Energy increase near black holes via the Penrose process and the Oberth maneuver

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1 Introduction

The theory of rocket trajectories has been generalized to the general relativity setting in [5]. In this text, we apply this theory to the study of the Oberth maneuver [7], performed near black holes. In Section 2, we follow [10] in a review of the Kerr metric, which describes the geometry around rotating uncharged black holes, along with some properties of their structure, namely the event horizons and the arise of a new surface that does not appear in the Schwarzschild metric, the ergosphere. Next, in Section 3, we delve into the Penrose process, a way in which energy can be extracted from a Kerr black hole that is made possible by the nature of the ergosphere. Through some geometric arguments, we find the maximal energy per unit mass gain that one can obtain from this process. For the remainder of the text, we focus on the study of maximal energy increase a rocket can obtain by performing the Oberth maneuver in different scenarios. In Subsection 4.1, we start by deriving the rocket equation in the context of Newton's theory, to then, in Subsection 4.2 compute the energy increase obtained from the Oberth maneuver performed near a spherically symmetric body. After this, in Subsection 4.3, we follow [5] to derive the rocket equation in the context of general relativity, which we then use to study how efficient the Oberth maneuver near black holes can be in order to increase the rockets energy per unit mass (in Subsection 4.4 the maneuver is performed for the Schwarzschild metric, and in Subsection 4.5 for the Kerr metric). In Subsection 4.6, following [5] and [6], we make a note on the optimality of the Oberth maneuver for instantaneous acceleration. In Subsection 4.7, we show that, when performed inside the ergosphere of a Kerr black hole, the Oberth maneuver can be seen as a Penrose process, as it allows the rocket to return to infinity with more total energy than it initially had. Subsection 4.8 contains a note on what can be regarded as an instantaneous acceleration, so that, in Subsection 4.9, we conclude the study of the Oberth maneuver with some computations for the suppermassive black hole at the center of the Milky Way, Sagittarius A*.

Throughout this text, we use geometrized units (in which the speed of light and Newton's gravitational constant are set equal to 1), except in the final Subsection, where the units used are explicit.

2 Kerr metric

We begin with a discussion of the metric that describes a rotating black hole, the Kerr metric. Such a black hole is characterised by two parameters: its mass, M, and angular momentum, J. We define $a := \frac{J}{M}$, which can be interpreted as the angular momentum per unit mass. In the coordinate system (t, r, θ, φ), which are called Boyer–Lindquist coordinates, the Kerr metric is written

$$\begin{aligned} g = &-\frac{\Delta - a^2 \sin^2\theta}{\rho^2} dt \otimes dt - 2a\frac{2Mr\sin^2\theta}{\rho^2} dt \otimes \ d\varphi \\ &+ \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\varphi \otimes d\varphi + \frac{\rho^2}{\Delta} dr \otimes dr + \rho^2 d\theta \otimes d\theta, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \Delta &:= r^2 - 2Mr + a^2, \\ \rho^2 &:= r^2 + a^2\cos^2\theta. \end{aligned} \tag{2}$$

At this point, we might already notice a couple things. Firstly, as one would expect, the Schwarzschild metric is a special case of the Kerr metric, obtained for $a = 0$. Secondly, the latter has an off-diagonal term, $g_{t\varphi}$, unlike the former, in which all non-vanishing terms are diagonal. The presence of this term introduces some effects on the trajectories of particles. If we consider a curve $c : I \subset \mathbb{R} \longrightarrow M$ (here, M represents a differentiable manifold which we equip with Kerr's metric), parametrized by the proper time, τ, the four-velocity vector, $U = \dot{c}(\tau)$ is given by

$$U = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi}\right). \tag{3}$$

Since the contravariant components of this vector are given by $U^\mu = g^{\mu\alpha}U_\alpha$, we have that

$$\begin{cases} U^\varphi = g^{\varphi\alpha}U_\alpha = g^{\varphi\varphi}U_\varphi + g^{\varphi t}U_t \\ U^t = g^{t\alpha}U_\alpha = g^{tt}U_t + g^{t\varphi}U_\varphi \end{cases} \tag{4}$$

If we consider a zero-angular-momentum particle, $U_\varphi = 0$, then, from (4),

$$\frac{d\varphi}{dt} = \frac{\frac{d\varphi}{d\tau}}{\frac{dt}{d\tau}} = \frac{g^{\varphi t}}{g^{tt}} := \omega(r, \theta), \tag{5}$$

and we know ω is a function of r and θ, since the terms of the metric only depend on this variables. By this definition, we interpret ω as the angular-velocity of a zero-angular-momentum particle.

Next, we consider two photons emitted in the equatorial plane ($\theta = \frac{\pi}{2}$) at some given r, initially going in the $\pm\varphi$ direction. Let the curve $c : I \subset \mathbb{R} \longrightarrow M$ represent the motion of such a photon, parametrized by t. Since photons travel along null geodesics, we have that $\langle \dot{c}(t), \dot{c}(t)\rangle = 0$. Moreover, because we are considering r and θ initially fixed, we have $\dot{c}(t) = \left(1, 0, 0, \frac{d\varphi}{dt}\right)$, therefore

$$\begin{aligned} \langle \dot{c}(t), \dot{c}(t) \rangle = 0 \Leftrightarrow g_{tt} + 2g_{t\varphi}\frac{d\varphi}{dt} + g_{\varphi\varphi}\left(\frac{d\varphi}{dt}\right)^2 = 0 \\ \Leftrightarrow \frac{d\varphi}{dt} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} \pm \sqrt{\left(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right)^2 - \frac{g_{tt}}{g_{\varphi\varphi}}}. \end{aligned} \tag{6}$$

If $g_{tt} = 0$, the solutions of the previous equation are

$$\frac{d\varphi}{dt} = 0 \tag{7}$$

and

$$\frac{d\varphi}{dt} = -\frac{2g_{t\varphi}}{g_{\varphi\varphi}}. \tag{8}$$

Regarding the first solution, it means the photon sent in the $-\varphi$ direction, opposite to the rotation of the hole, does not move at all. Therefore, massive particles, which move slower than photons, will be forced to rotate with the black hole regardless of how large their angular momentum is in the opposite direction. From (1), we can find where the term g_{tt} is zero

$$\begin{aligned} g_{tt} = 0 &\Leftrightarrow \Delta = a^2 \sin^2\theta \\ &\Leftrightarrow r^2 - 2Mr + a^2 = a^2 \sin^2\theta \\ &\Leftrightarrow r^2 - 2Mr + a^2\cos^2\theta = 0 \\ &\Leftrightarrow r = M \pm \sqrt{M^2 - a^2\cos^2\theta}. \end{aligned} \tag{9}$$

The surface with radius $r_0 = M + \sqrt{M^2 - a^2\cos^2\theta}$ is called the **ergosphere**. If $r < r_0$, then $g_{tt} > 0$, which means $\frac{\partial}{\partial t}$ is spacelike inside the ergosphere. Hence, inside this surface, all massive particles and photons must rotate with the hole.

The horizon for the Kerr metric is found where $g_{rr} = \infty$, i.e., where $\Delta = 0$. Solving this, we get

$$\begin{aligned} \Delta = 0 &\Leftrightarrow r^2 - 2Mr + a^2 = 0 \\ &\Leftrightarrow r = M \pm \sqrt{M^2 - a^2}. \end{aligned} \tag{10}$$

Since we get two surfaces where $g_{rr} = \infty$, a Kerr black hole admits two event horizons. The submanifold $r_+ = M + \sqrt{M^2 - a^2}$ is the **outer horizon** and the submanifold $r_- = M - \sqrt{M^2 - a^2}$ the **inner horizon** (since $r_+ > r_-$). Notice that the ergosphere lies outside the outer horizon, except at the poles (where $\theta = 0$ or $\theta = \pi$), where it is tangent to it, and we shall refer to the region between the two as the **ergoregion**.

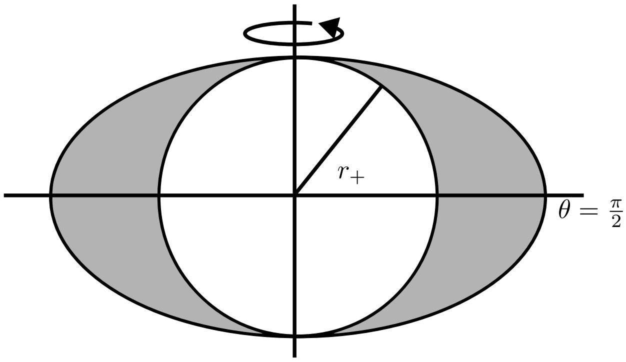

Figure 1: Illustration of a Kerr black hole, where the grey area corresponds to the ergoregion.

Being axisymmetric and stationary, the Kerr metric admits the following Killing vector fields

$$X = \frac{\partial}{\partial t} \text{ and } Y = \frac{\partial}{\partial \varphi}. \tag{11}$$

As a consequence, there are two conserved quantities associated to the motion of test particles, which we can find from the Euler-Lagrange equations. Firstly, the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} \left(g_{tt} \dot{t}^2 + 2 g_{t\varphi} \dot{t} \dot{\varphi} + g_{rr} \dot{r}^2 + g_{\theta\theta} \dot{\theta}^2 + g_{\varphi\varphi} \dot{\varphi}^2 \right). \tag{12}$$

Therefore, the Euler-Lagrange equation for $\frac{\partial}{\partial t}$ is

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{t}} \right) - \frac{\partial \mathcal{L}}{\partial t} = 0 \tag{13}$$

Since $\frac{\partial g_{\mu\nu}}{\partial t} = 0$, then we conclude $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{t}} \right) = 0$, meaning the quantity $\frac{\partial \mathcal{L}}{\partial \dot{t}}$ is preserved. By (12),

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{t}} &= g_{tt} \dot{t} + g_{t\varphi} \dot{\varphi} \\ &= g_{tt} U^t + g_{t\varphi} U^\varphi \\ &= g_{t\mu} U^\mu \\ &= U_t. \end{aligned} \tag{14}$$

We call $E \coloneqq -U_t$ the **energy**. Similarly, for $\frac{\partial}{\partial \varphi}$, we get

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \tag{15}$$

and since $\frac{\partial g_{\mu\nu}}{\partial\varphi} = 0$, then $\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right) = 0$, and so $\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}$ is also preserved. Moreover,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} &= g_{\varphi\varphi}\dot{\varphi} + g_{t\varphi}\dot{t} \\ &= g_{\varphi\varphi}U^{\varphi} + g_{t\varphi}U^{t} \\ &= g_{\varphi\mu}U^{\mu} \\ &= U_{\varphi},\end{aligned} \tag{16}$$

and we call $L := U_{\varphi}$ the **angular momentum per unit mass** (for massive particles).

3 Penrose process

Consider a unit rest mass particle in a trajectory from infinity towards a rotating black hole (at first, we shall consider the possibility that the particle is accelerating towards the hole). Let U represent the particle's 4-velocity vector and $E = -\left\langle U, \frac{\partial}{\partial t}\right\rangle$ its energy. Once inside the ergosphere, the particle breaks up in two particles, with 4-momenta p and q (for simplicity, we shall refer to these particles as P and Q, respectively). Since the breakup event must conserve the energy and momentum, we have

$$U = p + q. \tag{17}$$

Moreover, let $\{E_0, E_1, E_2, E_3\}$ be an orthonormal frame at the breakup event such that E_0 is future-pointing and E_1 is a negative multiple of $\frac{\partial}{\partial t}$. Hence, recall from Section 2 that E_1 is spacelike inside the ergoregion. It is possible that one of the particles (say, P) escapes the black hole, while the other falls towards the outer horizon (Q).

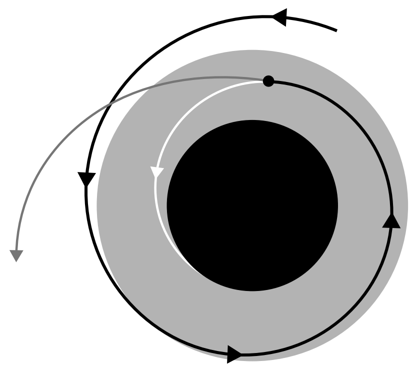

Figure 2: Illustration of the Penrose process. The black circle represents the region inside the outer horizon, while the grey area corresponds to the ergoregion. The black curve represents the trajectory of the incoming particle, the grey curve corresponds to the trajectory of particle P, which escapes the black hole, and the white curve to the trajectory of Q, the particle that heads towards the outer horizon.

We are interested in maximizing the energy of the outgoing particle P, which we denote by E_P. For that, we must maximize p^1, since

$$\begin{aligned} E_P &= -\left\langle p, \frac{\partial}{\partial t} \right\rangle \\ &= \left\langle p, \left| \frac{\partial}{\partial t} \right| E_1 \right\rangle \\ &= \left| \frac{\partial}{\partial t} \right| p^1. \end{aligned} \tag{18}$$

Notice that we are subject to some constraints:

- Firstly, as mentioned above, we must have $U = p + q$;
- Secondly, p and q must be timelike or null, otherwise, they could not be the 4-momenta of particles. Therefore, we shall impose $\langle p, p \rangle \leq 0$ and $\langle q, q \rangle \leq 0$;
- Thirdly, $p^0 \geq 0$ and $q^0 \geq 0$, so that p and q are future-pointing.

Taking p as the independent variable, the constraints become

- $\langle p, p \rangle \leq 0$;
- $\langle U - p, U - p \rangle \leq 0$;
- $0 \leq p^0 \leq U^0$.

Geometrically, we see that these constraints define a compact set which is the intersection of the past-cone of U (which we represent by $J^-(U)$) and the future-cone of 0 ($J^+(0)$), that is

$$C = J^-(U) \cap J^+(0), \tag{19}$$

as represented in Figure 3.

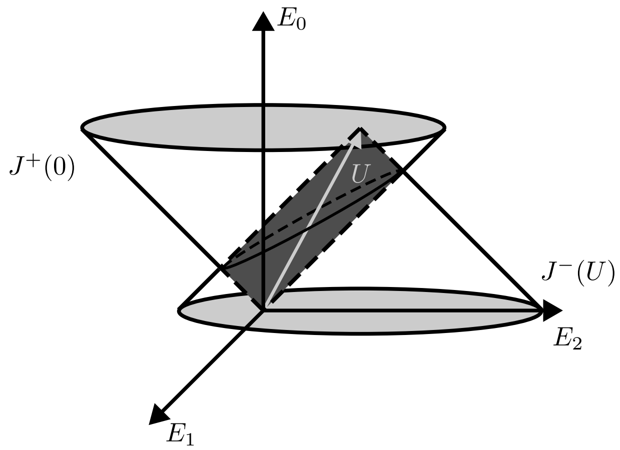

Figure 3: Representation of the compact set defined by the constraints for p.

Since p^1 does not have critical points (because $dp^1 = (0,1,0,0) = E_1$), its maximum must be on the boundary ∂C. Moreover, in one of the vertices, we would have $p = 0$, while on the other $p = U$, both of which cannot correspond to a breakup event. Now, if the maximum were on $\partial J^+(0) \setminus \partial J^-(U)$, then, by the Lagrange multiplier method, we would have

$$d\left(p^1 + \mu_1\langle p, p\rangle\right) = 0 \Leftrightarrow E_1 + 2\mu_1 p = 0. \tag{20}$$

However, this cannot be the case, since E_1 is spacelike and p must be timelike or null. Similarly, if the maximum were on $\partial J^-(U) \setminus \partial J^+(0)$, we would have

$$d\left(p^1 + \mu_2\langle U-p, U-p\rangle\right) = 0 \Leftrightarrow E_1 - 2\mu_2(U-p) = 0, \tag{21}$$

but, once again, E_1 is spacelike and $U-p$ must be timelike or null. Hence, we conclude the maximum of p^1 must be on $\partial J^-(U) \cap \partial J^+(0)$, where $\langle p, p\rangle = \langle U-p, U-p\rangle = 0$, and so

$$d\left(p^1 + \mu_1\langle p, p\rangle + \mu_2\langle U-p, U-p\rangle\right) = 0 \Leftrightarrow E_1 + 2\mu_1 p - 2\mu_2(U-p) = 0. \tag{22}$$

Because the maximum is on this intersection, p must be a null vector, and, from the previous expression, we also know it must be a linear combination of U and $\frac{\partial}{\partial t}$ (since E_1 is a multiple of $\frac{\partial}{\partial t}$). Furthermore, since $q = U - p$, then q must also be a null vector and a linear combination of U and $\frac{\partial}{\partial t}$. This means that, in order for p^1 to be maximal, P and Q must both be photons. To simplify our computations, we start by writing the linear combination $U + \beta\frac{\partial}{\partial t}$ and want to determine for which values of β it is a null vector. Hence, we have

$$\begin{aligned}\left\langle U+\beta\frac{\partial}{\partial t}, U+\beta\frac{\partial}{\partial t}\right\rangle = 0 \Leftrightarrow \langle U,U\rangle + 2\left\langle U,\beta\frac{\partial}{\partial t}\right\rangle + \left\langle \beta\frac{\partial}{\partial t},\beta\frac{\partial}{\partial t}\right\rangle = 0 \\ \Leftrightarrow -1 - 2\beta E + g_{tt}\beta^2 = 0 \\ \Leftrightarrow \beta = \beta_{\pm} = \frac{E \pm \sqrt{E^2+g_{tt}}}{g_{tt}},\end{aligned} \tag{23}$$

Notice that since we are considering the breakup event inside the ergoregion, then we have $g_{tt} = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right\rangle > 0$, i.e., $\frac{\partial}{\partial t}$ is spacelike. Thus, $\beta_+ > 0$ and $\beta_- < 0$. Writing p and q as

$$p = \alpha\left(U + \beta_-\frac{\partial}{\partial t}\right), \quad q = (1-\alpha)\left(U+\beta_+\frac{\partial}{\partial t}\right), \tag{24}$$

and, using the constraint $U = p + q$, we must have

$$\alpha\beta_- + (1-\alpha)\beta_+ = 0, \tag{25}$$

so we find that

$$\alpha = \frac{\beta_+}{\beta_+ - \beta_-} = \frac{E+\sqrt{E^2+g_{tt}}}{2\sqrt{E^2+g_{tt}}}. \tag{26}$$

Consequently, E_P is given by

$$\begin{aligned} E_P &= -\left\langle p, \frac{\partial}{\partial t}\right\rangle = -\left\langle \alpha\left(U+\beta_-\frac{\partial}{\partial t}\right), \frac{\partial}{\partial t}\right\rangle \\ &= \alpha\left(E - \beta_- g_{tt}\right) \\ &= \frac{1}{2}\left(E + \sqrt{E^2+g_{tt}}\right). \end{aligned} \tag{27}$$

The efficency of this process is then

$$\eta = \frac{E_P}{E} = \frac{1}{2}\left(1 + \sqrt{1+\frac{g_{tt}}{E^2}}\right). \tag{28}$$

From the previous expression, in order to maximize the efficency we must maximize g_{tt} and minimize E. For the latter, we should consider the case in which the incoming particle is simply dropped from rest at infinity, giving $E = 1$. Now, we must find the point inside the ergoregion where g_{tt} has a maximum (since the Kerr metric is axisymmetric, this is equivalent to finding r and θ that maximize g_{tt} in this region). For this, we recall that

$$g_{tt} = -\frac{\Delta - a^2\sin^2\theta}{\rho^2} = -1 + \frac{2Mr}{r^2 + a^2\cos^2\theta}, \tag{29}$$

so we must choose $\theta = \frac{\pi}{2}$ to maximize g_{tt}. Differentiating the previous expression with respect to r, we get

$$\frac{\partial g_{tt}}{\partial r} = -\frac{2M\left(r^2 - a^2\cos^2\theta\right)}{\left(r^2 + a^2\cos^2\theta\right)^2}. \tag{30}$$

Because we are inside the ergoregion, we have $r \geq r_+$. From the expression for the outer horizon, it is clear that $r_+ \geq M$, and, since $M \geq a$, we have $r \geq a$. With this, we conclude that $\frac{\partial g_{tt}}{\partial r} < 0$ in the ergoregion, and so we maximize g_{00} by choosing $r = r_+$.

For an arbitrary black hole, we find the maximum efficiency to be

$$\eta_{\max} = \frac{1}{2}\left(1 + \sqrt{\frac{2M}{r_+}}\right) = \frac{1}{2}\left(1 + \sqrt{\frac{2M}{M + \sqrt{M^2 - a^2}}}\right). \tag{31}$$

Depending on the parameters M and a, this maximum efficiency takes different values, and it is maximal when $M = a$, that is, for an extremal black hole. In that case, we have

$$\eta_{\max}^{\text{extremal}} = \frac{1 + \sqrt{2}}{2} \approx 1.207, \tag{32}$$

which means that, for an extremal black hole, if the breakup event occurs at $r = r_+$ and $\theta = \frac{\pi}{2}$, the outgoing particle will leave the ergoregion with, approximately, 20.7% more energy per unit mass than the initial incoming particle.

As for particle Q, its energy is given by

$$\begin{aligned} E_Q &= -\left\langle q, \frac{\partial}{\partial t}\right\rangle \\ &= -\left\langle (1-\alpha)\left(U + \beta_+\frac{\partial}{\partial t}\right), \frac{\partial}{\partial t}\right\rangle \\ &= (1-\alpha)\left(\left\langle U, \frac{\partial}{\partial t}\right\rangle + \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right\rangle\right) \\ &= \frac{1}{2}\left(E - \sqrt{E^2 + g_{tt}}\right). \end{aligned} \tag{33}$$

Since $g_{tt} > 0$, the term $\sqrt{E^2 + g_{tt}} > E$, hence $E_Q < 0$.

4 The Oberth maneuver in General Relativity

In this Section, we explore some aspects of rocket trajectories in general relativity, with the ultimate goal of applying the Oberth maneuver near a Kerr black hole and compare the energy gain to the Schwarzschild case. Firstly, we look at the motion of rockets according to Newton's theory, deriving the rocket equation and then performing the Oberth maneuver. After that, we write the generalization of the rocket equation to general relativity, and apply the Oberth maneuver in the presence of a gravitational field generated by the aforementioned black holes.

4.1 Rocket equation in Newton's theory

Consider a rocket at rest in empty space. From time t_0 to time t_1, the rocket will exhaust some of its mass in order to gain velocity. The mass of the rocket will, therefore, be a function of time, $m : [t_0, t_1] \subset \mathbb{R} \longrightarrow \mathbb{R}^+$. We assume the velocity of the exhaust particles is constant and given by $u \in \mathbb{R}^+$. Fixing an instant $t \in [t_0, t_1]$, and an interval of time Δt, the change in velocity of the rocket will be given by $\Delta v = v(t + \Delta t) - v(t)$, and the change in mass by $\Delta m = m(t + \Delta t) - m(t)$, with the former being positive, since the velocity increases, and the latter negative, because the mass decreases as some of it is lost to exhaustion. By the conservation of momentum, the following equality holds

$$\begin{aligned} m(t)v(t) &= m(t + \Delta t)v(t + \Delta t) + (-\Delta m)(v(t) - u) \\ &= (m(t) + \Delta m)(v(t) + \Delta v) + (-\Delta m)(v(t) - u). \end{aligned} \tag{34}$$

The left hand side of the previous equation is just the momentum at time t, while the right hand side corresponds to the momentum at $t + \Delta t$, where the first term accounts for the momentum of the rocket, and the second for the momentum of the exhaust particles. After canceling some terms, equation (34) becomes

$$m(t) + \Delta m + \frac{\Delta m}{\Delta v} u = 0. \tag{35}$$

In the limit $\Delta t \to 0$, we get $\Delta m \to 0$ and $\frac{\Delta m}{\Delta v} \to \frac{dm}{dv}$, and the previous equation becomes

$$m(t) + \frac{dm}{dv} u = 0 \Leftrightarrow -\frac{1}{u} dv = \frac{1}{m(t)} dm. \tag{36}$$

Integrating over the time interval $[t_0, t_1]$, we arrive at the classical rocket equation

$$v_1 - v_0 = u \log\left(\frac{m_0}{m_1}\right), \tag{37}$$

or, equivalently,

$$m_1 = m_0 \exp\left(-\frac{v_1 - v_0}{u}\right), \tag{38}$$

which establishes a relation between the final mass $m_1 \coloneqq m(t_1)$ and the initial mass $m_0 \coloneqq m(t_0)$, with $v_1 \coloneqq v(t_1)$ and $v_0 \coloneqq v(t_0)$. Thus, this equation implies that the velocity increase is fixed by the mass of exhaust available. In terms of the energy, if E_0 and E_1 denote the energy per unit mass of the rocket before and after the acceleration, respectively, then $E_0 = 0$ and $E_1 = \frac{1}{2}(v_1 - v_0)^2$.

4.2 Oberth maneuver in Newton's theory

We now take a look at the Oberth maneuver, which allows for an additional energy gain when compared to accelerating from rest in empty space. The idea consists of letting the rocket fall in a gravitational field, hence obtaining some velocity without resorting to exhaustion, and only after this will it accelerate.

Assume the rocket is at rest at infinity, so its initial mechanical energy is $E_0 = 0$. It then falls in the direction of a spherical body (for example, an idealized planet), with mass M, only by the effect of gravity. At a distance $r = r_0$ relative to the center of the body, its mechanical energy per unit mass will be given by

$$\frac{1}{2}v^2 - \frac{M}{r_0}, \tag{39}$$

where v is the norm of its velocity vector. Since the mechanical energy is conserved, we can find the velocity the rocket has once it gets to r_0

$$\frac{1}{2}v^2 - \frac{M}{r_0} = 0 \Leftrightarrow v = \sqrt{\frac{2M}{r_0}}. \tag{40}$$

As we saw from the rocket equation, if the rocket accelerates, its velocity can increase by an amount Δv, which is fixed by the available mass of exhaust. If the rocket accelerates at r_0, then its mechanical energy per unit mass will become

$$\begin{aligned} E_1 &= \frac{1}{2}(v + \Delta v)^2 - \frac{M}{r_0} = \frac{1}{2}v^2 - \frac{M}{r_0} + \frac{1}{2}(\Delta v)^2 + v\Delta v \\ &= \frac{1}{2}(\Delta v)^2 + v\Delta v, \end{aligned} \tag{41}$$

where we used equation (40) to cancel some terms. Comparing with the previous case, in which the velocity prior to acceleration was $v = 0$, there is an extra gain of $v\Delta v$ in the mechanical energy per unit mass. From (40), in order to maximize this quantity, r_0 must be as small as possible, that is, the radius of maximal approximation of the rocket to the center of the gravitational field.

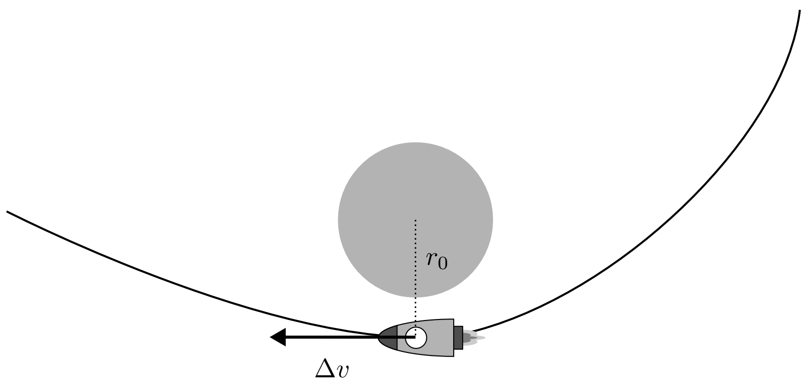

Figure 4: Illustration of a rocket performing the Oberth maneuver.

4.3 Rocket equation in General Relativity

We now move on to the general relativity setting. Consider a rocket moving along a curve $c : [\tau_0, \tau_1] \subset \mathbb{R} \longrightarrow M$, parametrized by the proper time τ, implying that $\langle \dot{c}, \dot{c} \rangle = -1$. During the interval $[\tau_0, \tau_1]$, the rocket will be accelerating by exhausting particles. Therefore, the rest mass of the rocket will be a nonincreasing function of the proper time, i.e., $m : [\tau_0, \tau_1] \subset \mathbb{R} \longrightarrow \mathbb{R}^+$ and, consequently, the 4-momentum vector of the rocket, given by $m\dot{c}$, will not be constant along c, that is

$$\nabla_{\dot{c}} (m\dot{c}) \neq 0.$$

Instead, we write

$$\nabla_{\dot{c}} (m\dot{c}) + K = 0 \Leftrightarrow \dot{m}\dot{c} + m\nabla_{\dot{c}}\dot{c} + K = 0 \Leftrightarrow K = -\dot{m}\dot{c} - m\nabla_{\dot{c}}\dot{c}, \tag{42}$$

where K is a vector field along c, representing the instantaneous rate, with respect to the proper time, at which energy-momentum is being carried away by the exhaust. Now, we must impose that K is timelike, otherwise, the rocket would be releasing particles faster than light, and future-pointing. Since the covariant acceleration $\nabla_{\dot{c}} (m\dot{c})$ is orthogonal to $\dot{c}$ and $\dot{c}$ is timelike, then $\nabla_{\dot{c}} (m\dot{c})$ is spacelike. Therefore, in order for K to be timelike, $\dot{m}\dot{c}$ must be timelike. For it to be future-pointing, it is necessary that $-\dot{m}\dot{c}$ is future-pointing, so we must have $\dot{m} \leq 0$ (which agrees with the fact that the rocket is ejecting some of its mass).

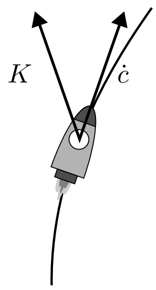

Figure 5: Rocket accelerating.

Notice that, if $\dot{m} = 0$, then K must also be 0, which happens if no particles are being ejected (if this were not the case, K would be spacelike whenever $\dot{m} = 0$). Hence, factoring out $\dot{m}$, we get

$$K = -\dot{m} (\dot{c} + V), \tag{43}$$

where V is a spacelike vector orthogonal to $\dot{c}$. The constraint that K must be timelike also implies that

$$\begin{aligned} \langle K, K \rangle \leq 0 &\Leftrightarrow \dot{m}^2 (\langle \dot{c}, V \rangle + \langle \dot{c}, \dot{c} \rangle + \langle V, V \rangle) \leq 0 \\ &\Leftrightarrow \dot{m}^2 (-1 + \langle V, V \rangle) \leq 0 \\ &\Rightarrow \langle V, V \rangle \leq 1. \end{aligned} \tag{44}$$

Comparing (42) and (43), we get

$$m\nabla_{\dot{c}}\dot{c} = \dot{m}V \Rightarrow \dot{m}|V| = -m|\nabla_{\dot{c}}\dot{c}|,$$

where the minus sign comes from the fact that $\dot{m} \leq 0$, and we can interpret $|V|$ as the instantaneous speed of exhaust particles measured by the rocket.

Consider the case in which the exhaust particles have constant speed, that is, $|V| = u$, with $0 < u \leq 1$. In the extreme case, $u = 1$, we have $\langle K, K \rangle = 0$, which corresponds to the exhaust particles being photons. Integrating, we obtain the rocket equation for a relativist rocket, which relates the final rest mass $m_1 := m(\tau_1)$ with the initial rest mass $m_0 := m(\tau_0)$

$$\begin{aligned}
&\frac{1}{u}\int_{\tau_0}^{\tau_1} |\nabla_{\dot{c}}\dot{c}| d\tau = -\int_{\tau_0}^{\tau_1} \frac{\dot{m}}{m} d\tau \\
&\Leftrightarrow \frac{1}{u}\int_{\tau_0}^{\tau_1} |\nabla_{\dot{c}}\dot{c}| d\tau = -\log m \Big|_{\tau_0}^{\tau_1} \\
&\Leftrightarrow m_1 = m_0 \exp\left(-\frac{1}{u}\int_{\tau_0}^{\tau_1} |\nabla_{\dot{c}}\dot{c}| d\tau\right) \\
&\Leftrightarrow m_1 = m_0 e^{-\frac{\psi}{u}},
\end{aligned} \tag{45}$$

where ψ is the hyperbolic angle between the initial and final 4-velocity vectors. This equation leads to the conclusion that for certain values of initial and final rest masses, there is a unique value of the integral of the covariant acceleration with respect to the proper time.

4.4 Oberth maneuver in Schwarzschild metric

In this section, we will apply the Oberth maneuver in the context of general relativity, as studied in [8]. For that, we shall consider a non-rotating black hole generating a gravitational field which is described by the Schwarzschild metric

$$g = -\left(1 - \frac{2M}{r}\right) dt \otimes dt + \left(1 - \frac{2M}{r}\right)^{-1} dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta d\varphi \otimes d\varphi. \tag{46}$$

Our aim is to find the energy gain by performing this maneuver and compare it to accelerating from rest in flat spacetime.

Consider a rocket with unit rest mass, approaching a non-rotating black hole with mass M, described by the metric above, starting from rest at infinity. For simplicity, we shall consider the motion in the equatorial plane, $\theta = \frac{\pi}{2}$. At the radius of closest approximation, which we shall denote as r_0, the rocket accelerates instantaneously and tangentially to its motion (in Subsection 4.6 we make a note on why this is the optimal direction for an instantaneous acceleration, and in Subsection 4.8 we will give an idea of what can be regarded as a good approximation for such an acceleration). Just before accelerating, the rocket's 4-velocity vector is given by

$$U_0 = U_0^t \frac{\partial}{\partial t} + U_0^\varphi \frac{\partial}{\partial \varphi}, \tag{47}$$

a linear combination of $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$, since $U_0^r = \dot{r}|_{r=r_0} = 0$. Dropping the rocket from rest at infinity implies that, before accelerating, its energy is

$$E_0 = -\left\langle U_0, \frac{\partial}{\partial t} \right\rangle = 1, \tag{48}$$

from which we get

$$U_0^t = \left(1 - \frac{2M}{r_0}\right)^{-1}. \tag{49}$$

To simplify the computations, let us define $f := \left(1 - \frac{2M}{r_0}\right)$, and so $U_0^t = f^{-1}$ (notice also that $g_{tt} = -f$ at $r = r_0$). Moreover, the 4-velocity satisfies the condition

$$\begin{aligned} &\langle U_0, U_0 \rangle = -1 \\ &\Leftrightarrow g_{tt}\left(U_0^t\right)^2 + g_{\varphi\varphi}\left(U_0^\varphi\right)^2 = -1 \\ &\Leftrightarrow U_0^\varphi = \pm \frac{1}{r_0}\sqrt{-1 + f^{-1}}. \end{aligned} \tag{50}$$

Let us choose the component $U_0^\varphi > 0$. Next, we find a unit spacelike vector $F_0 = F_0^t \frac{\partial}{\partial t} + F_0^\varphi \frac{\partial}{\partial \varphi}$, orthogonal to U_0, by solving the system

$$\begin{cases} \langle U_0, F_0 \rangle = 0 \\ \langle F_0, F_0 \rangle = 1 \end{cases} \tag{51}$$

This gives

$$F_0^t = \pm\sqrt{\left(f^{-1} - 1\right) f^{-1}}, \quad F_0^\varphi = \pm \frac{1}{r_0}\sqrt{f^{-1}} \tag{52}$$

and we choose the positive components (this is done so that the rocket is accelerating in the direction of its motion, otherwise, it would be decelerating). After the acceleration, the 4-velocity will be given by

$$U_1 = \cosh(\psi) U_0 + \sinh(\psi) F_0, \tag{53}$$

where ψ is the hyperbolic angle between U_0 and U_1, and the final energy will therefore be

$$\begin{aligned} E_1 = -\left\langle U_1, \frac{\partial}{\partial t} \right\rangle &= -\cosh(\psi)\left\langle U_0, \frac{\partial}{\partial t} \right\rangle - \sinh(\psi)\left\langle F_0, \frac{\partial}{\partial t} \right\rangle \\ &= \cosh(\psi) + \sinh(\psi) f \sqrt{\left(f^{-1} - 1\right) f^{-1}} \\ &= \cosh(\psi) + \sinh(\psi)\sqrt{\frac{2M}{r_0}}. \end{aligned} \tag{54}$$

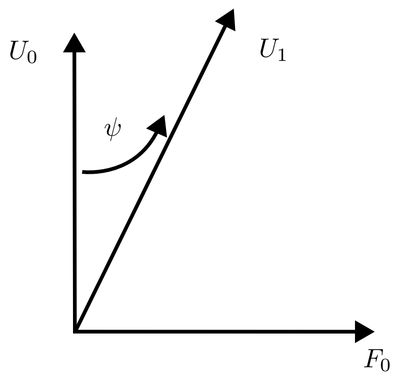

Figure 6: Frame with the initial 4-velocity of the rocket U_0, the orthonormal vector to it F_0, the final 4-velocity U_1 and the hyperbolic angle ψ.

If the acceleration had been performed away from any gravitational field (in flat spacetime), the energy gain would simply be $\cosh(\psi)$. However, performing this maneuver near a blackhole is more efficient, as it allows for an additional gain of $\sinh(\psi)\sqrt{\frac{2M}{r_0}}$. The efficiency is then

$$\eta = \frac{E_1}{E_0} = \cosh(\psi) + \sinh(\psi)\sqrt{\frac{2M}{r_0}}. \tag{55}$$

The gain is higher for smaller values of the radius. In the limit, the maximum energy will happen for the radius defining the event horizon, which, for this metric, is the Schwarzschild radius $r_\mathrm{S} := 2M$. In this case, we have

$$E_1^{\max} = \cosh(\psi) + \sinh(\psi). \tag{56}$$

and so

$$\eta_{\max} = \frac{E_1^{\max}}{E_0} = \cosh(\psi) + \sinh(\psi). \tag{57}$$

We end this Subsection with a generalization of equation (54). If, instead of considering the particle dropped from rest at infinity, we had considered it with some initial energy per unit mass $\Gamma > 1$, the final energy per unit mass would be given by

$$E_1 = \Gamma\cosh(\psi) + \sinh(\psi) f \sqrt{\frac{\Gamma f^{-1} - 1}{f - \Gamma + \Gamma^2}}. \tag{58}$$

4.5 Oberth maneuver in Kerr metric

Having studied the Oberth maneuver in the Schwarzschild metric, we now move on to the application of such a maneuver in the Kerr metric, that is, we consider that the body generating the gravitational field is a rotating black hole. Again, let M be the mass of the black hole, and let a denote its angular momentum per unit mass. The setting is analogous to the previous one: we consider a rocket with unit rest mass falling towards the black hole, starting from rest at infinity, with the motion restricted to the equatorial plane $\theta = \frac{\pi}{2}$. In this case, the components of the metric (cf. section 2) simplify to

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2M}{r}\right), \\ g_{t\varphi} = g_{\varphi t} &= -\frac{2M}{r}a, \\ g_{\varphi\varphi} &= r^2 + a^2 + \frac{2M}{r}a^2. \end{aligned} \tag{59}$$

Furthermore, we are particularly interested in performing the acceleration inside the ergoregion. Once again, if U_0 denotes the initial 4-velocity of the rocket, its initial energy must satisfy

$$\begin{aligned} E_0 = -\left\langle U_0, \frac{\partial}{\partial t}\right\rangle = 1 \\ \Leftrightarrow -g_{tt}U_0^t - g_{\varphi t}U_0^\varphi = 1. \end{aligned} \tag{60}$$

Moreover, the 4-velocity must satisfy

$$\begin{aligned} &\langle U_0, U_0\rangle = -1 \\ \Leftrightarrow &g_{tt}\left(U_0^t\right)^2 + 2g_{t\varphi}U_0^tU_0^\varphi + g_{\varphi\varphi}\left(U_0^\varphi\right)^2 = -1. \end{aligned} \tag{61}$$

Solving the system of equations (60) and (61), one finds that

$$\begin{aligned} U_0^\varphi &= \pm\sqrt{\frac{2M}{r_0\Delta}} \\ U_0^t &= \pm f^{-1}\left(1 - \frac{2Ma}{r_0}\sqrt{\frac{2M}{r_0\Delta}}\right). \end{aligned} \tag{62}$$

As we saw in Section 2, inside the ergoregion the rocket is forced to rotate in the same direction as the black hole. Therefore, we must choose $U_0^\varphi > 0$. With this choice, we also have

$$U_0^t = f^{-1}\left(1 - \frac{2Ma}{r_0}\sqrt{\frac{2M}{r_0\Delta}}\right), \tag{63}$$

hence,

$$U_0 = f^{-1}\left(1 - \frac{2Ma}{r_0}\sqrt{\frac{2M}{r_0\Delta}}\right)\frac{\partial}{\partial t} + \sqrt{\frac{2M}{r_0\Delta}}\frac{\partial}{\partial\varphi}. \tag{64}$$

Now, we complement U_0 with a unit vector F_0 orthogonal to it. This vector can be written as

$$F_0 = F_0^t\frac{\partial}{\partial t} + F_0^\varphi\frac{\partial}{\partial\varphi}, \tag{65}$$

and it must satisfy the following conditions

$$\begin{cases} \langle U_0, F_0 \rangle = 0 \\ \langle F_0, F_0 \rangle = 1 \end{cases} \Leftrightarrow \begin{cases} g_{tt} U_0^t F_0^t + g_{t\varphi} \left(F_0^t U_0^\varphi + U_0^t F_0^\varphi \right) + g_{\varphi\varphi} F_0^\varphi U_0^\varphi = 0 \\ g_{tt} \left(F_0^t \right)^2 + 2 g_{t\varphi} F_0^t F_0^\varphi + g_{\varphi\varphi} \left(F_0^\varphi \right)^2 = 1 \end{cases} . \tag{66}$$

Solving this system, after some algebra, we get

$$\begin{aligned} F_0^t &= \pm f^{-1} \left(\sqrt{\frac{2M}{r_0}} - \frac{2Ma}{r_0\sqrt{\Delta}} \right) \\ F_0^\varphi &= \pm \sqrt{\frac{1}{\Delta}}. \end{aligned} \tag{67}$$

Once again, there are two possible vectors, and we choose the one such that $F_0^\varphi > 0$, so that the acceleration has the same direction as the rocket's motion. Accelerating the rocket at r_0 would change the 4-velocity from U_0 to

$$U_1 = \cosh(\psi) U_0 + \sinh(\psi) F_0. \tag{68}$$

Hence, the energy after acceleration is given by

$$\begin{aligned} E_1 = -\left\langle U_1, \frac{\partial}{\partial t} \right\rangle &= -\cosh(\psi) \left\langle U_0, \frac{\partial}{\partial t} \right\rangle - \sinh(\psi) \left\langle F_0, \frac{\partial}{\partial t} \right\rangle \\ &= \cosh(\psi) - \sinh(\psi) \left(g_{tt} F_0^t + g_{t\varphi} F_0^\varphi \right) \\ &= \cosh(\psi) - \sinh(\psi) \left(-f f^{-1} \left(\sqrt{\frac{2M}{r_0}} - \frac{2Ma}{r_0\sqrt{\Delta}} \right) - \frac{2Ma}{r_0} \sqrt{\frac{1}{\Delta}} \right) \\ &= \cosh(\psi) + \sinh(\psi) \sqrt{\frac{2M}{r_0}}. \end{aligned} \tag{69}$$

We now note some things about this result, namely, we want to compare it to the previous result for the Schwarzschil metric, hence, we fix M as the mass of both black holes. Firstly, one would expect that taking $a = 0$ would reduce the previous expression to the Schwarzschild case. It turns out that the expression for the energy per unit mass after acceleration is exactly the same in both cases (cf. 54) and so is the efficiency, regardless of taking $a = 0$. The maximum energy, however, is where the dependence in the rotation parameter a appears. Recall that the maximum energy for the Schwarzschild black hole happens, in the limit, for acceleration in the Schwarzschild radius, since that is the smallest radius the rocket can escape from. For the Kerr black hole, the smallest radius corresponds to the outer horizon

$$r_+ = M + \sqrt{M^2 - a^2}, \tag{70}$$

which is smaller than $2M$, hence, $r_+ < r_S$. The maximum energy in Kerr would, therefore, be

$$E_1^{\max} = \cosh(\psi) + \sinh(\psi)\sqrt{\frac{2M}{M + \sqrt{M^2 - a^2}}} = \cosh(\psi) + \sinh(\psi)\sqrt{\frac{2}{1 + \sqrt{1 - \left(\frac{a}{M}\right)^2}}}. \tag{71}$$

Furthermore, the energy takes different values depending on the quotient $\frac{a}{M}$. Thus, the biggest value for $E_1^{\max}$ happens for an extremal black hole, $a = M$, giving

$$E_1^{\text{max, extremal}} = \cosh(\psi) + \sinh(\psi)\sqrt{2}. \tag{72}$$

To summarize, the rotation of the black hole does not explicitly affect the general expression for the final energy after performing the Oberth maneuver. However, its effect is implicit in the fact that it gives the black hole a different structure, which then allows the rocket to get closer to the center of the black hole, hence obtaining a bigger energy gain.

There is also another interesting effect that can arise performing this maneuver inside the ergoregion. Recall that, as mentioned in Section 3, the particle that falls into the outer horizon in the Penrose process has negative energy. Similarly, performing the Oberth maneuver in the ergoregion allows the exhaust to have negative energy. Let u denote the velocity of the exhaust relative to the rocket. The 4-velocity of the exhaust is then

$$W = U_0 - uF_0 \tag{73}$$

and its energy will, therefore, be given by

$$E_{\text{exhaust}} = -\left\langle W, \frac{\partial}{\partial t} \right\rangle = -\left\langle U_0, \frac{\partial}{\partial t} \right\rangle + u\left\langle F_0, \frac{\partial}{\partial t} \right\rangle = 1 - u\sqrt{\frac{2M}{r_0}}. \tag{74}$$

For this value to be negative, we must have

$$\begin{aligned} &1 - u\sqrt{\frac{2M}{r_0}} < 0 \\ &\Leftrightarrow \frac{1}{u^2} < \frac{2M}{r_0} \\ &\Leftrightarrow r_0 < 2Mu^2. \end{aligned} \tag{75}$$

Recall that, throughout this section, we are only considering motions restricted to the plane $\theta = \frac{\pi}{2}$. Moreover, the acceleration must occur outside the outer horizon, otherwise the rocket could not escape from the black hole, so we impose $r_0 > r_+$. Now, for fixed u, condition (75) defines circles in the equatorial plane with radius $2Mu^2$. For this circles to lie outside the outer horizon, we must have

$$
\begin{aligned}
&r_+ < 2Mu^2 \\
&\Leftrightarrow M + \sqrt{M^2 - a^2} < 2Mu^2 \\
&\Leftrightarrow u > 1 + \sqrt{1 - \left(\frac{a}{M}\right)^2}.
\end{aligned}
\tag{76}
$$

Therefore, provided u satisfies this condition, there is an annular region where acceleration can be performed so that exhaust is ejected with negative energy, defined by

$$
r_+ < r_0 < 2Mu^2. \tag{77}
$$

Notice that, for $u = 1$ (exhaust composed of photons), accelerating inside the ergoregion always results in exhaust with negative energy, since we get the condition

$$
r_+ < r_0 < 2M, \tag{78}
$$

which defines the ergoregion for $\theta = \frac{\pi}{2}$.

4.6 Optimality of the Oberth maneuver

In our study of the Oberth maneuver, we always considered the direction of the acceleration to be tangent to the motion of the rocket, and performed at the radius of closest approximation to the body generating the gravitational field. It turns out that these are, in fact, the optimal direction and place to accelerate, as it was studied in [6]. There, it was shown that the final energy is maximized when the instantaneous acceleration has the direction of the vector field P, the primer, given by

$$
P^\mu = -T^\mu - \langle T, U\rangle U^\mu, \tag{79}
$$

where T is the Killing vector field $\frac{\partial}{\partial t}$ (both a Killing vector field of the Schwarzschild and Kerr metrics), and performed when the primer's magnitude, ρ, is maximal. The rocket's energy per unit mass is then $E = -\langle U, T\rangle$. In our case, we have

$$
\begin{aligned}
\rho^2 &= \langle P, P\rangle \\
&= \langle -T + U, -T + U\rangle \\
&= \langle T, T\rangle - 2\langle U, T\rangle + \langle U, U\rangle \\
&= g_{tt} + 2 - 1 \\
&= g_{tt} + 1
\end{aligned}
\tag{80}
$$

and since $g_{tt} = -\left(1 - \frac{2M}{r}\right)$ (for both metrics, since we restrict the motion to the plane $\theta = \frac{\pi}{2}$), we simply get

$$
\rho^2 = \frac{2M}{r}, \tag{81}
$$

so the energy is maximized for the smallest radius. Moreover, since for the smallest radius we have $\dot{r} = 0$, the 4-velocity at that point is simply a linear combination of $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$. Therefore, by (79), P is also a linear combination of $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$, so we write

$$P = P^t \frac{\partial}{\partial t} + P^\varphi \frac{\partial}{\partial \varphi}, \tag{82}$$

and conclude that the optimal acceleration is tangent to the rocket's motion.

4.7 Is the Oberth maneuver a Penrose process?

From our study of the Oberth maneuver performed near a Schwarzschild or a Kerr black hole, we have seen that, in both cases, we can increase the energy per unit mass, that is, $E_1 > 1$. One might ask the question whether or not the total final energy can also be bigger that the initial energy. In other words, is it possible that $m_1 E_1 > m_0$? To answer this, recall that, from the rocket equation (45), there is a relation between the initial and final rest masses of the rocket, which we can write as

$$\frac{m_1}{m_0} = e^{-\frac{\psi}{u}}. \tag{83}$$

Hence, the goal is to analyse if it is possible that $\frac{m_1}{m_0} E_1 > 1$. Considering $u = 1$ (photon rocket), the left-hand side accounts for

$$\begin{aligned} \frac{m_1}{m_0} E_1 &= e^{-\psi} \left(\cosh\psi + \sinh\psi \sqrt{\frac{2M}{r_0}} \right) \\ &= \frac{1}{2} \left(1 + e^{-2\psi} + \left(1 - e^{-2\psi}\right) \sqrt{\frac{2M}{r_0}} \right) \\ &= \frac{1}{2} \left(1 + \sqrt{\frac{2M}{r_0}} + e^{-2\psi} \left(1 - \sqrt{\frac{2M}{r_0}} \right) \right). \end{aligned} \tag{84}$$

For $r_0 < 2M$, we have $\sqrt{\frac{2M}{r_0}} = 1 + \varepsilon$, for some $\varepsilon > 0$. Therefore, (84) becomes

$$\frac{m_1}{m_0} E_1 = \frac{1}{2} \left[1 + (1+\varepsilon) + e^{-2\psi} \left(1 - (1+\varepsilon)\right) \right] = 1 + \frac{1}{2} \varepsilon \left(1 - e^{-2\psi}\right). \tag{85}$$

Since $e^{-2\psi} < 1$, we get $\frac{m_1}{m_0} E_1 > 1$ and so the total final energy is, indeed, bigger than the initial one. Notice that this is impossible for the Schwarzschild balck hole, since in that case the rocket is not allowed to go beyond $r_0 = 2M$.

4.8 Note on instantaneous acceleration

In this Subsection, we make notice of what can be regarded as an instantaneous acceleration. Let $\Delta\tau$ denote the proper time interval of acceleration (the one measured inside the rocket). A good approximation of an instantaneous acceleration corresponds to values of $\Delta\tau$ for which the variation of the angle φ is much smaller than the variation of φ after an entire circular orbit around the black hole, i.e., 2π. This can be written as $\dot{\varphi}\Delta\tau \ll 2\pi$. As computed above, at the point of closest approximation, we have, for the Kerr metric

$$\dot{\varphi} = U_0^{\varphi} = \sqrt{\frac{2M}{r_0\Delta}} = \sqrt{\frac{2M}{r_0\left(r_0^2 - 2Mr_0 + a^2\right)}}, \tag{86}$$

therefore, we look for values of $\Delta\tau$ such that

$$\Delta\tau \ll 2\pi\sqrt{\frac{r_0\left(r_0^2 - 2Mr_0 + a^2\right)}{2M}}. \tag{87}$$

For instance, if the radius of closest approximation is $r_0 = 3M$ and the black hole is extremal, we get

$$\Delta\tau \ll 2\sqrt{6}\pi M, \tag{88}$$

and so a good approximation of an instantaneous acceleration would be one that lasted for much less than $2\sqrt{6}\pi M$ seconds.

4.9 Example: Oberth maneuver close to Sagittarius A*

In this final section, we apply the previous results to a known black hole to better understand how impactful the Oberth maneuver really is in maximizing the velocity a rocket is able to reach for a fixed amount of exhaust. For that, we will consider the black hole at the center of the Milky Way, Sagittarius A*. Its mass is, approximately, $M = 4 \times 10^6$ $\mathrm{M}_\odot$. According to [3], $a = (0.90 \pm 0.06)\,M$, thus, we make the approximation of considering this black hole as extremal. Taking $M_\odot = 1.5$ km, we get $M = 6 \times 10^6$ km $= \frac{6\times10^6}{3\times10^5}$ s $= 20$ s. By equation (88), if the radius of closest approximation is $r = 3M$, then the value for $\Delta\tau$ must be such that $\Delta\tau \ll 308$ s $=: \overline{\Delta\tau}$. Hence, let us consider $\Delta\tau = \frac{\overline{\Delta\tau}}{10}$ s, and that the acceleration the rocket is capable of reaching is $10g$. For this choice of parameters, if the rocket accelerates in flat spacetime, the velocity increase is simply

$$\Delta v = \Delta\tau \times 10g = \overline{\Delta\tau}g \approx 3 \text{ km/s}. \tag{89}$$

Now, we will see how the Oberth maneuver significantly improves this value. Consider that the rocket accelerates at the distance of closest approximation to the center of the black hole $r = 3M$. Since $\Delta v = \tanh(\psi)$ is small, we can take the approximation

$$\psi \approx \Delta v = \frac{3}{3 \times 10^5} \approx 10^{-5}. \tag{90}$$

We can also consider the approximations $\cosh(\psi) \approx 1$ and $\sinh(\psi) \approx \psi$, which, substituting in equation (69), gives

$$E_1 \approx 1 + \frac{\sqrt{2}}{2}\psi = 1 + \frac{\sqrt{2}}{2} \times 10^{-5}. \tag{91}$$

To find the velocity v the rocket reaches after acceleration, we use the fact that, in geometrized units, the energy per unit mass is simply

$$E_1 = \gamma = \frac{1}{\sqrt{1-v^2}} \approx 1 + \frac{v^2}{2}, \tag{92}$$

where γ is the Lorentz factor and the last approximation is due to the fact that v is much smaller than the speed of light. Thus, the velocity after acceleration is

$$v = \sqrt{2(E_1 - 1)} = \sqrt{\sqrt{2} \times 10^{-5}} \approx 3.8 \times 10^{-3}, \tag{93}$$

which corresponds to $v \approx 1140$ km/s. Notice that this value is much bigger than the 3 km/s one would get after accelerating in flat spacetime (380 times bigger!), and the acceleration was still performed at a large distance from the outter horizon.

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