degenerate at the photon sphere.

In the next Proposition, we follow the work of [21] to obtain an Integrated Local Energy Decay estimate, degenerating both at the event horizon and at the photon sphere of the black hole.

Proposition 4.4.2. ***(Degenerate Integrated Local Energy Decay estimate)*** *Let ϕ satisfy the wave equation with compactly supported initial data on Σ_0. Then, given $\tau_2 \geq \tau_1 \geq 0$ and $\delta > 0$, the following inequality holds:*

$$\int_{R_{\tau_1}^{\tau_2}} \left(\frac{1}{r^3} \left(1 - \frac{2m}{r}\right)^2 (\partial_r \phi)^2 + \frac{1}{r^3} \left(1 - \frac{3m}{r}\right)^2 \left(\frac{1}{r^\delta} (\partial_t \phi)^2 + |\nabla_{\mathbb{S}^2} \phi|^2 \right) + \frac{\phi^2}{r^4} \right) \mathrm{dVol}_{\mathcal{M}} \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) \,. \tag{4.70}$$

Proof. We will work with coordinates (t, r^*), since it makes the computations easier. Let us start by defining two modified energy currents:

$$\begin{aligned} J^{(1)}_\mu[\phi] &:= T_{\mu\nu}[\phi] T^\nu - \frac{1}{2} \left[\star d \left(\phi^2 \left(1 - \frac{2m}{r}\right) r \,\mathrm{dVol}_{\mathbb{S}^2} \right) \right]_\mu \,, \\ J^{(2)}_\mu[\phi] &:= T_{\mu\nu}[\phi] V^\nu + \left(\frac{f'}{4} + \frac{f}{2r} \left(1 - \frac{2m}{r}\right) \right) \partial_\mu (\phi^2) - \left[\partial_\mu \left(\frac{f'}{4} + \frac{f}{2r} \left(1 - \frac{2m}{r}\right) \right) - \frac{f'}{r} \partial_\mu r \right] \phi^2 \,, \end{aligned} \tag{4.71}$$

where

$$f = \left(1 - \frac{3m}{r}\right) \left(1 + \frac{6m}{r}\right)^{1/2} \qquad \text{and} \qquad V = f \partial_{r^*} . \tag{4.72}$$

It will become apparent later that the factor $\left(1 - \frac{3m}{r}\right)$ in the function f is necessary to obtain a positive semidefinite quadratic form, leading to the degeneracy phenomenon at the photon sphere described above. In equation (4.71) and in the context of this proof, the symbol $'$ will always denote differentiation with respect to r^* at constant t. The components of the first current are given by

$$\begin{aligned} J^{(1)}_t[\phi] &= \frac{1}{2r^2} \left((\partial_t \psi)^2 + (\partial_{r^*} \psi)^2 + \left(1 - \frac{2m}{r}\right) \frac{1}{r^2} |\nabla_{\mathbb{S}^2} \psi|^2 + \frac{2m}{r^3} \left(1 - \frac{2m}{r}\right) \psi^2 \right) , \\ J^{(1)}_\mu[\phi] &= \frac{1}{r^2} \partial_t \psi \, \partial_\mu \psi \,, \quad \mu = r^*, \theta, \varphi \,, \end{aligned} \tag{4.73}$$

where $\psi = r\phi$ is the radiation field (even though (4.70) does not mention the radiation field, we introduce it here since the computations become much easier). The components of the second current are

$$\begin{aligned} J^{(2)}_t[\phi] &= \frac{1}{r^2} \left(f \partial_t \psi \, \partial_{r^*} \psi + \frac{1}{2} f' \psi \, \partial_t \psi \right) , \\ J^{(2)}_{r^*}[\phi] &= \frac{1}{2r^2} \left(f (\partial_t \psi)^2 + f (\partial_{r^*} \psi)^2 - f \left(1 - \frac{2m}{r}\right) \frac{1}{r^2} |\nabla_{\mathbb{S}^2} \psi|^2 - \left(\frac{f''}{2} + \frac{2m}{r^3} \left(1 - \frac{2m}{r}\right) f \right) \psi^2 + f' \psi \, \partial_{r^*} \psi \right) , \\ J^{(2)}_\mu[\phi] &= \frac{1}{r^2} \partial_{r^*} \psi \, \partial_\mu \psi \,, \quad \mu = \theta, \varphi \,. \end{aligned} \tag{4.74}$$

We also define $K^{(i)}[\phi] = \nabla^\mu J^{(i)}_\mu[\phi]$ for $i = 1, 2$. For each current, we have

$$K^{(1)}[\phi] = 0\,,$$
$$K^{(2)}[\phi] = \frac{1}{r^2\left(1-\frac{2m}{r}\right)}\left(f'\left(\partial_{r^*}\psi\right)^2 - \frac{f}{2}\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)'\left|\nabla_{\mathbb{S}^2}\psi\right|^2 - \left(\frac{f'''}{4}+\frac{f}{2}\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)'\right)\psi^2\right) \tag{4.75}$$

The vanishing of the first bulk term is easily obtained using the following formula for the divergence of an arbitrary covector field ω:

$$\nabla^\mu \omega_\mu = -\star d \star \omega\,. \tag{4.76}$$

In addition, we define the energy flux through $\widetilde{\Sigma}_\tau$ as follows:

$$\begin{aligned}\widetilde{\mathbb{F}}^{(i)}[\phi](\tau) = &\int_{L_\tau} J^{(i)}_\mu[\phi]\left(\partial_t - \frac{2m}{r}\partial_{r^*}\right)^\mu r^2\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2} + \int_{S_\tau} J^{(i)}_\mu[\phi]\left(\partial_t\right)^\mu r^2\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2} + \\ &+\int_{N_\tau} J^{(i)}_\mu[\phi]\left(\partial_t + \partial_{r^*}\right) r^2\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2}\,, \quad i = 1,2\,.\end{aligned} \tag{4.77}$$

For $i = 1$, we have

$$\begin{aligned}\widetilde{\mathbb{F}}^{(1)}[\phi](\tau) =& \frac{1}{2}\int_{L_\tau}\left(\left(1+\frac{2m}{r}\right)\left(\partial_t\psi\right)^2 + \left(1-\frac{2m}{r}\right)\left(Y\psi\right)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\psi\right|^2 + \frac{2m}{r^3}\psi^2\right) dr\ \mathrm{dVol}_{\mathbb{S}^2} + \\ &+\frac{1}{2}\int_{S_\tau}\left(\left(1-\frac{2m}{r}\right)^{-1}\left(\partial_t\psi\right)^2 + \left(1-\frac{2m}{r}\right)\left(\partial_r\psi\right)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\psi\right|^2 + \frac{2m}{r^3}\psi^2\right) dr\ \mathrm{dVol}_{\mathbb{S}^2} + \\ &+\frac{1}{2}\int_{N_\tau}\left(4\left(1-\frac{2m}{r}\right)^{-1}\left(\partial_v\psi\right)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\psi\right|^2 + \frac{2m}{r^3}\psi^2\right) dr\ \mathrm{dVol}_{\mathbb{S}^2}\ .\end{aligned} \tag{4.78}$$

Before computing the energy for the second current, we compute the first two derivatives of f:

$$\begin{aligned}f' &= \left(1-\frac{2m}{r}\right)\frac{27m^2}{r^3}\left(1+\frac{6m}{r}\right)^{-1/2}\,, \\ f'' &= -\left(1-\frac{2m}{r}\right)\frac{81m^2r^2+189m^3r-1134m^4}{r^6\left(1+\frac{6m}{r}\right)^{3/2}}\,.\end{aligned} \tag{4.79}$$

Therefore, this function satisfies the following important conditions:

$$\begin{aligned}&|f| \lesssim 1\,, \\ &|f'| \lesssim \frac{1}{r^3}\left(1-\frac{2m}{r}\right)\,, \\ &|f''| \lesssim \frac{1}{r^4}\left(1-\frac{2m}{r}\right)\,.\end{aligned} \tag{4.80}$$

Using these conditions and applying Young's inequality, one can prove that

$$
\begin{aligned}
&\left|J^{(2)}_\mu[\phi]\left(\partial_t-\frac{2m}{r}\partial_{r^*}\right)^\mu\right|\lesssim J^{(1)}_\mu[\phi]\left(\partial_t-\frac{2m}{r}\partial_{r^*}\right)^\mu\,,\\
&\left|J^{(2)}_\mu[\phi]\,(\partial_t)^\mu\right|\lesssim J^{(1)}_\mu[\phi]\,(\partial_t)^\mu\,,\\
&\left|J^{(2)}_\mu[\phi]\,(\partial_t+\partial_{r^*})^\mu\right|\lesssim J^{(1)}_\mu[\phi]\,(\partial_t+\partial_{r^*})^\mu\,,
\end{aligned}
\tag{4.81}
$$

which implies that the energy fluxes are related as follows:

$$
\left|\widetilde{\mathbb{F}}^{(2)}[\phi](\tau)\right|\lesssim\widetilde{\mathbb{F}}^{(1)}[\phi](\tau)\,. \tag{4.82}
$$

Let us now define the energy flux of $2C_fJ^{(1)}[\phi]+2J^{(2)}[\phi]$ through $\widetilde{\Sigma}_\tau$, where $C_f>0$ is a positive constant:

$$
\widetilde{\mathbb{F}}[\phi](\tau)=2C_f\widetilde{\mathbb{F}}^{(1)}[\phi](\tau)+2\widetilde{\mathbb{F}}^{(2)}[\phi](\tau)\,. \tag{4.83}
$$

Equation (4.82) implies that we can choose C_f to be large enough so that

$$
\widetilde{\mathbb{F}}[\phi](\tau)\sim\widetilde{\mathbb{F}}^{(1)}[\phi](\tau)\geq 0\,. \tag{4.84}
$$

We also define the associated bulk term

$$
\begin{aligned}
&\mathbb{I}[\phi](\tau_1,\tau_2)\\
&=\int_{R^{\tau_2}_{\tau_1}}\nabla^\mu\left(2C_fJ^{(1)}_\mu[\phi]+2J^{(2)}_\mu[\phi]\right)\,\mathrm{dVol}_{\mathcal{M}}\\
&=\int_{R^{\tau_2}_{\tau_1}}\left(2f'\left(\partial_{r^*}\psi\right)^2-f\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)'|\nabla_{\mathbb{S}^2}\psi|^2-\left(\frac{f'''}{2}+f\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)'\right)\psi^2\right)\,dt\,dr^*\,\mathrm{dVol}_{\mathbb{S}^2}\,.
\end{aligned}
\tag{4.85}
$$

Applying the divergence theorem to the current $2C_fJ^{(1)}[\phi]+2J^{(2)}[\phi]$ over the region $R^{\tau_2}_{\tau_1}$, we obtain

$$
\begin{aligned}
&\mathbb{I}[\phi](\tau_1,\tau_2)+\int_{\mathscr{I}^+\cap\{\tau_1-r^*(R)\leq u\leq\tau_2-r^*(R)\}}\left(2C_fJ^{(1)}_\mu[\phi]+2J^{(2)}_\mu[\phi]\right)(\partial_u)^\mu\,r^2\,du\,\mathrm{dVol}_{\mathbb{S}^2}+\\
&+\int_{\mathscr{H}^+\cap\{\tau_1\leq t^*-2m\ln(r_0-2m)\leq\tau_2\}}\left(2C_fJ^{(1)}_\mu[\phi]+2J^{(2)}_\mu[\phi]\right)(\partial_v)^\mu\,r^2\,dv\,\mathrm{dVol}_{\mathbb{S}^2}=\widetilde{\mathbb{F}}[\phi](\tau_1)-\widetilde{\mathbb{F}}[\phi](\tau_2)\,.
\end{aligned}
\tag{4.86}
$$

Note that the integral over the event horizon is non-negative due to the third condition in (4.81) and the fact that C_f is taken large enough so that estimate (4.84) holds. Moreover, we also have

$$
\left|J^{(2)}_\mu[\phi]\,(\partial_t-\partial_{r^*})^\mu\right|\lesssim J^{(1)}_\mu[\phi]\,(\partial_t-\partial_{r^*})^\mu=\frac{2}{r^2}(\partial_u\psi)^2+\frac{1}{2r^2}\left(1-\frac{2m}{r}\right)\frac{1}{r^2}|\nabla_{\mathbb{S}^2}\psi|^2+\frac{m}{r^5}\left(1-\frac{2m}{r}\right)\psi^2\,, \tag{4.87}
$$

so we can take C_f large enough so that the integral at null infinity is also non-negative. Under these conditions, equation (4.86) implies that

$$
\mathbb{I}[\phi](\tau_1,\tau_2)\leq\widetilde{\mathbb{F}}[\phi](\tau_1)-\widetilde{\mathbb{F}}[\phi](\tau_2)\leq\widetilde{\mathbb{F}}[\phi](\tau_1)\lesssim\widetilde{\mathbb{F}}^{(1)}[\phi](\tau_1)\,. \tag{4.88}
$$

This last inequality resembles the Integrated Local Energy Decay estimate we want prove. Let us show that we can replace $\widetilde{\mathbb{F}}^{(1)}[\phi](\tau_1)$ by the flux of $J^T[\phi]$:

$$\begin{aligned}
&\int_{\widetilde{\Sigma}_\tau} \left(\left(1-\frac{2m}{r}\right)(\partial_\rho(r\phi))^2 + \frac{2m}{r^3}\phi^2 r^2 \right) dr\ \mathrm{dVol}_{\mathbb{S}^2} \\
&= \int_{\widetilde{\Sigma}_\tau} \left(1-\frac{2m}{r}\right)(\partial_\rho \phi)^2 r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2} + \int_{\widetilde{\Sigma}_\tau} \partial_\rho \left(\left(1-\frac{2m}{r}\right) r\phi^2 \right) dr\ \mathrm{dVol}_{\mathbb{S}^2} \\
&= \int_{\widetilde{\Sigma}_\tau} \left(1-\frac{2m}{r}\right)(\partial_\rho \phi)^2 r^2\, dr\ \mathrm{dVol}_{\mathbb{S}^2} + \lim_{r\to\infty} \int_{\mathbb{S}^2} \left(1-\frac{2m}{r}\right) r\phi^2(\tau,r,\omega)\ \mathrm{dVol}_{\mathbb{S}^2}\ .
\end{aligned} \tag{4.89}$$

As seen in the proof of Proposition 4.4.1, the limit can be bounded by $\widetilde{\mathbb{E}}^T[\phi](\tau)$. All the remaining derivatives in the definition of $\widetilde{\mathbb{F}}^{(1)}[\phi](\tau)$ commute with multiplication by r, so we obtain

$$\widetilde{\mathbb{F}}^{(1)}[\phi](\tau) \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau)\,. \tag{4.90}$$

To finish the proof, we need to relate $\mathbb{I}[\phi](\tau_1,\tau_2)$ with the spacetime integral in equation (4.70). Denote by ψ_0 the spherical average of ψ:

$$\psi_0 := \frac{1}{4\pi}\int_{\mathbb{S}^2} \psi\ \mathrm{dVol}_{\mathbb{S}^2}\ . \tag{4.91}$$

Let g be a function of the radius that will be chosen later, satisfying $|rg| \lesssim 1$. Setting $\psi = (\psi-\psi_0)+\psi_0$ in (4.85) and using the facts that $|\nabla_{\mathbb{S}^2}\psi_0|^2 = 0$ and that the integral of $\psi-\psi_0$ and of $\partial_{r^*}(\psi-\psi_0)$ over $\mathbb{S}^2$ vanishes, we can write $\mathbb{I}[\phi](\tau_1,\tau_2)$ as follows:

$$\begin{aligned}
\mathbb{I}[\phi](\tau_1,\tau_2) = &\int_{R^{\tau_2}_{\tau_1}} \left(2f' \left(\partial_{r^*}(\psi-\psi_0)\right)^2 - f\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)' |\nabla_{\mathbb{S}^2}(\psi-\psi_0)|^2 \right) dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} - \\
&- \int_{R^{\tau_2}_{\tau_1}} \left(\frac{f'''}{2} + f\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)' \right) (\psi-\psi_0)^2\, dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} + \\
&+ \int_{R^{\tau_2}_{\tau_1}} \left(2f'(\partial_{r^*}\psi_0 - g\,\psi_0)^2 + 2f' g\, \partial_{r^*}\left(\psi_0^2\right) \right) dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} - \\
&- \int_{R^{\tau_2}_{\tau_1}} \left(2f'g^2 + f\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)' + \frac{f'''}{2} \right) \psi_0^2\, dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2}\ .
\end{aligned} \tag{4.92}$$

We now wish to do an integration by parts on the term

$$\int_{R^{\tau_2}_{\tau_1}} 2f' g\, \partial_{r^*}\left(\psi_0^2\right)\ dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2}\ . \tag{4.93}$$

However, due to the geometry of $R^{\tau_2}_{\tau_1}$ it is convenient to replace the derivative ∂_{r^*} by $\tilde{Y} := (\partial_{r^*})_{t^*}$ in the region $C^{\tau_2}_{\tau_1}$ near the horizon and by ∂_v in the region $D^{\tau_2}_{\tau_1}$ near infinity. Near the horizon, we have

$$
\begin{aligned}
&\int_{C_{\tau_1}^{\tau_2}} 2f'g\,\partial_{r^*}\left(\psi_0^2\right)\,dt\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2}\\
&=\int_{C_{\tau_1}^{\tau_2}}\left(2f'g\,\tilde{Y}\left(\psi_0^2\right)+\frac{4m}{r}f'g\,\partial_{t^*}\left(\psi_0^2\right)\right)\,dt^*\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2}\\
&=-\int_{C_{\tau_1}^{\tau_2}} 2\,\tilde{Y}\,(f'g)\,\psi_0^2\,dt^*\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2}+\int_{\{r=r_0,\tau_1\le t\le\tau_2\}}2f'g\,\psi_0^2\,dt^*\ \mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{\mathscr{H}^+\cap\{\tau_1\le t^*-2m\ln(r_0-2m)\le\tau_2\}}2f'g\,\psi_0^2\,dt^*\ \mathrm{dVol}_{\mathbb{S}^2}+\int_{L_{\tau_2}}\frac{4m}{r}f'g\,\psi_0^2\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{L_{\tau_1}}\frac{4m}{r}f'g\,\psi_0^2\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2}\\
&=-\int_{C_{\tau_1}^{\tau_2}} 2\,\partial_{r^*}\,(f'g)\,\psi_0^2\,dt\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2}+\int_{\{r=r_0,\tau_1\le t\le\tau_2\}}2f'g\,\psi_0^2\,dt\ \mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{\mathscr{H}^+\cap\{\tau_1\le t^*-2m\ln(r_0-2m)\le\tau_2\}}2f'g\,\psi_0^2\,dt^*\ \mathrm{dVol}_{\mathbb{S}^2}+\int_{L_{\tau_2}}\frac{4m}{r}f'g\psi_0^2\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{L_{\tau_1}}\frac{4m}{r}f'g\,\psi_0^2\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2}\ .
\end{aligned}
\tag{4.94}
$$

Note that $|f'g|\lesssim\frac{1}{r^3}\left(1-\frac{2m}{r}\right)$, so the integral over the horizon vanishes. Regarding the boundary integrals over L_{τ_i}, one can use the following inequality

$$
\int_{\mathbb{S}^2}\psi_0^2\,\mathrm{dVol}_{\mathbb{S}^2}\le\int_{\mathbb{S}^2}\psi^2\,\mathrm{dVol}_{\mathbb{S}^2}\,,
\tag{4.95}
$$

to deduce that they can be bounded by $\widetilde{\mathbb{F}}^{(1)}[\phi](\tau_1)$:

$$
\begin{aligned}
&\left|\int_{L_{\tau_i}}\frac{4m}{r}f'g\psi_0^2\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2}\right|\\
&\lesssim\int_{L_{\tau_i}}\frac{1}{r^3}\left(1-\frac{2m}{r}\right)\psi_0^2\,dr^*\ \mathrm{dVol}_{\mathbb{S}^2}\\
&\le\int_{L_{\tau_i}}\frac{1}{r^3}\psi^2\,dr\ \mathrm{dVol}_{\mathbb{S}^2}\\
&\lesssim\widetilde{\mathbb{F}}^{(1)}[\phi](\tau_i)\le\widetilde{\mathbb{F}}^{(1)}[\phi](\tau_1)\,,\quad i=1,2\,.
\end{aligned}
\tag{4.96}
$$

The last inequality comes from a direct application of the divergence theorem to the first energy current over $R_{\tau_1}^{\tau_2}$. A similar procedure can be applied in the region $D_{\tau_1}^{\tau_2}$:

$$\begin{aligned}
&\int_{D_{\tau_1}^{\tau_2}} 2f'g\partial_{r^*}\left(\psi_0^2\right)\,dt\,dr^*\,\mathrm{dVol}_{\mathbb{S}^2}\\
&=\int_{D_{\tau_1}^{\tau_2}}\left(2f'g\partial_v\left(\psi_0^2\right)-2f'g\partial_u\left(\psi_0^2\right)\right)\frac{1}{2}\,du\,dv\,\mathrm{dVol}_{\mathbb{S}^2}\\
&=-\int_{D_{\tau_1}^{\tau_2}}\left(\partial_v(2f'g)-\partial_u(2f'g)\right)\psi_0^2\frac{1}{2}\,du\,dv\,\mathrm{dVol}_{\mathbb{S}^2}+\int_{\mathscr{I}^+\cap\{\tau_1-R\le u\le\tau_2-R\}}2f'g\psi_0^2\frac{1}{2}\,du\,\mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{\{r=R,\tau_1\le t\le\tau_2\}}2f'g\psi_0^2\frac{1}{2}\,du\,\mathrm{dVol}_{\mathbb{S}^2}+\int_{N_{\tau_1}}2f'g\psi_0^2\frac{1}{2}\,dv\,\mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{N_{\tau_2}}2f'g\psi_0^2\frac{1}{2}\,dv\,\mathrm{dVol}_{\mathbb{S}^2}-\int_{\{r=R,\tau_1\le t\le\tau_2\}}2f'g\psi_0^2\frac{1}{2}\,dv\,\mathrm{dVol}_{\mathbb{S}^2}\\
&=-\int_{D_{\tau_1}^{\tau_2}}2\partial_{r^*}(f'g)\psi_0^2\,dt\,dr^*\,\mathrm{dVol}_{\mathbb{S}^2}+\int_{\mathscr{I}^+\cap\{\tau_1-R\le u\le\tau_2-R\}}2f'g\psi_0^2\frac{1}{2}\,du\,\mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{\{r=R,\tau_1\le t\le\tau_2\}}2f'g\psi_0^2\,dt\,\mathrm{dVol}_{\mathbb{S}^2}-\int_{N_{\tau_2}}2f'g\psi_0^2\frac{1}{2}\,dv\,\mathrm{dVol}_{\mathbb{S}^2}+\int_{N_{\tau_1}}2f'g\psi_0^2\frac{1}{2}\,dv\,\mathrm{dVol}_{\mathbb{S}^2}\;.
\end{aligned}\tag{4.97}$$

Once again, the integrals over null infinity and L_{τ_i} can be bounded by $\widetilde{\mathbb{F}}^{(1)}[\phi](\tau_1)$. For the intermediate region, the integration by parts gives

$$\begin{aligned}
&\int_{R_{\tau_1}^{\tau_2}\cap\{r_0\le r\le R\}}2f'g\partial_{r^*}\left(\psi_0^2\right)\,dt\,dr^*\,\mathrm{dVol}_{\mathbb{S}^2}\\
&=-\int_{R_{\tau_1}^{\tau_2}\cap\{r_0\le r\le R\}}\partial_{r^*}(2f'g)\psi_0^2\,dt\,dr^*\,\mathrm{dVol}_{\mathbb{S}^2}+\int_{\{r=R,\tau_1\le t\le\tau_2\}}2f'g\psi_0^2\,dt\,\mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{\{r=r_0,\tau_1\le t\le\tau_2\}}2f'g\psi_0^2\,dt\,\mathrm{dVol}_{\mathbb{S}^2}\;.
\end{aligned}\tag{4.98}$$

Therefore, we can write

$$\int_{R_{\tau_1}^{\tau_2}}2f'g\partial_{r^*}\left(\psi_0^2\right)\,dt\,dr^*\,\mathrm{dVol}_{\mathbb{S}^2}=-\int_{R_{\tau_1}^{\tau_2}}2\partial_{r^*}(f'g)\psi_0^2\,dt\,dr^*\,\mathrm{dVol}_{\mathbb{S}^2}+\text{boundary terms}\,,\tag{4.99}$$

where the absolute value of the boundary terms can be bounded by $\widetilde{\mathbb{F}}^{(1)}[\phi](\tau_1)$. Hence, using equation (4.88), we have

$$\widetilde{\mathbb{I}}[\phi](\tau_1,\tau_2)\lesssim\widetilde{\mathbb{F}}^{(1)}[\phi](\tau_1)+\text{ boundary terms}\lesssim\widetilde{\mathbb{F}}^{(1)}[\phi](\tau_1)\lesssim\widetilde{\mathbb{E}}^{T}[\phi](\tau_1)\,,\tag{4.100}$$

where $\widetilde{\mathbb{I}}[\phi](\tau_1,\tau_2)$ is the spacetime integral resulting from the integration by parts in $\mathbb{I}[\phi](\tau_1,\tau_2)$:

$$
\begin{aligned}
\widetilde{\mathbb{I}}[\phi](\tau_1,\tau_2) = \int_{R_{\tau_1}^{\tau_2}} & \left(2f'\left(\partial_{r^*}(\psi-\psi_0)\right)^2 - f\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)' |\nabla_{\mathbb{S}^2}(\psi-\psi_0)|^2\right) dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} - \\
& - \int_{R_{\tau_1}^{\tau_2}} \left(\frac{f'''}{2} + f\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)'\right)(\psi-\psi_0)^2\, dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} + \\
& + \int_{R_{\tau_1}^{\tau_2}} 2f'(\partial_{r^*}\psi_0 - g\psi_0)^2\, dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} - \\
& - \int_{R_{\tau_1}^{\tau_2}} \left(2(f'g)' + 2f'g^2 + f\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)' + \frac{f'''}{2}\right)\psi_0^2\, dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2}\ .
\end{aligned}
\tag{4.101}
$$

Consider now the following estimate:

$$
-f\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)' = \frac{2}{r^3}\left(1-\frac{3m}{r}\right)^2\left(1+\frac{6m}{r}\right)^{1/2}\left(1-\frac{2m}{r}\right) \gtrsim \frac{1}{r^3}\left(1-\frac{3m}{r}\right)^2\left(1-\frac{2m}{r}\right)\ .
\tag{4.102}
$$

Let us also consider the Poincaré inequality on the unit sphere (see for instance [3]) applied to the function $\psi - \psi_0$:

$$
\int_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2}(\psi-\psi_0)|^2\ \mathrm{dVol}_{\mathbb{S}^2} \geq 2\int_{\mathbb{S}^2}(\psi-\psi_0)^2\ \mathrm{dVol}_{\mathbb{S}^2}\ ,
\tag{4.103}
$$

We now introduce a parameter $0 < \varepsilon < 1$, whose usefulness will become clear later. Applying the Poincaré inequality, we obtain the following estimate:

$$
\begin{aligned}
& \int_{R_{\tau_1}^{\tau_2}} -f\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)' |\nabla_{\mathbb{S}^2}(\psi-\psi_0)|^2\ dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} \\
&= \int_{R_{\tau_1}^{\tau_2}} -\varepsilon f\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)' |\nabla_{\mathbb{S}^2}\psi|^2\ dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} + \\
&+ \int_{R_{\tau_1}^{\tau_2}} -(1-\varepsilon)f\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)' |\nabla_{\mathbb{S}^2}(\psi-\psi_0)|^2\ dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} \\
&\gtrsim \int_{R_{\tau_1}^{\tau_2}} \frac{1}{r^3}\left(1-\frac{3m}{r}\right)^2\left(1-\frac{2m}{r}\right)|\nabla_{\mathbb{S}^2}\psi|^2\ dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} + \\
&+ \int_{R_{\tau_1}^{\tau_2}} -2(1-\varepsilon)f\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)'(\psi-\psi_0)^2\, dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2}\ .
\end{aligned}
\tag{4.104}
$$

We also have

$$
\begin{aligned}
& \int_{R_{\tau_1}^{\tau_2}} 2f'(\partial_{r^*}\psi_0 - g\psi_0)^2\, dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} \\
&\geq \int_{R_{\tau_1}^{\tau_2}} \left(2\varepsilon f'(\partial_{r^*}\psi_0)^2 - 2\varepsilon f' g\partial_{r^*}(\psi_0^2)\right)\ dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} \\
&= \int_{R_{\tau_1}^{\tau_2}} \left(2\varepsilon f'(\partial_{r^*}\psi_0)^2 + 2\varepsilon(f'g)'\psi_0^2\right)\ dt\, dr^*\ \mathrm{dVol}_{\mathbb{S}^2} + \text{boundary terms}\,.
\end{aligned}
\tag{4.105}
$$

The boundary terms in the previous equation are the same as before, so their absolute value can be bounded by $\widetilde{\mathbb{E}}^T[\phi](\tau_1)$. A simple computation shows that

$$\int_{\mathbb{S}^2}\left((\partial_{r^*}(\psi-\psi_0))^2+(\partial_{r^*}\psi_0)^2\right)\mathrm{dVol}_{\mathbb{S}^2}=\int_{\mathbb{S}^2}(\partial_{r^*}\psi)^2\,\mathrm{dVol}_{\mathbb{S}^2}\ . \tag{4.106}$$

Putting all these assertions together, and using that $f'\sim\frac{1}{r^3}\left(1-\frac{2m}{r}\right)$, we obtain

$$\begin{aligned}
&\widetilde{\mathbb{E}}^T[\phi](\tau_1)\gtrsim\widetilde{\mathbb{I}}[\phi](\tau_1,\tau_2)\\
&\gtrsim\int_{R^{\tau_2}_{\tau_1}}\left(\frac{1}{r^3}(\partial_{r^*}\psi)^2+\frac{1}{r^3}\left(1-\frac{3m}{r}\right)^2|\nabla_{\mathbb{S}^2}\psi|^2\right)\left(1-\frac{2m}{r}\right)dt\,dr^*\,\mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{R^{\tau_2}_{\tau_1}}\left(2(1-\varepsilon)f\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)'+\frac{f'''}{2}+f\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)'\right)(\psi-\psi_0)^2\,dt\,dr^*\,\mathrm{dVol}_{\mathbb{S}^2}-\\
&\quad-\int_{R^{\tau_2}_{\tau_1}}\left(2f'g^2+2(1-\varepsilon)(f'g)'+\frac{f'''}{2}+f\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)'\right)\psi_0^2\,dt\,dr^*\,\mathrm{dVol}_{\mathbb{S}^2}\ .
\end{aligned} \tag{4.107}$$

Note the importance of the parameter ε, which allowed the introduction of the terms proportional to $(1-\varepsilon)$; eventually we will chose ε sufficiently small. After a lengthy computation, we get

$$\begin{aligned}
&2f\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)'+\frac{f'''}{2}+f\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)'\\
&=\frac{\left(1-\frac{2m}{r}\right)\left(8r^6+108mr^5-104m^2r^4-3342m^3r^3+2889m^4r^2+36180m^5r-54108m^6\right)}{2\left(1+\frac{6m}{r}\right)^{1/2}r^7(r+6m)^2}\ .
\end{aligned} \tag{4.108}$$

The polynomial on the numerator is strictly positive for $r\geq 2m$, and so we obtain, for such values of r,

$$\begin{aligned}
&2f\left(\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\right)'+\frac{f'''}{2}+f\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)'\\
&\gtrsim\frac{\left(1-\frac{2m}{r}\right)r^6}{2\left(1+\frac{6m}{r}\right)^{1/2}r^7(r+6m)^2}\gtrsim\frac{1}{r^3}\left(1-\frac{2m}{r}\right)\gtrsim\frac{1}{r^4}\left(1-\frac{2m}{r}\right)\ .
\end{aligned} \tag{4.109}$$

Let us now set

$$g(r)=\frac{1}{2}\left(1-\frac{2m}{r}\right)-\frac{1}{2}\frac{m^2}{r^3}\ . \tag{4.110}$$

For this choice of g, we have

$$\begin{aligned}
&2f'g^2+2(f'g)'+\frac{f'''}{2}+f\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)'\\
&=\frac{m\left(1-\frac{2m}{r}\right)\left(12r^7+13mr^6-48m^2r^5+1215m^3r^4+702m^4r^3-16551m^5r^2+21060m^6r-972m^7\right)}{2\left(1+\frac{6m}{r}\right)^{-1/2}r^8(r+6m)^3}\ .
\end{aligned} \tag{4.111}$$

Once again, the polynomial on the numerator is strictly positive for $r\geq 2m$, so, in this region, we have

$$
\begin{aligned}
&2f'g^2 + 2(f'g)' + \frac{f'''}{2} + f\left(\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)\right)' \\
&\gtrsim \frac{m\left(1-\frac{2m}{r}\right)r^7}{2\left(1+\frac{6m}{r}\right)^{-1/2} r^8 (r+6m)^3} \gtrsim \frac{1}{r^4}\left(1-\frac{2m}{r}\right) .
\end{aligned}
\tag{4.112}
$$

Taking $\varepsilon > 0$ sufficiently small, equation (4.107) then implies that

$$
\begin{aligned}
\widetilde{\mathbb{E}}^T[\phi](\tau_1) \gtrsim &\int_{R_{\tau_1}^{\tau_2}} \left(\frac{1}{r^3}\left(\partial_{r^*}\psi\right)^2 + \frac{1}{r^3}\left(1-\frac{3m}{r}\right)^2 |\nabla_{\mathbb{S}^2}\psi|^2\right)\left(1-\frac{2m}{r}\right) dt\, dr^*\, \mathrm{dVol}_{\mathbb{S}^2} + \\
&+ \int_{R_{\tau_1}^{\tau_2}} \frac{1}{r^4}\left((\psi-\psi_0)^2 + \psi_0^2\right)\left(1-\frac{2m}{r}\right) dt\, dr^*\, \mathrm{dVol}_{\mathbb{S}^2} .
\end{aligned}
\tag{4.113}
$$

Since the spherical average of ψ satisfies

$$
\int_{\mathbb{S}^2} \left((\psi-\psi_0)^2 + \psi_0^2\right) \mathrm{dVol}_{\mathbb{S}^2} = \int_{\mathbb{S}^2} \psi^2\, \mathrm{dVol}_{\mathbb{S}^2} ,
\tag{4.114}
$$

we finally obtain

$$
\int_{R_{\tau_1}^{\tau_2}} \left(\frac{1}{r^3}\left(\partial_{r^*}\psi\right)^2 + \frac{1}{r^3}\left(1-\frac{3m}{r}\right)^2 |\nabla_{\mathbb{S}^2}\psi|^2 + \frac{1}{r^4}\psi^2\right)\left(1-\frac{2m}{r}\right) dt\, dr^*\, \mathrm{dVol}_{\mathbb{S}^2} \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) .
\tag{4.115}
$$

To get the desired result we still need to recover the time derivative and to replace the radiation field ψ by ϕ in the previous inequality. Note that multiplication by r commutes with all derivatives except the radial one, so the only difficulty when replacing ψ by ϕ arises with this derivative.

$$
\begin{aligned}
&\int_{R_{\tau_1}^{\tau_2}} \frac{1}{r^3}\left(1-\frac{2m}{r}\right)\left(\partial_{r^*}(r\phi)\right)^2\, dt\, dr^*\, \mathrm{dVol}_{\mathbb{S}^2} \\
&= \int_{R_{\tau_1}^{\tau_2}} \frac{1}{r^3}\left(1-\frac{2m}{r}\right)\left(\partial_{r^*}\phi\right)^2 r^2\, dt\, dr^*\, \mathrm{dVol}_{\mathbb{S}^2} + \int_{R_{\tau_1}^{\tau_2}} \frac{1}{r^3}\left(1-\frac{2m}{r}\right)^2 \partial_{r^*}(r\phi^2)\, dt\, dr^*\, \mathrm{dVol}_{\mathbb{S}^2} \\
&= \int_{R_{\tau_1}^{\tau_2}} \frac{1}{r^3}\left(1-\frac{2m}{r}\right)\left(\partial_{r^*}\phi\right)^2 r^2\, dt\, dr^*\, \mathrm{dVol}_{\mathbb{S}^2} - \\
&\quad - \int_{R_{\tau_1}^{\tau_2}} \left(1-\frac{2m}{r}\right)\left(-\frac{3}{r^4}+\frac{16m}{r^5}-\frac{20m^2}{r^6}\right)\frac{\psi^2}{r}\, dt\, dr^*\, \mathrm{dVol}_{\mathbb{S}^2} + \text{boundary terms} ,
\end{aligned}
\tag{4.116}
$$

where in the last step we performed an integration by parts similar to what was done before. Note that the absolute value of the integrand function on the boundary terms is bounded by $\frac{1}{r^3}\left(1-\frac{2m}{r}\right)\psi^2$, so these terms can be bounded by $\widetilde{\mathbb{F}}^{(1)}[\phi](\tau_1)$, and consequently by $\widetilde{\mathbb{E}}^T[\phi](\tau_1)$, just like in the integration by parts done before. Similarly, we have

$$
\left|\left(1-\frac{2m}{r}\right)\left(-\frac{3}{r^4}+\frac{16m}{r^5}-\frac{20m^2}{r^6}\right)\frac{\psi^2}{r}\right| \lesssim \frac{1}{r^5}\left(1-\frac{2m}{r}\right)\psi^2 \lesssim \frac{1}{r^4}\left(1-\frac{2m}{r}\right)\psi^2 ,
\tag{4.117}
$$

so, by applying equation (4.115), we have

$$\left| \int_{R_{\tau_1}^{\tau_2}} \left(1 - \frac{2m}{r}\right) \left(-\frac{3}{r^4} + \frac{16m}{r^5} - \frac{20m^2}{r^6}\right) \frac{\psi^2}{r} \, dt \, dr^* \, \mathrm{dVol}_{\mathbb{S}^2} \right| \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) \,. \tag{4.118}$$

These considerations and equation (4.115) imply that

$$\int_{R_{\tau_1}^{\tau_2}} \left(\frac{1}{r^3} \left(\partial_{r^*}\phi\right)^2 + \frac{1}{r^3} \left(1 - \frac{3m}{r}\right)^2 \left|\nabla_{\mathbb{S}^2}\phi\right|^2 + \frac{1}{r^4}\phi^2 \right) \mathrm{dVol}_{\mathcal{M}} \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) \,. \tag{4.119}$$

To recover the time derivative we will apply the vector field method with the non-modified current $J^V_\mu[\phi]$ and make use of equation (4.119) to control the terms involving other derivatives. Consider the vector field V defined in (4.72), but now for a different choice of f. For such a vector field, the bulk term is given by

$$K^V[\phi] = \left(\frac{1}{2}\frac{f'}{1-\frac{2m}{r}} + \frac{f}{r}\right)\left(\partial_t\phi\right)^2 + \left(\frac{1}{2}\frac{f'}{1-\frac{2m}{r}} - \frac{f}{r}\right)\left(\partial_{r^*}\phi\right)^2 - \left(\frac{f'}{2} + \frac{m}{r^2}f\right)\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 \,. \tag{4.120}$$

Given $\delta > 0$, we want f to satisfy the following differential equation

$$\frac{1}{2}\frac{df}{dr} + \frac{f}{r} = \frac{1}{r^{3+\delta}}\left(1 - \frac{3m}{r}\right)^2 \,, \tag{4.121}$$

such that the coefficient of the time derivative in the bulk term is the desired one. Solving the ordinary differential equation (4.121), we obtain

$$f = \frac{C}{r^2} - \frac{2}{\delta r^{2+\delta}} + \frac{12m}{(1+\delta)r^{3+\delta}} - \frac{18m^2}{(2+\delta)r^{4+\delta}} \,, \tag{4.122}$$

where C is a real constant to be fixed later. If we took $\delta = 0$, the solution to the differential equation would have a term proportional to $\ln(r)/r^2$, which would be an obstruction to using equation (4.119) to control the non-temporal derivatives. For such a choice of f, it is immediate that the coefficient of the radial derivative satisfies

$$\left|\frac{1}{2}\frac{f'}{1-\frac{2m}{r}} - \frac{f}{r}\right| \lesssim \frac{1}{r^3} \,. \tag{4.123}$$

Regarding the term with the angular derivatives, we have

$$-\left(\frac{f'}{2} + \frac{m}{r^2}f\right) = \left(1 - \frac{3m}{r}\right)\left(\frac{C}{r^3} - \left(1 + \frac{2}{\delta}\right)\frac{1}{r^{3+\delta}} + \left(5 + \frac{12}{\delta}\right)\frac{m}{r^{4+\delta}} - \left(6 + \frac{18}{\delta}\right)\frac{m^2}{r^{5+\delta}}\right) \,. \tag{4.124}$$

However, in order to make use of equation (4.119), this coefficient must vanish quadratically at $r = 3m$. Therefore, we choose

$$C = \left(1 + \frac{2}{\delta}\right)\frac{1}{(3m)^\delta} - \left(5 + \frac{12}{\delta}\right)\frac{m}{(3m)^{1+\delta}} + \left(6 + \frac{18}{\delta}\right)\frac{m^2}{(3m)^{2+\delta}} \,, \tag{4.125}$$

so that the following inequality is satisfied:

$$\left|\frac{f'}{2}+\frac{m}{r^2}f\right|\lesssim\frac{1}{r^3}\left(1-\frac{3m}{r}\right)^2 . \tag{4.126}$$

Thus, applying the divergence theorem in the region $R_{\tau_1}^{\tau_2}$ to the current $J^V[\phi]$ we obtain

$$\begin{aligned}
&\int_{R_{\tau_1}^{\tau_2}}\frac{1}{r^{3+\delta}}\left(1-\frac{3m}{r}\right)^2(\partial_t\phi)^2\,\mathrm{dVol}_{\mathcal{M}}\\
&=\widetilde{\mathbb{E}}^V[\phi](\tau_1)-\widetilde{\mathbb{E}}^V[\phi](\tau_2)-\int_{R_{\tau_1}^{\tau_2}}\left(\frac{1}{2}\frac{f'}{1-\frac{2m}{r}}-\frac{f}{r}\right)(\partial_{r^*}\phi)^2\,\mathrm{dVol}_{\mathcal{M}}+\\
&\quad+\int_{R_{\tau_1}^{\tau_2}}\left(\frac{f'}{2}+\frac{m}{r^2}f\right)\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\mathrm{dVol}_{\mathcal{M}}\\
&\lesssim\widetilde{\mathbb{E}}^V[\phi](\tau_1)-\widetilde{\mathbb{E}}^V[\phi](\tau_2)+\int_{R_{\tau_1}^{\tau_2}}\left(\frac{1}{r^3}(\partial_{r^*}\phi)^2+\frac{1}{r^3}\left(1-\frac{3m}{r}\right)^2\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right)\mathrm{dVol}_{\mathcal{M}}\\
&\lesssim\widetilde{\mathbb{E}}^V[\phi](\tau_1)-\widetilde{\mathbb{E}}^V[\phi](\tau_2)+\widetilde{\mathbb{E}}^T[\phi](\tau_1)\,.
\end{aligned}\tag{4.127}$$

In the last step, we used equation (4.119) to bound the spacetime integral over $R_{\tau_1}^{\tau_2}$. To finish the proof, one needs to show that the absolute value of the energy flux of V can be bounded by the energy flux of T. Regarding the energy flux through L_τ, we can use the fact that f is bounded and Young's inequality as follows:

$$\begin{aligned}
&\left|J_\mu^V[\phi]\left(\left(1+\frac{2m}{r}\right)T^\mu-\frac{2m}{r}Y^\mu\right)\right|\\
&=|f|\left|\frac{m}{r}\left(1+\frac{2m}{r}\right)(T\phi)^2+\left(1-\frac{4m^2}{r^2}\right)(T\phi)(Y\phi)-\frac{m}{r}\left(1-\frac{2m}{r}\right)(Y\phi)^2+\frac{m}{r}\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right|\\
&\le|f|\left(\frac{m}{r}\left(1+\frac{2m}{r}\right)(T\phi)^2+\frac{1}{2}\left(1-\frac{2m}{r}\right)\left(1+\frac{2m}{r}\right)\left((T\phi)^2+(Y\phi)^2\right)\right)+\\
&\quad+|f|\left(\frac{m}{r}\left(1-\frac{2m}{r}\right)(Y\phi)^2+\frac{m}{r}\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right)\\
&=|f|\left(\frac{1}{2}\left(1+\frac{2m}{r}\right)(T\phi)^2+\left(\frac{1}{2}+\frac{2m}{r}\right)\left(1-\frac{2m}{r}\right)(Y\phi)^2+\frac{m}{r}\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right)\\
&\lesssim\left(\frac{1}{2}\left(1+\frac{2m}{r}\right)(T\phi)^2+\frac{1}{2}\left(1-\frac{2m}{r}\right)(Y\phi)^2+\frac{1}{2}\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right).
\end{aligned}\tag{4.128}$$

For the energy flux through S_τ, we have

$$\begin{aligned}
\left|J_\mu^V[\phi]T^\mu\left(1-\frac{2m}{r}\right)^{-1}\right|&=|f(T\phi)(\partial_r\phi)|\\
&\le\frac{|f|}{2}\left(\left(1-\frac{2m}{r}\right)^{-1}(T\phi)^2+\left(1-\frac{2m}{r}\right)(\partial_r\phi)^2\right)\\
&\lesssim\frac{1}{2}\left(1-\frac{2m}{r}\right)^{-1}(T\phi)^2+\frac{1}{2}\left(1-\frac{2m}{r}\right)(\partial_r\phi)^2\,.
\end{aligned}\tag{4.129}$$

In addition, we can easily bound the flux through N_τ:

$$\left|fJ_\mu^V[\phi](\partial_v)^\mu\right|=|f|\left|(\partial_v)^2-\frac{1}{4}\left(1-\frac{2m}{r}\right)\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right|\lesssim(\partial_v)^2+\frac{1}{4}\left(1-\frac{2m}{r}\right)\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 . \tag{4.130}$$

The previous three inequalities imply that

$$\left|\widetilde{\mathbb{E}}^V[\phi](\tau)\right| \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau)\,, \tag{4.131}$$

so equation (4.127) allows to conclude that

$$\int_{R_{\tau_1}^{\tau_2}} \frac{1}{r^{3+\delta}}\left(1-\frac{3m}{r}\right)^2 (\partial_t\phi)^2 \,\mathrm{dVol}_{\mathcal{M}} \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1)\,. \tag{4.132}$$

□

If necessary, we now redefine r_0 and R such that $r_0 < 3m < R$. An immediate consequence of Proposition 4.4.2 is that we can control a nondegenerate energy on spatially compact regions away from the horizon and not intersecting the photon sphere. This will be particularly useful when proving the Dafermos-Rodnianski hierarchy in Section 4.4.2.

Corollary 4.4.3. *Let ϕ be a solution of the wave equation with compactly supported initial data on Σ_0. Then, given $\tau_1 \geq 0$ and a spatially compact region K contained in $R_{\tau_1}^{\infty}$ such that $K\cap\{r=2m\} = K\cap\{r=3m\}=\varnothing$, the following inequality holds:*

$$\int_K \left((\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 + \phi^2\right)\mathrm{dVol}_{\mathcal{M}} \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1)\,, \tag{4.133}$$

where the constant in the previous inequality is independent of τ_1, but depends on the maximum and minimum values of the radius function on K.

The next step is to obtain an Integrated Local Energy Decay estimate that does not degenerate on the event horizon, thus allowing to control the radial derivative on the whole spacetime. For this purpose, we will now apply some of the properties of the redshift vector field deduced previously. As expected, in this case the spacetime integral will now be bounded by the energy flux $\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau)$ associated to the redshift vector field, since it also does not degenerate on $\mathscr{H}^+$. In what follows, we also show that we can replace the radial derivative $\partial_r\phi$ by the radial derivative ∂_ρ tangent to the hypersurfaces $\widetilde{\Sigma}_\tau$.

Proposition 4.4.4. ***(Non-degenerate Integrated Local Energy Decay estimate)*** *Let ϕ be a solution of the wave equation with compactly supported initial data Σ_0, and let ∂_ρ be the radial derivative tangent to $\widetilde{\Sigma}_\tau$ defined previously. Then, given $\tau_2 \geq \tau_1 \geq 0$ and $\delta > 0$, the following inequality holds:*

$$\int_{R_{\tau_1}^{\tau_2}} \left(\frac{1}{r^3}(\partial_\rho\phi)^2 + \frac{1}{r^3}\left(1-\frac{3m}{r}\right)^2\left(\frac{1}{r^\delta}(\partial_t\phi)^2 + \left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) + \frac{\phi^2}{r^4}\right)\mathrm{dVol}_{\mathcal{M}} \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1)\,. \tag{4.134}$$

Proof. Equation (4.41) implies that

$$\int_{R_{\tau_1}^{\tau_2}\cap\{r\leq 0.9\,r_0\}} J_\mu^N[\phi]n_L^\mu \,\mathrm{dVol}_{\mathcal{M}} \lesssim \int_{R_{\tau_1}^{\tau_2}\cap\{0.9\,r_0\leq r\leq r_0\}} J_\mu^N[\phi]n_L^\mu \,\mathrm{dVol}_{\mathcal{M}} + \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1)\,. \tag{4.135}$$

Note that for $0.9\,r_0 \leq r \leq r_0$,

$$J_\mu^N[\phi]n_L^\mu \sim (\partial_t\phi)^2 + (\partial_r\phi)^2 + |\nabla_{\mathbb{S}^2}\phi|^2 \,. \tag{4.136}$$

Therefore, applying the degenerate Integrated Local Energy Decay estimate (4.70), we have

$$\int_{R_{\tau_1}^{\tau_2}\cap\{r\leq 0.9\,r_0\}} J_\mu^N[\phi]n_L^\mu \,\mathrm{dVol}_{\mathcal{M}} \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1)\,. \tag{4.137}$$

On the other hand, the redshift vector fields satisfies

$$J_\mu^N[\phi]n_L^\mu \sim (\partial_t\phi)^2 + (Y\phi)^2 + |\nabla_{\mathbb{S}^2}\phi|^2 \,, \tag{4.138}$$

for $r \leq r_0$, so equation (4.137) implies

$$\int_{R_{\tau_1}^{\tau_2}\cap\{r\leq 0.9\,r_0\}} \left((\partial_t\phi)^2 + (Y\phi)^2 + |\nabla_{\mathbb{S}^2}\phi|^2\right) \mathrm{dVol}_{\mathcal{M}} \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1)\,. \tag{4.139}$$

Noting that for $0.9\,r_0 \leq r \leq r_0$ we have

$$(T\phi)^2 + (Y\phi)^2 \sim (T\phi)^2 + (\partial_r\phi)^2\,, \tag{4.140}$$

the degenerate Integrated Local Energy Decay estimate (4.70) implies that

$$\int_{R_{\tau_1}^{\tau_2}\cap\{0.9\,r_0\leq r\leq r_0\}} \left((\partial_t\phi)^2 + (Y\phi)^2 + |\nabla_{\mathbb{S}^2}\phi|^2\right) \mathrm{dVol}_{\mathcal{M}} \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1)\,. \tag{4.141}$$

Since $(T\phi)^2 + (\partial_\rho\phi)^2 \sim (T\phi)^2 + (\partial_r\phi)^2$, for $r \geq R$, adding equations (4.139) and (4.141) to the degenerate Integrated Local Energy Decay estimate yields the desired result.

□

We end the discussion about Integrated Local Energy Decay estimates with a fully nondegenerate result (that is, one that also does not degenerate at the photon sphere), which will also be relevant in the proof of the Dafermos-Rodnianski hierarchy in order to obtain energy decay estimates. Since the Schwarzschild spacetime is static and spherically symmetric, the Integrated Local Energy Decay estimate also holds with $\Omega_i\phi$ and $T\phi$ in place of ϕ. Hence, considering that $\sum_i(\Omega_i\phi)^2 = |\nabla_{\mathbb{S}^2}\phi|^2$, we obtain the following result:

Proposition 4.4.5. *Let ϕ be a solution of the wave equation with compactly supported initial data on Σ_0. Then, given $\tau_2 \geq \tau_1 \geq 0$, the following inequality holds:*

$$\int_{R_{\tau_1}^{\tau_2}} \frac{1}{r^4}\left((\partial_t\phi)^2 + (\partial_\rho\phi)^2 + |\nabla_{\mathbb{S}^2}\phi|^2\right) \mathrm{dVol}_{\mathcal{M}} \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) + \widetilde{\mathbb{E}}^{\tilde{N}}[T\phi](\tau_1) + \sum_i \widetilde{\mathbb{E}}^{\tilde{N}}[\Omega_i\phi](\tau_1)\,. \tag{4.142}$$

Remark 4.4.6. *Note that the function $T\phi$ seems to be ill defined on Σ_0. Nevertheless, we can take $T\phi$ to be the solution of the wave equation with initial conditions $T\phi\big|_{\Sigma_0} = \phi_1$ and $T(T\phi)\big|_{\Sigma_0}$, where the latter can be determined from the wave equation as follows:*

$$T(T\phi)\big|_{\Sigma_0} = \left(1-\frac{2m}{r}\right)\frac{1}{r^2}\partial_r\left(r^2\left(1-\frac{2m}{r}\right)\phi_0\right) + \left(1-\frac{2m}{r}\right)\frac{1}{r^2}\Delta_{\mathbb{S}^2}\phi_0\,. \tag{4.143}$$

As a corollary, we can now control the nondegenerate energy on arbitrary spatially compact regions, which can intersect both the event horizon and the photon sphere, at the cost of adding extra energy fluxes to the right-hand side of the inequality. Nonetheless, since T and Ω_i are Killing fields, these extra terms are also bounded and will also satisfy a weak decay result that will allow us to deduce the main final decay estimate.

Corollary 4.4.7. *Let ϕ satisfy the wave equation with compactly supported initial data on Σ_0. Then, given $\tau_2 \geq \tau_1 \geq 0$ and $R_0 > 2m$, we have*

$$\int_{R_{\tau_1}^{\tau_2}\cap\{r\leq R_0\}} \left((\partial_t\phi)^2 + (\partial_\rho\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) \mathrm{dVol}_{\mathcal{M}} \lesssim_{R_0} \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) + \widetilde{\mathbb{E}}^{\tilde{N}}[T\phi](\tau_1) + \sum_i \widetilde{\mathbb{E}}^{\tilde{N}}[\Omega_i\phi](\tau_1)\,. \tag{4.144}$$

4.4.2 Dafermos-Rodnianski hierarchy

Just like in the case of the Minkowski spacetime, we prove the exact same r-weighted inequality, which will be used to prove the Dafermos-Rodnianski hierarchy for the Schwarzschild spacetime and, consequently, an energy decay result. The main difference is that, in this case, the inequality only holds for $p \leq 3$, which is a necessary condition for some extra terms arising in the proof to have the correct sign. We emphasize that the proof of all results in this Section are quite similar to the Minkowski spacetime case, as we again follow the approach introduced in [9] to deduce energy decay estimates.

Proposition 4.4.8. *Let ϕ satisfy be a solution of the wave equation with compactly supported initial data on Σ_0. Given $p \leq 3$ and $\tau_2 \geq \tau_1 \geq 0$, we have*

$$\begin{aligned}\int_{N_{\tau_2}} r^{p-2}\left(\partial_v\psi\right)^2 r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2} + \int_{D_{\tau_1}^{\tau_2}} \left(p\, r^{p-3}\left(\partial_v\psi\right)^2 + \frac{2-p}{4} r^{p-1}\frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2\right) r^2\, dt\, dr\ \mathrm{dVol}_{\mathbb{S}^2} \\ \lesssim \widetilde{\mathbb{E}}^{T}[\phi](\tau_1) + \int_{N_{\tau_1}} r^{p-2}\left(\partial_v\psi\right)^2 r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2}\ .\end{aligned} \tag{4.145}$$

Proof. Let ζ be a smooth cut-off function depending only on the radius, satisfying $\zeta(r) = 0$ for $r \leq R+1/2$ and $\zeta(r) = 1$ for $r \geq R+1$. Let V be a vector field defined as

$$V = r^q\partial_v\,, \tag{4.146}$$

where $q = p-2$. Applying the divergence theorem to the current $J_\mu^V[\zeta\psi]$ in the region $D_{\tau_1}^{\tau_2}$, we have

$$\int_{D_{\tau_1}^{\tau_2}} \left(K^V[\zeta\psi] + \Box_g(\zeta\psi)(V(\zeta\psi))\right) r^2\, dt\, dr\ \mathrm{dVol}_{\mathbb{S}^2}$$
$$= \int_{N_{\tau_1}} J^V_\mu[\zeta\psi]\,(\partial_v)^\mu\ dv\ \mathrm{dVol}_{\mathbb{S}^2} - \int_{N_{\tau_2}} J^V_\mu[\zeta\psi]\,(\partial_v)^\mu\ dv\ \mathrm{dVol}_{\mathbb{S}^2} - \tag{4.147}$$
$$- \int_{\mathscr{I}^+\cap\{\tau_1 - R\le u\le \tau_2 - R\}} J^V_\mu[\zeta\psi]\,(\partial_u)^\mu\ du\ \mathrm{dVol}_{\mathbb{S}^2}\ .$$

Let us start by computing the first bulk term. The deformation tensor of V is given by

$$\pi^V = \frac{1}{4}\left(1-\frac{2m}{r}\right)^2 qr^{q-1}\,du^2 - \left(\frac{m}{2r^2}\left(1-\frac{2m}{r}\right)r^q + \frac{1}{4}\left(1-\frac{2m}{r}\right)^2 qr^{q-1}\right) du\,dv + \\ + \left(1-\frac{2m}{r}\right)\frac{r^{q+1}}{2}\,d\Omega^2\,, \tag{4.148}$$

so that the first bulk term is

$$K^V[\zeta\psi] = 2r^{q-1}\partial_u(\zeta\psi)\partial_v(\zeta\psi) + qr^{q-1}\left(\partial_v(\zeta\psi)\right)^2 - \left(\frac{2m}{r^2}r^q + \left(1-\frac{2m}{r}\right)qr^{q-1}\right)\frac{1}{4r^2}\left|\nabla_{\mathbb{S}^2}\zeta\psi\right|^2\ . \tag{4.149}$$

This can be written as the corresponding bulk term for ψ plus an error term:

$$K^V[\zeta\psi] = K^V[\psi] + Z_1[\zeta,\psi]\,, \tag{4.150}$$

where $Z_1[\zeta,\psi]$ is a quadratic form on $(\psi,\partial_\mu\psi)$ with smooth coefficients supported on $\{r\le R+1\}$. Therefore, by applying Young's inequality, we have, for $R\le r\le R+1$,

$$\begin{aligned} |Z_1[\zeta,\psi]| &\lesssim_R (\partial_u\psi)^2 + (\partial_v\psi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\psi\right|^2 + \phi^2 \\ &\lesssim_R (\partial_u\phi)^2 + (\partial_v\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 + \phi^2 \\ &\sim (\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 + \phi^2\,. \end{aligned} \tag{4.151}$$

We now use a similar procedure for the second bulk term. First note that, since ϕ satisfies the wave equation, ψ satisfies

$$-\frac{4}{1-\frac{2m}{r}}\partial_u\partial_v\psi + \frac{1}{r^2}\Delta_{\mathbb{S}^2}\psi - \frac{2m}{r^3}\psi = 0\,, \tag{4.152}$$

so that we can write the d'Alembertian of ψ as

$$\Box_g\psi = \frac{2}{r}(\partial_v\psi - \partial_u\psi) + \frac{2m}{r^3}\psi\,. \tag{4.153}$$

Hence, we can write the second bulk term as

$$\Box_g(\zeta\psi)\,(V(\zeta\psi)) = 2r^{q-1}(\partial_v\psi)^2 - 2r^{q-1}\partial_u\psi\partial_v\psi + 2mr^{q-3}\psi\partial_v\psi + Z_2[\zeta,\psi]\,, \tag{4.154}$$

where $Z_2[\zeta,\psi]$ is an error term supported on $\{r \le R+1\}$ which is also a quadratic form on $(\psi, \partial_\mu\psi)$ with smooth coefficients, so that one also has, for $R \le r \le R+1$,

$$|Z_2[\zeta,\psi]| \lesssim_R (\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}|\nabla_{\mathbb{S}^2}\phi|^2 + \phi^2 \,. \tag{4.155}$$

Using the Integrated Local Energy Decay estimate in Corollary 4.4.3, one proves that the error terms can be bounded by the initial energy:

$$\begin{aligned}
&\int_{D_{\tau_1}^{\tau_2}} \left(|Z_1[\zeta,\psi]| + |Z_2[\zeta,\psi]|\right) r^2\, dt\, dr\, \mathrm{dVol}_{\mathbb{S}^2} \\
&\lesssim_R \int_{D_{\tau_1}^{\tau_2}\cap\{R+1/2\le r\le R+1\}} \left((\partial_t\phi)^2 + (\partial_r\phi)^2 + \frac{1}{r^2}|\nabla_{\mathbb{S}^2}\phi|^2 + \phi^2\right) r^2\, dt\, dr\, \mathrm{dVol}_{\mathbb{S}^2} \\
&\lesssim_R \widetilde{\mathbb{E}}^T[\phi](\tau_1)\,,
\end{aligned} \tag{4.156}$$

where we used the fact that $R > 3m$, so that the degeneracy at the photon sphere plays no role in the previous estimate. The remaining part of the bulk term is given by

$$\begin{aligned}
&K^V[\psi] + 2r^{q-1}(\partial_v\psi)^2 - 2r^{q-1}\partial_u\psi\partial_v\psi + 2mr^{q-3}\psi\partial_v\psi \\
&= (q+2)r^{q-1}(\partial_v\psi)^2 + mr^{q-3}\partial_v\left(\psi^2\right) - \left(\frac{2m}{r^2}r^q + \left(1-\frac{2m}{r}\right)qr^{q+1}\right)\frac{1}{4r^2}|\nabla_{\mathbb{S}^2}\phi|^2 \,.
\end{aligned} \tag{4.157}$$

To control the error terms arising in the boundary integrals, we use a similar procedure:

$$J^V_\mu[\zeta\psi](\partial_v)^\mu = r^q(\partial_v\psi)^2 + Z_3[\zeta,\psi]\,, \tag{4.158}$$

where $Z_3[\zeta,\psi]$ is a quadratic form on $(\psi,\partial_v\psi)$ with smooth coefficients supported in $\{r \le R+1\}$, so we have that, for $R \le r \le R+1$,

$$|Z_3[\zeta,\psi]| \lesssim_R (\partial_v\phi)^2 + \frac{\phi^2}{r^2}\,. \tag{4.159}$$

Using equations (4.64) and (4.21), we have

$$\begin{aligned}
&\int_{N_{\tau_i}} |Z_3[\zeta,\psi]| r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2} \\
&\lesssim_R \int_{N_{\tau_i}\cap\{R+1/2\le r\le R+1\}} \left((\partial_v\phi)^2 + \frac{\phi^2}{r^2}\right) r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2} \\
&\lesssim_R \widetilde{\mathbb{E}}^T[\phi](\tau_i) \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1)\,,
\end{aligned} \tag{4.160}$$

for $i = 1, 2$. Moreover, notice that the integral over $\mathscr{I}^+$ is non-negative:

$$\begin{aligned}
J^V_\mu[\zeta\psi](\partial_u)^\mu &= \partial_u(\zeta\psi)\partial_v(\zeta\psi) - \frac{1}{2}\left(-\frac{4}{1-\frac{2m}{r}}\partial_u(\zeta\psi)\partial_v(\zeta\psi) + \frac{1}{r^2}|\nabla_{\mathbb{S}^2}\zeta\psi|^2\right)\left(-\frac{1}{2}\right)\left(1-\frac{2m}{r}\right) \\
&= \left(1-\frac{2m}{r}\right)\frac{1}{4r^2}|\nabla_{\mathbb{S}^2}\zeta\psi|^2 \ge 0\,.
\end{aligned} \tag{4.161}$$

All these assertions allow us to conclude that

$$\int_{D_{\tau_1}^{\tau_2}} \left((q+2)r^{q-1}(\partial_v\psi)^2 + mr^{q-3}\partial_v\left(\psi^2\right) - \left(\frac{2m}{r^2}r^q + \left(1-\frac{2m}{r}\right)qr^{q+1}\right)\frac{1}{4r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 \right) r^2\, dt\, dr\ \mathrm{dVol}_{\mathbb{S}^2} +$$
$$+ \int_{N_{\tau_2}} r^q\left(\partial_v\psi\right)^2 r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2} \leq C_R\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) + \int_{N_{\tau_2}} r^q\left(\partial_v\psi\right)^2 r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2}\ . \tag{4.162}$$

Since $p \leq 3$, we have $q \leq 1$, so we can apply the degenerate Integrated Local Energy Decay estimate as follows:

$$\int_{D_{\tau_1}^{\tau_2}} \frac{2m}{r^2}r^q\frac{1}{4r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 \mathrm{dVol}_{\mathcal{M}} \lesssim \int_{D_{\tau_1}^{\tau_2}} \frac{1}{r^3}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 \mathrm{dVol}_{\mathcal{M}} \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1)\,, \tag{4.163}$$

where we used $D_{\tau_1}^{\tau_2} \cap \{r = 3m\} = \varnothing$ to apply the result in Corollary 4.4.3 with non-vanishing coefficient at the photon sphere. Additionally, we have $1 - \frac{2m}{r} \sim 1$ for $r \geq R$, so equation (4.162) implies that

$$\begin{aligned}&\int_{D_{\tau_1}^{\tau_2}} \left((q+2)r^{q-1}(\partial_v\psi)^2 + mr^{q-3}\partial_v\left(\psi^2\right) - qr^{q+1}\frac{1}{4r^2}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 \right) r^2\, dt\, dr\ \mathrm{dVol}_{\mathbb{S}^2} +\\ &+ \int_{N_{\tau_2}} r^q\left(\partial_v\psi\right)^2 r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2} \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{N_{\tau_1}} r^q\left(\partial_v\psi\right)^2 r^2\, dv\ \mathrm{dVol}_{\mathbb{S}^2}\ .\end{aligned} \tag{4.164}$$

Finally, we want to do an integration by parts on the term second term in the first integral of the previous inequality. However, this would lead to a boundary term at $\{r = R\}$ which we cannot control, so instead we write this term as

$$mr^{q-3}\partial_v\left(\psi^2\right) = mr^{q-3}\partial_v\left(\zeta\psi^2\right) + Z_4[\zeta,\psi]\,, \tag{4.165}$$

where $Z_4[\zeta,\psi]$ is a quadratic form on (ψ,∂_v) with smooth coefficients supported on $\{r \leq R+1\}$. Therefore we have, for $R \leq r \leq R+1$,

$$\left|Z_4[\zeta,\psi]\right| \lesssim_R \left(\partial_v\phi\right)^2 + \phi^2\,, \tag{4.166}$$

and its integral can be bounded by $\widetilde{\mathbb{E}}^T[\phi](\tau_1)$ using Corollary 4.4.3 in the same way as was done for the previous error terms. For the remaining term in (4.165), we have

$$\begin{aligned}&\int_{D_{\tau_1}^{\tau_2}} r^{q-1}\partial_v\left(\zeta\psi^2\right)\, dt\, dr\ \mathrm{dVol}_{\mathbb{S}^2}\\ &\sim \int_{D_{\tau_1}^{\tau_2}} r^{q-1}\partial_v\left(\zeta\psi^2\right)\, du\, dv\ \mathrm{dVol}_{\mathbb{S}^2}\\ &= -\int_{D_{\tau_1}^{\tau_2}} (q-1)r^{q-2}\frac{1}{2}\left(1-\frac{2m}{r}\right)\zeta\psi^2\, du\, dv\ \mathrm{dVol}_{\mathbb{S}^2} + \int_{\mathscr{I}^+\cap\{\tau_1-r^*(R)\leq u\leq\tau_2-r^*(R)\}} r^{q-1}\zeta\psi^2 \geq 0\,,\end{aligned} \tag{4.167}$$

where, in the last step, we used that $q \leq 1$ (since $p \leq 3$). The r-weighted inequality now follows from equations (4.164) and (4.167).

□

This r-weighted inequality allows us to prove the Dafermos-Rodnianski hierarchy for the Schwarzschild spacetime. This hierarchy is slightly different from the corresponding hierarchy for the Minkowski spacetime, because the only fully nondegenerate Integrated Local Energy Decay estimate is the one in Proposition 4.4.5, which has some extra terms on the right-hand side when compared with the estimate on Proposition 3.2.2.

Proposition 4.4.9. ***(Dafermos-Rodnianski hierarchy)*** *Let ϕ be a solution of the wave equation with compactly supported initial data on Σ_0, and let $\psi = r\phi$ be the radiation field. Given $\tau_2 \geq \tau_1 \geq 0$, we have*

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau)\,d\tau \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) + \widetilde{\mathbb{E}}^{\tilde{N}}[T\phi](\tau_1) + \sum_i \widetilde{\mathbb{E}}^{\tilde{N}}[\Omega_i\phi](\tau_1) + \int_{N_{\tau_1}} \frac{(\partial_v\psi)^2}{r} r^2\,dv\;\mathrm{dVol}_{\mathbb{S}^2}\,, \\
&\int_{\tau_1}^{\tau_2} \left(\int_{N_\tau} \frac{(\partial_v\psi)^2}{r} r^2\,dv\;\mathrm{dVol}_{\mathbb{S}^2} \right) d\tau \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) + \int_{N_{\tau_1}} (\partial_v\psi)^2\, r^2\,dv\;\mathrm{dVol}_{\mathbb{S}^2}\;.
\end{aligned} \tag{4.168}$$

Proof. The second inequality of the Proposition follows trivially from setting $p = 2$ in equation (4.145). Regarding the first inequality, we set $p = 1$ in equation (4.145) to obtain

$$\int_{D_{\tau_1}^{\tau_2}} \left((\partial_v\psi)^2 + \frac{1}{4}\left|\nabla_{\mathbb{S}^2}\phi\right|^2 \right) dt\,dr\;\mathrm{dVol}_{\mathbb{S}^2} \lesssim \widetilde{\mathbb{E}}^{T}[\phi](\tau_1) + \int_{N_{\tau_1}} \frac{(\partial_v\psi)^2}{r} r^2\,dv\;\mathrm{dVol}_{\mathbb{S}^2}\;. \tag{4.169}$$

To finish the proof we have to perform an additional computation to replace ψ by ϕ on the left-hand side of the inequality:

$$\begin{aligned}
&\int_{D_{\tau_1}^{\tau_2}} (\partial_v\psi)^2\;dt\,dr\;\mathrm{dVol}_{\mathbb{S}^2} \\
&\sim \int_{\tau_1-r^*(R)}^{\tau_2-r^*(R)} \int_{\mathbb{S}^2} \int_{u+2r^*(R)}^{\infty} (\partial_v(r\phi))^2\;dv\;\mathrm{dVol}_{\mathbb{S}^2}\;du \\
&= \int_{\tau_1-r^*(R)}^{\tau_2-r^*(R)} \int_{\mathbb{S}^2} \int_{u+2r^*(R)}^{\infty} \left((\partial_v\phi)^2 r^2 + \left(1-\frac{2m}{r}\right) r\phi\,\partial_v\phi + \frac{1}{4}\left(1-\frac{2m}{r}\right)^2 \phi^2 \right) dv\;\mathrm{dVol}_{\mathbb{S}^2}\;du \\
&\sim \int_{\tau_1-r^*(R)}^{\tau_2-r^*(R)} \int_{\mathbb{S}^2} \int_{u+2r^*(R)}^{\infty} \left((\partial_v\phi)^2 r^2 + r\partial_v(\phi^2) + \frac{1}{2}\left(1-\frac{2m}{r}\right)\phi^2 \right) dv\;\mathrm{dVol}_{\mathbb{S}^2}\;du \\
&= \int_{\tau_1-r^*(R)}^{\tau_2-r^*(R)} \int_{\mathbb{S}^2} \int_{u+2r^*(R)}^{\infty} \left((\partial_v\phi)^2 r^2 + \partial_v(r\phi^2) \right) dv\;\mathrm{dVol}_{\mathbb{S}^2}\;du \\
&= \int_{\tau_1-r^*(R)}^{\tau_2-r^*(R)} \int_{\mathbb{S}^2} \int_{u+2r^*(R)}^{\infty} \left((\partial_v\phi)^2 r^2 + \partial_v(r(1-\zeta)\phi^2) + \partial_v(r\zeta\phi^2) \right) dv\;\mathrm{dVol}_{\mathbb{S}^2}\;du \\
&= \int_{\tau_1-r^*(R)}^{\tau_2-r^*(R)} \int_{\mathbb{S}^2} \int_{u+2r^*(R)}^{\infty} \left((\partial_v\phi)^2 r^2 + \partial_v(r(1-\zeta)\phi^2) \right) dv\;\mathrm{dVol}_{\mathbb{S}^2}\;du + \\
&\quad + \int_{\mathbb{S}^2} \int_{u+2r^*(R)}^{\infty} \lim_{v\to\infty} (r\phi^2)\;\mathrm{dVol}_{\mathbb{S}^2}\;du \\
&\geq \int_{\tau_1-r^*(R)}^{\tau_2-r^*(R)} \int_{\mathbb{S}^2} \int_{u+2r^*(R)}^{\infty} \left((\partial_v\phi)^2 r^2 + \partial_v(r(1-\zeta)\phi^2) \right) dv\;\mathrm{dVol}_{\mathbb{S}^2}\;du
\end{aligned} \tag{4.170}$$

The second term is a quadratic form of $(\phi, \partial_v\phi)$ with smooth coefficients supported on $\{r \leq R+1\}$, so we can use Corollary 4.4.3 to bound it by the initial energy, just like we did for the error terms in the proof of equation (4.145), and obtain

$$\int_{\tau_1}^{\tau_2} \left(\int_{L_\tau} \left((\partial_v \phi)^2 + \frac{1}{4r^2} |\nabla_{\mathbb{S}^2} \phi|^2 \right) r^2 \, dv \, \mathrm{dVol}_{\mathbb{S}^2} \right) d\tau \lesssim \widetilde{\mathbb{E}}^T[\phi](\tau_1) + \int_{N_{\tau_1}} \frac{(\partial_v \psi)^2}{r} r^2 \, dv \, \mathrm{dVol}_{\mathbb{S}^2} \, . \tag{4.171}$$

Since we have $|\nabla t^*| \sim 1$ for $r \leq r_0$ and $|\nabla t| \sim 1$ for $r_0 \leq r \leq R$, we can add equation (4.144) with $R_0 = R$ to the previous inequality and use the coarea formula to obtain the desired inequality. □

4.4.3 Energy decay and pointwise estimate

Despite the slight change in the hierarchy when compared to the Minkowski spacetime case, it can still be used to prove energy decay.

Theorem 4.4.10. *If ϕ a solution of the wave equation with compactly supported initial data on Σ_0, then there exists a constant $C > 0$ such that*

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau) \leq \frac{C}{\tau^2} \, , \quad \forall \tau \geq 0 \, . \tag{4.172}$$

Proof. Using the same notation as for the Minkowski spacetime, we define:

$$\begin{aligned} f_1(\tau) &= \int_{N_\tau} \frac{(\partial_v \psi)^2}{r} r^2 \, dv \, \mathrm{dVol}_{\mathbb{S}^2} \, , \\ f_2(\tau) &= \int_{N_\tau} (\partial_v \psi)^2 r^2 \, dv \, \mathrm{dVol}_{\mathbb{S}^2} \, . \end{aligned} \tag{4.173}$$

Since the support of the initial data is contained in the region $\{r \leq R\}$, these functions satisfy $f_1(0) = f_2(0) = 0$. Then the Dafermos-Rodnianski hierarchy can be expressed as

$$\begin{aligned} &\int_{\tau_1}^{\tau_2} \widetilde{\mathbb{E}}^{\tilde{N}}[\phi] \, d\tau \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) + \widetilde{\mathbb{E}}^{\tilde{N}}[T\phi](\tau_1) + \sum_i \widetilde{\mathbb{E}}^{\tilde{N}}[\Omega_i \phi](\tau_1) + f_1(\tau_1) \, , \\ &\int_{\tau_1}^{\tau_2} f_1(\tau) \, d\tau \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_1) + f_2(\tau_1) \, . \end{aligned} \tag{4.174}$$

First note that the energy decays with $1/\tau$:

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau) = \frac{1}{\tau} \int_0^\tau \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau) \, d\tau' \lesssim \frac{1}{\tau} \int_0^\tau \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau') \, d\tau' \lesssim \frac{\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](0) + \widetilde{\mathbb{E}}^{\tilde{N}}[T\phi](0) + \sum_i \widetilde{\mathbb{E}}^{\tilde{N}}[\Omega_i \phi](0)}{\tau} \, . \tag{4.175}$$

The second inequality of the hierarchy implies that the function f_1 is integrable on $[0, \infty)$:

$$\int_0^\infty f_1(\tau) \, d\tau \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](0) < \infty \, . \tag{4.176}$$

Define a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ such that $\tau_n \in [2^n, 2^{n+1})$ and

$$f_1(\tau_n) = \frac{\int_{2^n}^{2^{n+1}} f_1(\tau)\, d\tau}{2^n}, \quad \forall n \in \mathbb{N}\,. \tag{4.177}$$

Therefore, f_1 satisfies

$$f_1(\tau_n) \leq \frac{C\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](0)}{2^n} \lesssim \frac{1}{2^{n+1}} \leq \frac{1}{\tau_n}\,, \tag{4.178}$$

which immediately implies that

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_{n+2}) \lesssim \frac{\int_{\tau_n}^{\tau_{n+2}} \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau)\, d\tau}{\tau_{n+2}-\tau_n} \lesssim \frac{\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_n) + \widetilde{\mathbb{E}}^{\tilde{N}}[T\phi](\tau_n) + \sum_i \widetilde{\mathbb{E}}^{\tilde{N}}[\Omega_i\phi](\tau_n) + f_1(\tau_n)}{2^{n+1}-2^n}\,. \tag{4.179}$$

Since equation (4.175) also holds with $T\phi$ and $\Omega_i\phi$ in place of ϕ, estimate (4.179) implies that

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_{n+2}) \lesssim \frac{1}{\tau_n 2^n} \lesssim \frac{1}{\tau_{n+2}^2}\,. \tag{4.180}$$

Given $\tau > 0$, let $n \in \mathbb{N}$ be such that $\tau \in [2^n, 2^{n+1})$. Then, we have

$$\widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau) \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](2^n) \lesssim \widetilde{\mathbb{E}}^{\tilde{N}}[\phi](\tau_{n-1}) \lesssim \frac{1}{\tau_{n-1}^2} \lesssim \frac{1}{\tau_{n+1}^2} \leq \frac{1}{\tau^2}\,. \tag{4.181}$$

□

Finally, the pointwise decay result follows immediately from the energy decay estimate.

Theorem 4.4.11. *Let ϕ be a solution of the wave equation with compactly supported initial data on Σ_0. Then there exists a constant $C > 0$ such that*

$$\sup_{x\in\widetilde{\Sigma}_\tau} |\phi(x)| \leq \frac{C}{\tau}\,, \quad \forall \tau \geq 0\,. \tag{4.182}$$

Proof. The proof is exactly the same as that of Theorem 4.3.6, but in this case one uses the fact that the energy decays with τ^{-2}, instead of just using that it is bounded. □

Conclusions

5

In this work, we were able to prove that the solutions of the wave equation with smooth and compactly supported initial data are bounded, and moreover decay in time, both for the Minkowski and Schwarzschild spacetimes. For this purpose, we made use of the procedure introduced in [9], called the Dafermos-Rodnianski hierachy, which proved to be a robust energy method that allows the derivation of energy decay estimates. We also followed the work [21] to obtain an Integrated Local Energy Decay estimate for the Schwarzschild spacetime, by adapting it slightly to the hypersurfaces that we considered in our proof.

Moreover, we note that in the case of the Schwarzshild spacetime two energy degeneracy phenomena take place, associated to the presence of the event horizon and to the trapping of light rays on the photon sphere of the black hole (which is an obstruction to the decay of energy near this surface). The first degeneracy was resolved, following [2], by constructing a vector field with a corresponding nondegenerate energy flux, which relies heavily on the fact that the surface gravity of the Schwarzschild black hole is positive. Hence, the redshift effect occurring in the vicinity of the black hole turned out to be crucial for obtaining the aforementioned results. Additionally, a simple application of the spherical symmetry of the Schwarzschild spacetime allowed us to overcome the degeneracy at the photon sphere.

The results presented in this thesis could be further improved by weakening the conditions to be satisfied by the initial data. For instance, this proof did not require the initial data to be smooth, but rather to satisfy $\phi\big|_{\Sigma_0} \in H^3(\Sigma_0)$ and $\partial_t\phi\big|_{\Sigma_0} \in H^2(\Sigma_0)$. Additionally, one could also have considered initial data supported away from the bifurcation sphere, instead of having compact support, provided that $\phi\big|_{\Sigma_0} \to 0$ as $r \to \infty$, since this condition is indispensable in our proof. This could be done, for example, by imposing initial data on the hypersurface $\{t^* = 0\}$, as this hypersurface crosses the event horizon instead of approaching the bifurcation sphere.

The methods employed throughout this work relied heavily on the spherical symmetry of the Minkowski and Schwarzschild spacetimes, especially for obtaining pointwise bounds from energy estimates. The next goal would naturally be the study of the wave equation on the Kerr spacetime, which, however, is only axysymmetric. Therefore, this would require different techniques to deduce analogous results. Moreover, the trapping phenomenon in this spacetime is far more complicated, since light rays can be trapped at different values of the radial coordinate, thus leading to additional issues that need a careful analysis to be solved. Nevertheless, many of the techniques presented here can still be appropriately adapted to the Kerr spacetime, leading to proofs of boundedness and decay for the solutions of the wave equation (see for instance [9, 25]).

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