Spinorial proof of the positive mass theorem

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Abstract

This report is about Witten's proof of the positive mass theorem and it was done in the context of a one semester course on geometric analysis.

1 Introduction

Defining mass in general relativity (GR) is a very subtle issue. Roughly speaking, in Minkowski spacetime, we say that the energy content is given by an integral of the energy density of all the fields being considered over a constant t hypersurface. The problem is that in GR, while this construction can still be carried out for matter fields, it fails for the gravitational field. This is because the gravitational field is now described by the curvature of spacetime and there is no well-defined local concept of energy density in that setting.

This is where the concept of ADM mass comes into play. In 1961, while developing a Hamiltonian formulation of GR, Arnowitt-Deser-Misner [ADM60; ADM61] defined the notion of ADM energy-momentum, a four vector in spacetime which can be associated to each spacelike hypersurface of an asymptotically flat (AF) spacetime (for a definition of AF spacetime manifold see Ch. 11 of [Wal84])[1]. The norm of this four vector is called the ADM mass and it is easy to see that it exhibits some properties one would expect from a definition of total mass of the spacetime: *(i)* it is independent of the choice of hypersurface and *(ii)* in cases with more symmetry (like an everywhere timelike Killing vector field which allows for a more "intuitive" definition of mass) it coincides with other methods for calculating the total mass in the system. There is still one fundamental property of mass to recover – being positive!

Establishing the positivity of the ADM mass is of the utmost importance, namely because it is easy to see that the ADM mass is identically zero for Minkowski spacetime. Thus the stability of Minkowski spacetime as the ground state of GR (that is the only state with zero mass) would be dependent on the positivity of the ADM mass and on Minkowski space being the only spacetime with zero mass. Thus, the positive mass theorem is one of the most important results in mathematical relativity as it establishes the positivity of the ADM mass under some global assumptions about the curvature/matter content of an asymptotically flat spacetime. Throughout the 1960's and 1970's there were many attempts at proving what was then called the positive mass conjecture, but the conditions were very restrictive on the spacetime being considered. The first robust proof was put forward by Schoen and Yau [SY79; SY81] using minimal surface theory. However, in this report I will be sketching the proof of the positive mass theorem outlined by Witten in 1981 [Wit81] and rigorously completed by Parker and Taubes in 1982 [PT82].

The main reference for this work was Ch. 5 of Lee's book "Geometric Relativity" [Lee19]. I also consulted Natário's lecture notes [Nat20] for the definition of ADM mass and the standard book "Spin Geometry" by Lawson and Michelsohn [LM89] for the part of constructing spin structures. I also consulted [GN14] for the sign conventions of the Riemann curvature tensor. This document is organized as follows: in sec. 2 I will introduce the concept of ADM mass and give some motivation for the importance of positive mass theorems; next in sec. 3 I will introduce the concept of spinors and spin manifolds, which will lead us to introduce the Dirac operator in sec. 4 and then prove the important Schrodinger-Lichnerowicz formula. Finally, in sec. 5 I will prove the positive mass theorem for spin manifolds.

[1] In broad terms, an asymptotically flat spacetime in GR is a description of an isolated gravitating system.

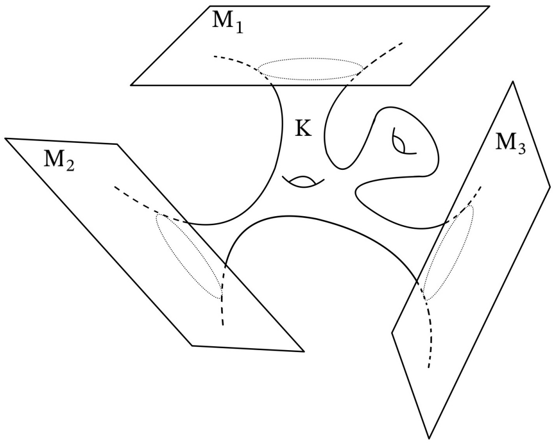

Figure 1: Pictorial representation of an asymptotically flat manifold M with three ends M_1, M_2, M_3 and bounded region K, whose boundary is represented by the dotted lines.

2 The ADM mass

Definition 2.1 (Asymptotically flat Riemannian manifold)**.** *A Riemannian manifold (M, g) of dimension $n \geq 3$ is asymptotically flat if $\exists\, K \subset M$ bounded such that $M \setminus K$ is a finite union of ends $M_1, ..., M_\ell$ such that for each end M_k there $\exists$ diffeomorphism (asymptotically flat coordinate chart)*

$$\Phi_k : M_k \to \mathbb{R}^n$$
$$p \mapsto (x^1(p), ..., x^n(p)) ,$$

such that in $(x^1, ..., x^n)$ coordinates, the metric on that end can be written as

$$g_{ij} = \delta_{ij} + o_2(|x|^{-\alpha}) , \quad \alpha > q := \frac{n-2}{2} ,$$

where $|x|$ is the radial function in the asymptotically flat coordinate chart. Furthermore we require the scalar curvature be integrable over the whole manifold, so on each end we must have

$$R \textit{ is } o_0(|x|^{-n}) .$$

A function $f : M \to \mathbb{R}$ is said to be $o_p(|x|^{-\alpha})$ if

$$\sum_{i=0}^{p} |x|^i |\partial^i f| < \epsilon |x|^{-\alpha} , \quad \forall\, \epsilon > 0 ,$$

where ∂ refers to derivatives with respect to the asymptotically flat coordinates. An example of an asymptotically flat manifold is represented pictorially in figure 1.

Definition 2.2 (ADM mass)**.** *Given an asymptotically flat manifold (M, g) of dimension n with ends $M_1, ..., M_\ell$, we say that the ADM mass of the end M_k is*

$$m_{ADM}(M_k, g) = \lim_{\rho \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_\rho} \sum_{i,j=1}^{n} \left(\frac{\partial g_{ij}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^j} \right) \frac{x^j}{|x|} d\bar{\mu}_{S_\rho} ,$$

where S_ρ is a $n-1$ coordinate sphere of radius ρ in the asymptotic chart on M_k, $d\bar{\mu}_{S_\rho}$ is the volume element induced by the Euclidean metric and ω_{n-1} is the volume of the $n-1$ unit sphere.

Remark. *Despite having been defined in 1961, it was only in 1986 that a rigorous proof was given by Bartnik showing that the ADM mass does not depend on the choice of asymptotic coordinate chart [Bar86].*

3 Spin structures on Riemannian manifolds

We now focus on the topic of constructing a spin structure on our Riemannian manifold (M, g). From now on we always assume $n \geq 3$.

Definition 3.1 (Oriented orthonormal frame bundle)**.** *Given an oriented n-dimensional manifold (M, g) we define the oriented orthonormal frame bundle as the principal $SO(n)$-bundle over M consisting of all oriented orthonormal frames $(e_1, ..., e_n)$ of the tangent space T_pM for all $p \in M$, that is*

$$F_{SO} = \bigcup_{p \in M} \{(e_1, ..., e_n) \in T_pM : g(e_i, e_j) = \delta_{ij} \; \forall i, j = 1, ..., n \; \textit{ and } \; (e_1, ..., e_n) \; \textit{ is positive}\} \, .$$

Since $SO(n)$ is not simply connected (in fact its fundamental group is $\pi_1(SO(n)) = \mathbb{Z}_2$) we can construct its simply connected double cover – this will be the Lie group $Spin(n)$. Before that though, we need to define the Clifford algebra of $\mathbb{R}^n$.

Definition 3.2 (Clifford algebra of $\mathbb{R}^n$)**.** *The Clifford algebra of $\mathbb{R}^n$ is defined as*

$$Cl(n) = \left(\bigoplus_{r=0}^{\infty} \mathbb{R}^n \otimes ... \otimes \mathbb{R}^n \right) / I \, ,$$

where we have r copies of $\mathbb{R}^n$ and I is the ideal generated by the relation

$$v \otimes v = -|v|^2 \, , \quad \forall \, v \in \mathbb{R}^n \, .$$

Given an o.n. basis $(e_1, ..., e_n)$ of $\mathbb{R}^n$ this relation can be obtained for all vectors by imposing it on the basis vectors, that is,

$$e_i e_j + e_j e_i = -2\delta_{ij} \, , \quad \forall \, i, j = 1, ..., n$$

(the operation induced by $\otimes$ is called the Clifford product, and the tensor product sign is usually ommited).

Definition 3.3 (Spin(n))**.** *The Lie group $Spin(n) \subset Cl(n)$ is the set of all products of the form*

$$(v_1 w_1)...(v_k w_k) \in Cl(n) \, , \quad v_i, w_i \in \mathbb{R}^n \, , \, |v_i| = |w_i| = 1 \, , \quad \forall \, i = 1, ..., k \, .$$

The group operation is just the Clifford product. It can be shown that $\exists$ surjective, $2 \mapsto 1$ group homomorphism $h : Spin(n) \to SO(n)$ which establishes $Spin(n)$ to be the simply connected double cover of $SO(n)$.

In general it is not easy to define this group homomorphism, and it is also not central for this work, but to give an idea of what is happening we will look at the particular case of $n = 3$. To do so consider the real vector space generated by the four matrices in $M_{2\times2}(\mathbb{C})$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad u_x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad u_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad u_z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \, .$$

It is easy to check that $u_k^2 = u_x u_y u_x = -\mathbb{1}$, $\forall \, k = 1, 2, 3$. Thus elements of this vector space

$$U = a\mathbb{1} + bu_x + cu_y + du_z \, , \quad a, b, c, d \in \mathbb{R} \, ,$$

are in one to one correspondence with quaternions $z = a + bi + cj + dk$. More importantly, restricting to the case $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$ we get an identification of matrices in $SU(2)$ with unit quaternions. Indeed, it is easy to see that any matrix in $SU(2)$ can be written in this form. Note further that we have

$$u_i u_j + u_j u_i = -2\delta_{ij}\mathbb{1} \, ,$$

which means that by identifying vectors of $\mathbb{R}^3$ like $v = \sum_i v^i e_i$, where (e_1, e_2, e_3) is an o.n. basis of $\mathbb{R}^3$, with matrices $U_v = \sum_i v^i u_i$ we get a realization of $Cl(3)$ where the Clifford product is just the usual matrix product.

Now take a unit vector $n \in \mathbb{R}^3$, $|n| = 1$ and an angle $\theta \in [0, 2\pi)$. Then any matrix in $SU(2)$ can be written as

$$(n, \theta) \mapsto U = e^{n\theta/2} = \cos(\theta/2)\mathbb{1} + \sin(\theta/2)U_n \, ,$$

so (n, θ) is a parameterization of $SU(2)$. We can define an action of $SU(2)$ on vectors of $\mathbb{R}^3$ as follows:

$$\begin{aligned} A : SU(2) \times \mathbb{R}^3 &\to \mathbb{R}^3 \\ ((n, \theta), v) &\mapsto e^{n\theta/2} U_v e^{-n\theta/2} = R_n^\theta v \end{aligned}$$

where in the last equality we claim that this action is the same as acting on v with $R_n^\theta \in SO(3)$, the matrix describing a rotation by an angle θ around the direction defined by n (this claim is easy to prove). Since any rotation can be written in such a way we can construct the map

$$h : SU(2) \to SO(3)$$
$$(n, \theta) \mapsto R_n^\theta .$$

Such a map is clearly a group homomorphism (because of the way it was constructed from the action of $SU(2)$ on vectors of $\mathbb{R}^3$), it is also surjective (for the reasons we pointed out above) and a $2 \mapsto 1$ correspondence because $R_n^\theta = R_{-n}^{2\pi-\theta}$. This is concludes a sketch of the proof that $Spin(3) = SU(2)$.

Definition 3.4 (Spin manifold)**.** *An oriented (M, g) is said to be spin if F_{SO} can be lifted to F_{Spin}, a principal $Spin(n)$-bundle over M.*

The idea is that all the transition functions of F_{SO}, which are in $SO(n)$, can be lifted to $Spin(n)$ in a consistent way that fits together, thus allowing to construct F_{Spin}.

Remark. *This is a purely topological condition and is equivalent to the vanishing of the second Stiefel-Whitney class.*

We give some examples below.

- All compact and oriented smooth manifolds with $n \leq 3$ are spin.
- $\mathbb{S}^n$ are all spin.
- $\mathbb{CP}^{2n-1}$ are all spin.
- $\mathbb{CP}^{2n}$ are all not spin.

Since $Spin(n)$ is the simply connected double cover of $SO(n)$ it will have representations which do not descend to $SO(n)$. Elements of a vector space S where one such representation exists are called *spinors*, as opposed to vectors. Now take such a vector space S, and assume that the representation of $Spin(n) \subset Cl(n)$ can be extended to a representation of all of $Cl(n)$, thus endowing S with the structure of a real module over $Cl(n)$. This means we can define a product $\cdot$ of elements of $Cl(n)$ on spinors,

$$v_1 ... v_k \cdot s \,, \quad v_i \in \mathbb{R}^n \,, \; s \in S \,.$$

Such a product can be used to define a hermitian structure in S such that unit vectors of $\mathbb{R}^n \subset Cl(n)$ act orthogonally on S.

$$\exists \langle \, , \, \rangle_S \quad \text{such that} \quad \langle v \cdot s_1, v \cdot s_2 \rangle_S = \langle s_1, s_2 \rangle_S \,, \quad \forall v \in \mathbb{R}^n \text{ with } |v| = 1 \,, \; \forall \, s_1, s_2 \in S \,.$$

Definition 3.5 (Spinor bundle)**.** *Given a spin manifold (M, g) and a representation of $Spin(n)$ on a vector space S which carries the structure of a real module over $Cl(n)$, the spinor bundle $S(M)$ over M is the associated vector bundle constructed from S and the principal $Spin(n)$-bundle F_{Spin}.*

Remark. *Using the natural action of $SO(n)$ on $Cl(n)$ we can define the Clifford bundle $Cl(M)$ as the associated bundle constructed from $Cl(T_pM) \cong Cl(n)$ and the principal $SO(n)$-bundle F_{SO}. The fibers of $Cl(M)$ act on the fibers of $S(M)$ with the product $\cdot$ in the same way that $Cl(n)$ acts on S.*

The last ingredient we need before we start doing some analysis is the notion of parallel transport on $S(M)$. From now on, we will always work with a local trivialization of all the bundles we are considering and this is done by choosing an o.n. frame $(e_1, ..., e_n)$ on $U \subset M$. Then we will have for $p \in U$:

$$F_{SO} \ni x = (p, g) \,, \; p = \pi(x) \,, \;\; g \in SO(n) \; : \; x = g(e_1, ..., e_n)_p \,,$$
$$F_{Spin} \ni \tilde{x} = (p, \tilde{g}) \,, \; \tilde{g} \in Spin(n) \,,$$
$$S(M) \ni \psi = (p, s) \,, \; s \in S \,,$$

where $\tilde{g}$ is the lift of g. Note also that in this trivialization we would have $(e_1, ..., e_n)_p = (p, \mathbb{1}_{SO(n)})$.

Definition 3.6 (Constant spinor)**.** *A spinor $\psi \in \Gamma(S(M))$ is said to be constant with respect to a local frame $(e_1, ..., e_n)$ if, in the trivialization associated to it we have*

$$\psi(p) = (p, s) \,, \quad \forall \, p \in U \,,$$

for some constant $s \in S$.

The idea now to define parallel transport is as follows: we use the notion of parallel transport on TM with the Levi-Civita connection, lift that to F_{SO}, lift again to F_{Spin}, and then use that and the action of $Spin(n)$ on S to define parallel transport on $S(M)$. We will not make this calculation here, but in the end, the covariant derivative of a spinor $\psi \in \Gamma(S(M))$ that is constant with respect to the given frame $(e_1, ..., e_n)$ is

$$\nabla\psi = -\frac{1}{4}\sum_{i,j=1}^{n} \omega^i_j e_i e_j \cdot \psi \;\in\; T^*M \otimes \Gamma(S(M)) \;, \tag{1}$$

where $\omega^i_j \in T^*M$ are the connection 1-forms associated to $(e_1, ..., e_n)$. It is a straightforward exercise to show that a connection on $S(M)$ can be defined from this expression and the requirement that it obeys the Leibniz rule, and that moreover this connection is compatible with the Clifford product of vector fields by spinors and also with the hermitian metric defined on $S(M)$ by the hermitian inner product $\langle\, ,\, \rangle_S$ on the fibers. We leave this exercise for the reader.

4 The Dirac operator

Definition 4.1 (Dirac operator). *The Dirac operator is a differential operator $\mathcal{D} : \Gamma(S(M)) \to \Gamma(S(M))$ and in a local frame $(e_1, ..., e_n)$ it can be written as*

$$\mathcal{D}\psi = \sum_{i=1}^{n} e_i \cdot \nabla_i \psi \,, \quad \forall\, \psi \in \Gamma(S(M)) \;,$$

where $\nabla_i \equiv \nabla_{e_i}$. It is straightforward to show that the Dirac operator is well defined in the sense that it does not depend on the choice of local frame and that it is formally self adjoint. Spinors solving the equation $\mathcal{D}\psi = 0$ are called harmonic spinors.

The Dirac operator will be central in proving the positive mass theorem because we can relate it to the scalar curvature of the manifold through the Schrödinger-Lichnerowicz formula which we prove below. This result and the corollary that follows it were originally found by Lichnerowicz in 1963 [Lic63].

Theorem 4.1 (Schrödinger-Lichnerowicz formula). *Let (M, g) be a Riemannian spin manifold. Then for any $\psi \in \Gamma(S(M))$ we have*

$$\mathcal{D}^2\psi = \nabla^*\nabla\psi + \frac{1}{4}R\psi,$$

where ∇^* is the formal adjoint of the covariant derivative ∇ on $S(M)$, and R is the scalar curvature.

Proof. Take a local frame $(e_1, ..., e_n)$ on $U \subset M$ such that at $p \in U$ we have $\nabla_i e_j = 0\ \forall i, j = 1, ..., n$. Then

$$\begin{aligned}
\mathcal{D}^2\psi &= \sum_{i,j=1}^{n} e_i \cdot \nabla_i (e_j \cdot \nabla_j \psi) \\
&= \sum_{i,j=1}^{n} e_i e_j \cdot (\nabla_i \nabla_j \psi) \\
&= -\sum_{i=1}^{n} \nabla_i \nabla_i \psi + \sum_{i<j=1}^{n} e_i e_j \cdot (\nabla_i \nabla_j - \nabla_j \nabla_i)\, \psi \\
&= \nabla^*\nabla\psi + \sum_{i<j=1}^{n} e_i e_j \cdot R^S(e_i, e_j)\psi
\end{aligned}$$

where $R^S(e_i, e_j) \in End(\Gamma(S(M)))$ is the curvature of the spinor bundle. This is a zeroth order operator on $\Gamma(S(M))$, just as the Riemann curvature tensor is on TM. Thus the result is the same if ψ is constant or not w.r.t. the given frame. We have

$$\begin{aligned}
R^S(e_i,e_j)\psi &= (\nabla_i\nabla_j - \nabla_j\nabla_i)\,\psi \\
&= \nabla_i\left(-\frac{1}{4}\sum_{k,\ell=1}^{n}\omega^k_\ell(e_j)e_k e_\ell\cdot\psi\right) - (i\leftrightarrow j) \\
&= \left(-\frac{1}{4}\sum_{k,\ell=1}^{n}\left(\nabla_i\omega^k_\ell(e_j)\right)e_k e_\ell\cdot\psi + \frac{1}{16}\sum_{k,\ell,p,q=1}^{n}\omega^k_\ell(e_j)\omega^p_q(e_i)e_k e_\ell e_p e_q\cdot\psi\right) - (i\leftrightarrow j) \\
&= -\frac{1}{4}\sum_{k,\ell=1}^{n} d\omega^k_\ell(e_i,e_j)e_k e_\ell\cdot\psi\,,
\end{aligned}$$

where $(i \leftrightarrow j)$ means the same expression as the previous term but swapping i for j and we used the fact that the frame $(e_1, ..., e_n)$ is parallel at p so $\omega^i_j = 0\;\forall\, i,j = 1,...,n$. Replacing this result in the expression for R^S we get

$$R^S(\cdot,\cdot) = -\frac{1}{4}\sum_{k,\ell=1}^{n} d\omega^k_\ell e_k e_\ell = -\frac{1}{4}\sum_{k,\ell=1}^{n}\Omega^k_\ell e_k e_\ell = \frac{1}{4}\sum_{k,\ell=1}^{n} Riem(\cdot,\cdot,e_k,e_\ell)e_k e_\ell\,,$$

where Ω^k_ℓ are the curvature forms and $Riem$ is the curvature tensor and where we are using the convention of [GN14]. Plugging this back into the expression for $\mathcal{D}^2\psi$ we find

$$\mathcal{D}^2\psi = \nabla^*\nabla\psi + \frac{1}{8}\sum_{i,j,k,l=1}^{n} R_{ijkl}e_i e_j e_k e_l\cdot\psi = \nabla^*\nabla\psi + \frac{1}{4}R\psi\,,$$

where R_{ijkl} are just the components of the curvature tensor in the given basis and we just have to use the symmetries of curvature tensor and the Clifford algebra to obtain the scalar curvature R. □

Corollary 4.1. *If (M, g) is a compact spin manifold and $R \geq 0$ everywhere and nonzero somewhere then there are no harmonic spinors.*

Proof. Take the inner product of the Schrödinger-Lichnerwicz formula with ψ and integrate over M:

$$\int_M\left(\langle\mathcal{D}^2\psi,\psi\rangle - \langle\nabla^*\nabla\psi,\psi\rangle - \frac{1}{4}R|\psi|^2\right)d\mu_M = 0 \iff$$
$$\int_M\left(|\mathcal{D}\psi|^2 - |\nabla\psi|^2\right)d\mu_M \geq 0\,,$$

where we use the the fact that the Dirac operator is formally self-adjoint. If ψ is a harmonic spinor this implies that ψ is parallel, which leads to a clear contradiction. □

Remark. *Just like with the Bochner theorem for 1-forms we saw in class this gives us some topological information about the manifold. Indeed, the non-existence of harmonic spinors is equivalent to the vanishing of the Hirzebruch $\hat{A}$ genus [24a].*

5 The positive mass theorem for spin manifolds

Theorem 5.1 (Positive mass theorem for spin manifolds)**.** *Let (M, g) be a complete, asymptotically flat spin manifold with nonnegative scalar curvature. Then:*

1. *The ADM mass of each end is nonnegative.*
2. *If the ADM mass of any end is zero, then (M,g) is Euclidean space.*

The second result is called a rigidity result, and it is fundamental in establishing the stability of Minkowski space, as discussed in Sec. 1

Before proving the main theorem stated above, we will need four propositions which we state and prove below.

Proposition 5.1. *Take $\Omega \subset M$ a bounded subset of the spin manifold M with $\partial\Omega$ smooth. Then for any $\psi \in \Gamma\left(S(M)\right)$*

$$\int_\Omega \left(|\nabla\psi|^2 - |\mathcal{D}\psi|^2 + \frac{1}{4}R|\psi|^2 \right) d\mu_M = \int_{\partial\Omega} \sum_{i=1}^n \langle \psi, L_i\psi\rangle \nu^j d\mu_{\partial\Omega}\,,$$

where $L_i = \sum_j (\delta_{ij} + e_i e_j) \cdot \nabla_j$, ν^j is the unit outward pointing normal to $\partial\Omega$ and $d\mu_{\partial\Omega}$ is the volume element induced on $\partial\Omega$ by the Riemannian volume element on M.

Proof. Assuming again that the frame $(e_1, ..., e_n)$ is parallel at $p \in M$, we have

$$\begin{aligned} -|\mathcal{D}\psi|^2 &= \sum_{i,j=1}^n \langle -e_i \cdot \nabla_i\psi, e_j \cdot \nabla_j\psi\rangle \\ &= \sum_{i,j=1}^n \langle \nabla_i\psi, e_i e_j \cdot \nabla_j\psi\rangle \\ &= \sum_{i,j=1}^n \nabla_i \langle \psi, e_i e_j \cdot \nabla_j\psi\rangle - \langle \psi, \mathcal{D}^2\psi\rangle\,, \end{aligned}$$

and

$$\begin{aligned} |\nabla\psi|^2 &= \sum_{i=1}^n \langle \nabla_i\psi, \nabla_i\psi\rangle \\ &= \sum_{i=1}^n \left(\nabla_i\langle\psi, \nabla_i\psi\rangle - \langle\psi, \nabla_i\nabla_i\psi\rangle\right) \\ &= \sum_{i=1}^n \nabla_i\langle\psi, \nabla_i\psi\rangle + \langle\psi, \nabla^*\nabla\psi\rangle\,. \end{aligned}$$

Substituting this into the left-hand side of the expression in the statement of the proposition and then using the Schrödinger-Lichnerowicz formula and the divergence theorem yields the desired result. □

Proposition 5.2. *Let (M, g) be an asymptotically flat spin manifold and take $(e_1, ..., e_n)$ to be an o.n. frame on one of the ends, M_k. Now take $\psi_0 \in \Gamma\left(S(M)\right)$ to be constant with respect to the given frame on M_k. Then*

$$\lim_{\rho\to\infty} \int_{S_\rho} \sum_{i=1}^n \langle \psi_0, L_i\psi_0\rangle \nu^i d\mu_{S_\rho} = \frac{1}{2}(n-1)\omega_{n-1}|\psi_0|^2 m_{ADM}(M_k, g)\,,$$

where S_ρ is a coordinate sphere in M_k.

Proof. We can construct the o.n. frame $(e_1, ..., e_n)$ by using Gram-Schmidt on the standard basis induced by the asymptotically flat coordinate chart $(\partial_1, ..., \partial_n)$. Recalling the decay properties of the metric from the definition of asymptotically flat manifold we easily find

$$e_i = \partial_i - \frac{1}{2}\sum_{j=1}^n h_{ij}\partial_j + o_1(|x|^{-2q})\,,$$

where $q = \frac{n-2}{2}$. Using this result, a straightforward calculation yields

$$\omega^i_j(e_k) = \frac{1}{2}(\partial_j g_{ik} - \partial_i g_{jk}) + o(|x|^{-2q-1})\,.$$

Thus, we find

$$
\begin{aligned}
\sum_{i=1}^{n}\langle\psi, L_i\psi\rangle\nu^i &= -\frac{1}{4}\sum_{\substack{i\neq j\\ k\neq\ell}}\omega^k_\ell(e_j)\langle\psi_0, e_ie_je_ke_\ell\psi_0\rangle\nu^i \\
&= -\frac{1}{4}\sum_{\substack{i\neq j\\ k\neq\ell}}\omega^k_\ell(e_j)\langle\psi_0,\psi_0\rangle\left(\delta_{i\ell}\delta_{jk}-\delta_{ik}\delta_{j\ell}\right)\nu^i + o(|x|^{-2q-1}) \\
&= \frac{1}{4}|\psi_0|^2\sum_{i\neq j}\left(\omega^i_j(e_j)-\omega^j_i(e_j)\right)\nu^i + o(|x|^{-2q-1}) \\
&= \frac{1}{8}|\psi_0|^2\sum_{i\neq j}\left(\partial_j g_{ij}-\partial_i g_{jj}-\partial_i g_{jj}+\partial_j g_{ij}\right)\nu^i + o(|x|^{-2q-1}) \\
&= \frac{1}{4}|\psi_0|^2\sum_{i\neq j}\left(\partial_j g_{ij}-\partial_i g_{jj}\right)\nu^i + o(|x|^{-2q-1})\,.
\end{aligned}
$$

From the first to the second line the Clifford algebra relations together with the symmetries of the expression for $\omega^i_j(e_k)$ under index swapping are used. Replacing this result in the integral above the lower order terms all go to zero when the limit is taken. Cmparing with the expression for the ADM mass given in 2.2 and noting $\nu^i = x^i/|x|$ completes the proof. $\square$

Definition 5.1 (Weighted function spaces)**.** *Take (M,g) to be a complete, asymptotically flat n-dimensional spin manifold. Choose $r \in C^\infty(M)$ such that $r > 0$ everywhere on M and $r = |x|$ in the asymptotic coordinate chart on each end M_k. Then, for $u \in \Gamma(S(M))$ we define:*

- *Weighted $L^p_s(S(M))$ norm:*

$$
\|u\|_{L^p_s(S(M))} := \left(\int_M |u|^p r^{-sp-n} d\mu_M\right)^{1/p}\,.
$$

- *Weighted $W^{k,p}_s(S(M))$ norm:*

$$
\|u\|_{W^{k,p}_s(S(M))} := \sum_{i=0}^{k}\|\nabla^i u\|_{L^p_{s-i}(S(M))}\,.
$$

We can now state a result which will leave us very close to proving the positive mass theorem.

Proposition 5.3. *Let (M,g) be a complete, asymptotically flat spin manifold and take $(e_1,...,e_n)$ to be an o.n. basis on some end M_k. Now take $\psi_0 \in \Gamma(S(M))$ to be a spinor that is constant with respect to the given frame in M_k and identically zero in all other ends. Take also a spinor $\psi \in \Gamma(S(M))$ such that $\psi - \psi_0 \in W^{1,2}_{-q}(\Gamma(S(M)))$. Then*

$$
\int_M \left(|\nabla\psi|^2 - |\mathcal{D}\psi|^2 + \frac{1}{4}R|\psi|^2\right)d\mu_M = \frac{1}{2}(n-1)\omega_{n-1}|\psi_0|^2 m_{ADM}(M_k,g)\,.
$$

Proof. The proof is straightforward, so we will only sketch it here. First one has to prove that Proposition 5.2 still holds for a spinor ψ like that described above. This is not hard to do because the fact that $\psi - \psi_0 \in W^{1,2}_{-q}$ means we have control of the derivatives of $\psi - \psi_0$ in the asymptotic region where the integral is taken. Thus we can show that the only non-zero term gives the same as for a constant spinor.

Then we consider the result of Proposition 5.1 with this spinor ψ and taking the region Ω to be the bounded region K appearing in the definition of asymptotically flat manifold. Now start expanding Ω into the asymptotic regions and deform the boundaries so that they coincide with coordinate spheres. Finally take the limit where the radii of these sphere goes to infinity. Thus the result of Proposition 5.2 will be valid for all the ends, yielding the ADM mass multiplied by some constants in M_k and zero in all other ends. Meanwhile on the left hand side of the equation $\Omega \to M$, thus yielding the desired result. $\square$

Now the idea is very simple: if we can find a harmonic spinor which satisfies the conditions of the previous proposition we are done. To establish the existence of such a spinor we need one more result.

Proposition 5.4. *Let (M, g) be a complete, asymptotically flat spin manifold with $R \geq 0$ and let $q = \frac{n-2}{2}$. Then*

$$\mathcal{D} : W^{1,2}_{-q}(S(M)) \to L^2_{-q-1}(S(M)) \equiv L^2(S(M))$$

is an isomorphism.

Proof. To prove the statement above we must show that $\mathcal{D} : W^{1,2}_{-q} \to L^2$ is a linear, bounded and bijective operator.

The linearity is inherited from the covariant derivative. The boundedness is obtained from Proposition 5.3: taking $\psi_0 = 0$ we get for any $\varphi \in W^{1,2}_{-q}(S(M))$

$$\begin{aligned} &\int_M \left(|\nabla\varphi|^2 - |\mathcal{D}\varphi|^2 + \frac{1}{4}R|\varphi|^2 \right) d\mu_M = 0 \\ \iff &\int_M |\mathcal{D}\varphi|^2 d\mu_M = \int_M \left(|\nabla\varphi|^2 + \frac{1}{4}R|\varphi|^2 \right) d\mu_M \\ \iff &\|\mathcal{D}\varphi\|^2_{L^2} \leq \|\nabla\varphi\|^2_{L^2} + \|\varphi\|^2_{L^2_{-q}} \\ \iff &\|\mathcal{D}\varphi\|_{L^2} \leq \|\varphi\|_{W^{1,2}_{-q}} , \end{aligned}$$

where we used the fact that the scalar curvature decays much faster than the weight function r in all ends (see 2.1 and 5.1). This proves that the Dirac operator acting on $W^{1,2}_{-q}(S(M))$ is bounded.

Now we have to prove it is injective. It follows directly from Proposition 5.3 that

$$\|\nabla\varphi\|_{L^2} \leq \|\mathcal{D}\varphi\|_{L^2} .$$

Then we have to use a weighted Poincaré inequality, which is proved in [Lee19], and which states $\exists\, C > 0$ for which the following holds:

$$\|\varphi\|_{L^2_{-q}} \leq C\|\nabla\varphi\|_{L^2} . \tag{2}$$

Putting these two results together we get

$$\begin{aligned} \|\varphi\|_{W^{1,2}_{-q}} &= \|\varphi\|_{L^2_{-q}} + \|\nabla\varphi\|_{L^2} \\ &\leq (C+1)\|\nabla\varphi\|_{L^2} \\ &\leq (C+1)\|\mathcal{D}\varphi\|_{L^2} . \end{aligned}$$

This is enough to prove injectivity.

Finally, we must prove that the operator is also surjective, that is, that $\forall\, \eta \in L^2$, $\exists\, \xi \in W^{1,2}_{-q}$ such that $\mathcal{D}\xi = \eta$. We start by considering the case where $\eta \in \Gamma_0(S(M)) \subset L^2$ is compactly supported.

Take $\phi, \omega \in W^{1,2}_{-q}$. We showed $\|\mathcal{D}\varphi\|_{L^2} \leq \|\varphi\|_{W^{1,2}_{-q}} \leq (C+1)\|\mathcal{D}\varphi\|_{L^2}$, implying that the $W^{1,2}_{-q}$ inner product on the Hilbert space H of spinors $\varphi \in W^{1,2}_{-q}$ is equivalent to the pairing

$$\langle \omega, \varphi \rangle_H \coloneqq \langle \mathcal{D}\omega, \mathcal{D}\varphi \rangle_{L^2} .$$

Thus take the function

$$\begin{aligned} f : W^{1,2}_{-q} &\to \mathbb{C} \\ \varphi &\mapsto \langle \eta, \varphi \rangle_{L^2} , \end{aligned}$$

which is easily seen to be linear and bounded from the previous calculations. Then, we can apply the Riesz representation theorem [24b], which tells us that $\exists\, \omega \in W^{1,2}_{-q}$ such that

$$f(\varphi) = \langle \omega, \varphi \rangle_H \iff \langle \eta, \varphi \rangle_{L^2} = \langle \mathcal{D}\omega, \mathcal{D}\varphi \rangle_{L^2}$$

Now we claim that $\xi = \mathcal{D}\omega$ is the desired solution. However we only know that $\xi \in L^2$ which is not enough. That being the case, consider a sequence $\{\xi_j\}_{\in W^{1,2}_{-q}} \rightharpoonup \xi$ converging weakly to ξ in L^2. Then $\forall\, \varphi \in W^{1,2}_{-q}$ we have

$$\lim_{j\to\infty} \langle \mathcal{D}\xi_j, \varphi \rangle_{L^2} = \lim_{j\to\infty} \langle \xi_j, \mathcal{D}\varphi \rangle_{L^2} = \langle \xi, \mathcal{D}\varphi \rangle_{L^2} \equiv \langle \eta, \varphi \rangle_{L^2}$$

which allows us to say $\{\mathcal{D}\xi_j\}_{\in L^2} \to \eta$. Then by the injectivity we proved before we conclude that we must have $\{\xi_j\}_{\in W^{1,2}_{-q}} \to \xi$. So we get that $\xi \in W^{1,2}_{-q}$ and it satisfies

$$\langle \eta, \varphi \rangle_{L^2} = \langle \mathcal{D}\xi, \varphi \rangle_{L^2} , \quad \forall\, \varphi \in W^{1,2}_{-q} .$$

So it is a solution of $\mathcal{D}\xi = \eta$ in $W^{1,2}_{-q}$, which proves that there always is a solution for compactly supported spinors η. By a density argument, the result can be extended to include all $\eta \in L^2$, thus completing the proof. □

Putting all these pieces together, we can finally formalize the proof of the positive mass theorem for spin manifolds.

Proof. (Positive mass theorem for spin manifolds – Theorem 5.1)

We will start by proving the first statement, and we do it by steps:

- Take an end M_k and an o.n. frame $(e_1, ..., e_n)$ on that end. Now choose $\psi_0 \in \Gamma(S(M))$ such that ψ_0 is constant with $|\psi_0| = 1$ in M_k and $\psi_0 = 0$ in all other ends.
- Take $\eta = -\mathcal{D}\psi_0$ which is easily seen to be in L^2.
- By Proposition 5.4, $\exists\, \xi \in W^{1,2}_{-q}$ such that $\mathcal{D}\xi = \eta$.
- Then the spinor $\psi = \psi_0 + \xi$ satisfies $\psi - \psi_0 \in W^{1,2}_{-q}$ and it also has $\mathcal{D}\psi = 0$.
- Then by Proposition 5.3 we get
$$m_{ADM}(M_k, g) \geq 0\,.$$

To prove the second statement we have to make a slightly more complicated construction. Assuming that $m_{ADM}(M_k, g) = 0$, we immediately get that $\nabla\psi = 0$ everywhere by Proposition 5.3. But now note that any other choice of constant spinor ψ_0 in M_k will yield a different spinor ψ satisfying $\nabla\psi = 0$. Concretely, for $i = 1, ..., n$, we can choose define ψ_i to be the harmonic spinor which asymptotes to $e_i \cdot \psi_0$ and has $\nabla\psi_i = 0$. Now we define the vector fields V_i such that

$$\langle V_i, w\rangle_{TM} = \langle w \cdot \psi, \psi_i\rangle_{S(M)}\,,$$

for any $w \in T_pM$ and $\forall\, p \in M$. This vector field asymptotes to e_i:

$$\begin{aligned}
\langle V_i, w\rangle &= \langle w \cdot \psi, \psi_i\rangle \\
&\sim \langle w \cdot \psi_0, e_i \cdot \psi_0\rangle \\
&= \frac{1}{2}\left(\langle w \cdot \psi_0, e_i \cdot \psi_0\rangle + \overline{\langle w \cdot \psi_0, e_i \cdot \psi_0\rangle}\right) \\
&= \frac{1}{2}\left(\langle w \cdot \psi_0, e_i \cdot \psi_0\rangle + \langle e_i \cdot \psi_0, w \cdot \psi_0\rangle\right) \\
&= -\frac{1}{2}\langle (we_i + e_i e) \cdot \psi_0, \psi_0\rangle \\
&= \langle e_i, w\rangle |\psi_0|^2 \\
&= \langle e_i, w\rangle\,,
\end{aligned}$$

where we used the fact that we know $\langle V_i, w\rangle \in \mathbb{R}$. Moreover it satisfies $\nabla V_i = 0$ everywhere :

$$\begin{aligned}
\langle \nabla V_i, w\rangle &= \nabla\langle V_i, w\rangle - \langle V_i, \nabla w\rangle \\
&= \nabla\langle w \cdot \psi, \psi_i\rangle - \langle \nabla w \cdot \psi, \psi_i\rangle \\
&= 0\,,
\end{aligned}$$

since $\nabla\psi = \nabla\psi_i = 0$. This means that $(V_1, ..., V_n)$ is a global frame of parallel vector fields, so (M, g) must be flat. Since it is also asymptotically flat, we conclude from the Killing-Hopf theorem that it must be the Euclidean space. □

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