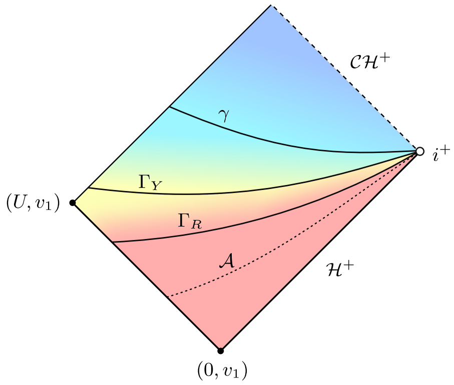
UNIVERSIDADE DE LISBOA
INSTITUTO SUPERIOR TÉCNICO



Stability of the Cauchy Horizon for Cosmological Black Holes in Spherical Symmetry

Flavio Rossetti

Supervisor: Dr. José António Maciel Natário
Co-supervisor: Dr. João Lopes Costa

Thesis specifically prepared to obtain the PhD Degree in

Mathematics

Draft

October 2024

[…] omnia quapropter debent per inane quietum aeque ponderibus non aequis concita ferri.

[…] wherefore all [bodies], with equal speed, though equal not in weight, must rush, borne downward through the still inane.

Lucretius, De Rerum Natura, book II

(translation by William Ellery Leonard)

Acknowledgments

I would like to express my gratitude to:

My supervisors José Natário and João Costa for teaching me about general relativity with patience and contagious enthusiasm, no matter if I was in Lisbon or (literally) anywhere else in the world. Thank you for your support and advice.

The mathematics department (and, especially, the GR group) at Princeton university, for the warm hospitality during my stay.

The people who kindly shared their insights and perspectives on general relativity with me over the past years. In particular: Gabriele Benomio, Mihalis Dafermos, Peter Hintz, Jesús Oliver, Rita Teixeira da Costa, Maxime Van de Moortel, Alex Vañó-Viñuales.

My family and friends in Italy, who always encouraged me.

This thesis project was supported by FCT/Portugal through the PhD scholarship UI/BD/152068/2021; partially supported by FCT/Portugal through CAMGSD, IST-ID, projects UIDB/04459/2020 and UIDP/04459/2020; and partially supported by the H2020-MSCA-2022-SE project Einstein–Waves, GA No. 101131233.

Abstract

This thesis is concerned with the problem of global uniqueness of solutions to the initial value problem for the Einstein equations. The statement of global uniqueness is encoded in the Strong Cosmic Censorship Conjecture (SCCC), whose negative resolution implies a violation of classical determinism in General Relativity. The present work provides rigorous results that suggest the failure of certain formulations of the SCCC in the case of spacetime solutions given by cosmological black holes.

The mathematical framework is the following: we investigate the interior of a dynamical black hole as described by the Einstein-Maxwell-charged-Klein-Gordon system of equations with a cosmological constant, under spherical symmetry. In particular, we consider a characteristic initial value problem where, on the outgoing initial hypersurface, interpreted as the event horizon $\mathcal{H}^+$ of a dynamical black hole, we prescribe: a) initial data asymptotically approaching a fixed sub-extremal Reissner-Nordström-de Sitter solution; b) an exponential Price law upper bound for the charged scalar field.

After showing local well-posedness for the corresponding first-order system of partial differential equations, we establish the existence of a Cauchy horizon $\mathcal{CH}^+$ for the evolved spacetime, extending the bootstrap methods used in the case $\Lambda = 0$ by Van de Moortel [79]. In this context, we show the existence of C^0 spacetime extensions beyond $\mathcal{CH}^+$. Moreover, if the scalar field decays at a sufficiently fast rate along $\mathcal{H}^+$, we show that the renormalized Hawking mass remains bounded for a large set of initial data. With respect to the analogous model concerning an uncharged and massless scalar field, we are able to extend the known range of parameters for which mass inflation is prevented, up to the optimal threshold suggested by the linear analyses by Costa-Franzen [18] and Hintz-Vasy [60].

In this no-mass-inflation scenario, which includes near-extremal solutions, we further prove that the spacetime can be extended across the Cauchy horizon with continuous metric, Christoffel symbols in L^2_{loc} and scalar field in H^1_{loc}.

By generalizing the work by Costa-Girão-Natário-Silva [22] to the case of a charged and massive scalar field, our results reveal a potential failure of the Christodoulou-Chruściel version of the strong cosmic censorship under spherical symmetry.

Keywords: Black holes, strong cosmic censorship, stability, Cauchy horizon, blueshift instability

Resumo

A presente tese trata do problema da unicidade global de soluções para o problema de valor inicial das equações de Einstein. A afirmação da unicidade global está codificada na Censura Cósmica Forte (CCF), cuja resolução negativa implica uma violação do determinismo clássico da Relatividade Geral. O presente trabalho fornece resultados rigorosos que sugerem uma violação de certas formulações da CCF no caso de soluções das equações de Einstein que modelam buracos negros cosmológicos.

O enquadramento matemático é o seguinte: investigamos o interior de um buraco negro dinâmico descrito pelo sistema de equações de Einstein com uma constante cosmológica, acoplado às equações de Maxwell e Klein-Gordon para um campo escalar carregado, em simetria esférica. Em particular, consideramos um problema de valor inicial caraterístico em que, numa das hipersuperfícies iniciais, interpretada como o horizonte de eventos $\mathcal{H}^+$ de um buraco negro dinâmico, prescrevemos: a) dados iniciais que se aproximam assintoticamente de uma solução não extrema de Reissner-Nordström-de Sitter; b) um limite superior para o campo escalar carregado dado por uma lei de Price exponencial.

Depois de mostrarmos existência local e unicidade para o sistema relevante de equações diferenciais parciais de primeira ordem, estabelecemos a existência de um horizonte de Cauchy $\mathcal{CH}^+$ para o espaço-tempo, estendendo os métodos de bootstrap usados no caso $\Lambda = 0$ por Van de Moortel [79]. Neste contexto, mostramos a existência de extensões C^0 do espaço-tempo para além de $\mathcal{CH}^+$. Além disso, se o campo escalar decair suficientemente rápido ao longo de $\mathcal{H}^+$, mostramos que a massa de Hawking renormalizada permanece limitada para um conjunto aberto de dados iniciais. No que diz respeito ao modelo análogo relativo a um campo escalar sem carga e sem massa, alargamos a gama conhecida de parâmetros para os quais a divergência da massa de Hawking é evitada, até ao limiar ótimo sugerido pelas análises lineares de Costa-Franzen [18] e Hintz-Vasy [60].

Neste cenário sem divergência de massa, que inclui soluções quase extremas, provamos ainda que o espaço-tempo pode ser estendido através do horizonte de Cauchy com métrica contínua, símbolos de Christoffel em L^2_{loc} e campo escalar em H^1_{loc}.

Generalizando o trabalho de Costa-Girão-Natário-Silva [22] para o caso de um campo escalar carregado e massivo, os nossos resultados revelam uma potencial falha da versão de Christodoulou-Chruściel da CCF em simetria esférica.

Palavras-chave: Buracos negros, conjectura de censura cósmica forte, estabilidade, horizonte de Cauchy, desvio para o azul

Preface

This thesis was written to fulfil the requirements for the completion of the doctoral programme at the Centre for Mathematical Analysis, Geometry and Dynamical Systems (CAMGSD) in the Mathematics department of Instituto Superior Técnico, University of Lisbon. During the doctoral programme, a 6-month period (November 2023 to April 2024) was spent as a visiting student research collaborator (VSRC) at the Mathematics department of Princeton University, Princeton, NJ USA.

This thesis is based on the following paper to appear in *Annales Henri Poincaré*:

- F. Rossetti, *Strong Cosmic Censorship for the Spherically Symmetric Einstein–Maxwell-Charged-Klein–Gordon System with Positive* Λ*: Stability of the Cauchy Horizon and* H^1 *Extensions.* Ann. Henri Poincaré (2024). https://doi.org/10.1007/s00023-024-01454-x.

The following two articles were published during the completion of the doctoral programme and are not discussed in the present thesis:

- J. Natário and F. Rossetti, *Explicit formulas and decay rates for the solution of the wave equation in cosmological spacetimes.* J. Math. Phys. 64 (3): 032504, 2023,

- F. Rossetti and A. Vañó-Viñuales, *Decay of solutions of the wave equation in cosmological spacetimes—a numerical analysis.* Class. Quantum Grav. 40 175004, 2023.

Contents

List of Figures

Chapter 1

Black holes: an introduction

1.1 Black hole solutions in general relativity

Approximately 150 millions of kilometres away from us, thermonuclear reactions are occurring inside the core of our nearest star, the Sun. Fast-forward 5 billion years, these inner nuclear fusions will cease due to the exhaustion of hydrogen. Such a shortage will allow the Sun to become a red giant, its size extending beyond the current location of the Earth. After this stage, the red giant will keep expelling gas while its inner core will become denser under the effect of gravity. What will be left from this process is a faint remaining of the star core, i.e. a white dwarf, surrounded by gas in the form of a planetary nebula [42, 43].

For masses larger than about 10 times the solar mass, a similar mechanism leads to the formation of red supergiants. They are ultimately destined to explode through a supernova blast. If the remaining core, after the explosion, is about at least three times more massive than our Sun, gravity induces further shrinking until, ultimately, reaching the state in which not even light can escape from the surrounding area. This final state is called a stellar black hole. Some black holes can be millions of times more massive than the Sun: these are denoted *supermassive* and their genesis is thought to be more convoluted than that of their stellar counterparts.

A supermassive black hole exists in our galaxy [1] and actually is thought to exist in most galaxies of our universe. Objects with infinite escape velocities have been theorized at least since Laplace [55], however such astrophysical objects have gained more interest since their striking prediction given by the theory of general relativity at the beginning of the 20th century. From the 1960s onwards, black holes have been studied systematically both in the mathematics and physics literature. The present work will focus on black hole spacetimes from a mathematical point of view.

In Einstein's theory of relativity, whose remarkable accuracy has been verified in a plethora of experiments [83, 56, 94], black hole regions are mathematically described in a way that embodies the layman notion of "being unable to escape to infinity". In particular, a solution to the Einstein equations

$$\mathrm{Ric}(g) - \frac{1}{2}\mathrm{R}g + \Lambda g = 8\pi T,$$

is said to contain a non-trivial black hole region if $J^-(\mathscr{I}^+)$, i.e. the causal past of future null infinity, has a non-empty complement, whenever such a quantity is well-defined.[1]

When the cosmological constant Λ is set equal to zero, the Kerr-Newman black holes are the unique[2] real-analytic, stationary, asymptotically flat solutions to the Einstein-Maxwell equations. They are parametrized in terms of a black hole mass M, a charge Q and angular momentum a. Subsets of this family are given by the Reissner-Nordström ($a = 0$), the Kerr ($Q = 0$) and Schwarzschild ($Q = 0$, $a = 0$) black holes. Analogous definitions hold in the case $\Lambda > 0$ (Kerr-Newman-de Sitter family) and $\Lambda < 0$ (Kerr-Newman-Anti de Sitter family).

The main features of black hole spacetimes are best understood when the Einstein equations are posed as an initial value problem (**IVP**). This setting comes naturally as a consequence of the hyperbolic nature of the Einstein equations. However, due to the geometric character of the PDE system (the unknown quantity being the ambient metric itself), such a formulation requires caution. A rigorous framework for the IVP was first advanced by Choquet–Bruhat [46] and then Choquet–Bruhat–Geroch [12]. In particular, a Riemannian manifold $(\mathcal{R}, h)$ and a symmetric 2-tensor field K on $\mathcal{R}$ satisfying the constraint equations

$$\begin{cases} \bar{\mathrm{R}} + \left(K^i{}_i\right)^2 + K_{ij}K^{ij} - 2\Lambda = 16\pi T_{00}, \\ \bar{\nabla}_i K^j{}_j - \bar{\nabla}_j K^j{}_i = -8\pi T_{0i}, \end{cases}$$

with $\bar{\mathrm{R}}$ the Ricci scalar of $\mathcal{R}$, uniquely determine (up to isometries) a globally hyperbolic Lorentzian manifold $(\mathcal{M}, g)$ that satisfies the Einstein equations. A posteriori, h can be seen to be the restriction of the Lorentzian metric g to $\mathcal{R}$ and K can be seen to be the second fundamental form of $\mathcal{R}$. Analogous IVPs in terms of characteristic initial hypersurfaces are common in general relativity [90] and allow to exploit the null structure of the Einstein equations.

1.2 The uniqueness problem and strong cosmic censorship

A natural question when analysing an initial value problem for a system of differential equations is whether it is well-posed in the sense of Hadamard. Existence and uniqueness for solutions to the Einstein equations have non-trivial implications of physical relevance. In the following, we focus on the uniqueness problem, which is the central topic of this thesis and is related to notable black hole solutions (see also section 1.4 for a discussion of the existence problem and weak cosmic censorship).

Let us consider a black hole interior as in Fig. 1.1. If starting from an initial value problem, such a diagram can be drawn a posteriori, since the spacetime is constructed during the dynamics. Since the event horizon $\mathcal{H}^+$ is a null hypersurface, it is natural to prescribe initial data on it (and on an additional null hypersurface) and obtain such a spacetime from a characteristic IVP. One of the main points of this construction is that $\mathcal{H}^+$ is a future-inextendible null hypersurface, i.e. future timelike infinity i^+ plays a physical role in this conformal diagram.

[1]The notion of causal past, here, is defined with respect to a larger manifold with boundary in which the original spacetime is conformally embedded. In particular, future null infinity (which is not necessarily a null hypersurface, for instance when $\Lambda > 0$) is contained in the boundary of the extended spacetime.

[2]In a suitable sense [17].

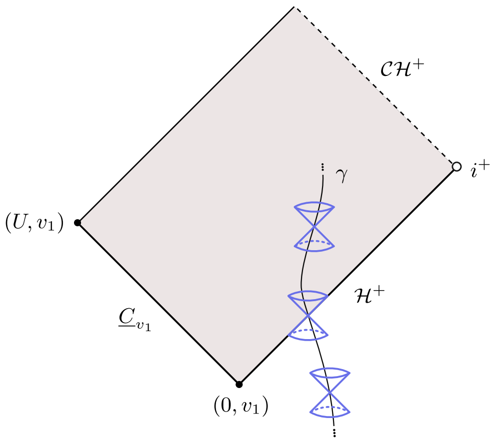


Figure 1.1: A local region of the interior of a Reissner-Nordström(-de Sitter) black hole. For each point in this conformal diagram, it is possible to distinguish between the causal future and the causal past of such a point. The curve γ describes the motion of an observer entering the black hole region. At the moment of crossing the exterior boundary $\mathcal{H}^+$ of the black hole (the event horizon), the causal future of the observer is contained in the black hole interior. For visualization purposes, the causal cones are drawn in three dimensions.

Notable black hole solutions, like those belonging to the Reissner-Nordström(-de Sitter) and, more generally, to the Kerr-Newman(-de Sitter) family can be constructed via a characteristic IVP of this sort. For such solutions[3], the metric is regular along the boundary of the domain of dependence of the initial hypersurfaces (i.e. the Cauchy horizon $\mathcal{CH}^+$) and can be smoothly extended to the future (in infinitely many ways) beyond it (see Fig. 1.2). This is a paradigmatic example of the failure of uniqueness (without loss of regularity) in the IVP for the Einstein equations.

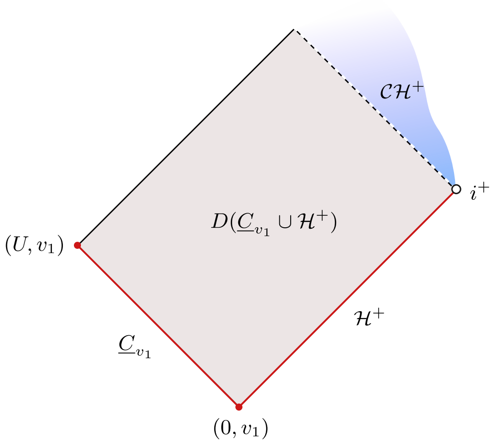


Figure 1.2: For Kerr-Newman(-de Sitter) black holes with non-trivial charge or non-trivial angular momentum, the spacetime metric can be smoothly extended in infinite ways beyond the domain of dependence of the initial hypersurfaces.

The strong cosmic censorship conjecture (SCCC) states that such extensions are unstable under perturbations. The instability statement is often meant as follows: for a (generic) set of initial data posed on a Cauchy hypersurface, a unique (up to isometries) spacetime solution exists and, for such a solution, no extensions (belonging to a certain regularity class) can be constructed. The genericity

[3]Provided that either the charge Q or the angular momentum a is non-zero. In the case $a = 0$, $Q = 0$ (Schwarzschild spacetime) no Cauchy horizon forms.

assumptions on the initial data and the dependence on the regularity of solutions are crucial points that will be explored in the thesis.

The present work shows that, starting from a characteristic IVP describing spherically symmetric charged black holes with a positive cosmological constant, geometric extensions beyond the domain of dependence can be built for near-extremal[4] initial data. In particular, such metric extensions belong to H^1 in a neighbourhood of the Cauchy horizon and solve the Einstein equations in a weak sense. This construction suggests a violation of the H^1 formulation of the SCCC in the case of a positive cosmological constant. In the following, we explain why this result is expected for this family of black holes.

1.3 The blueshift mechanism and the case $\Lambda > 0$

In the 1960s, Penrose suggested that Cauchy horizons of black hole spacetimes, such as that of a Reissner-Nordström spacetime, should suffer from an instability mechanism due to the blueshift effect, that forbids observers from crossing the inner horizon of such black hole interiors (in the spirit of the SCCC, see already Fig. 2.1). In the following, we will see that, when the cosmological constant Λ is positive, this phenomenon might fail to render geometric extensions unstable.

A formal description of the blueshift mechanism can be given as follows (see also [54]). Let ϕ be a solution of the linear wave equation

$$\Box_g \phi = 0,$$

where g is a Reissner-Nordström (interior) background. As it will be clear from the next steps, this construction (in fact, the blueshift mechanism) does not depend on the sign of the cosmological constant. In a suitable geometric optics approximation, we can think of ϕ as modelling some radiation propagating in the black hole interior. In double null (u, v) coordinates, the metric g becomes

$$g = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\sigma_{S^2} = 4f(r(u,v))dudv + r^2\sigma_{S^2},$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2} = \frac{(r_+ - r)(r - r_-)}{r^2},$$

σ_{S^2} is the standard metric on the unit round 2-sphere and $r_\pm = M \pm \sqrt{M^2 - e^2}$. Here, we used that

$$\begin{cases} u = \frac{r^* - t}{2}, \\ v = \frac{r^* + t}{2}, \end{cases}$$

with r^* such that $dr^* = f(r)^{-1}dr$. Notice that the above Eddington-Finkelstein coordinates are such that $u \in (-\infty, +\infty)$ and $v \in (-\infty, +\infty)$.

Let $X = X^u \partial_u + X^v \partial_v$ be the 4-velocity of a family of (radially) free-falling observers in the black hole interior, reaching the (right) Cauchy horizon (see also Fig. 1.1). The geodesic motion gives the

[4] The near-extremality condition can be interpreted as requiring very charged black holes. In the uncharged and rotating case, which is only briefly discussed in this work, near-extremality corresponds to rapidly-spinning solutions.

condition $g(X,X) = -1$ and thus

$$X^u = -\frac{1}{4f(r)X^v},$$

whenever the above coefficients are non-zero. On the other hand, since $K := \partial_u + \partial_v$ is a Killing vector for g, we have the additional condition $g(K,X) = E = const$. We can set $E = 1$ by choosing suitable initial condition (for instance, using (v, r, θ, ϕ) coordinates, we can set $\frac{dr}{d\lambda}(r_+) = 1$ along the event horizon, where λ is the affine parameter of the timelike geodesic). It follows that:

$$X^v = \frac{1 + \sqrt{1 - 4f(r)}}{4f(r)} \tag{1.1}$$

and thus

$$X^u = \frac{1 - \sqrt{1 - 4f(r)}}{4f(r)}. \tag{1.2}$$

Since the right Cauchy horizon is formally identified with the set $\{v = +\infty\}$, it is useful to understand the asymptotic behaviour of r^*. From the definitions of f and of r^*, the latter satisfies

$$r^* = -\frac{1}{2K_-} \log |r - r_-| + O(1), \qquad \text{as } r \to r_-,$$

where $K_\pm = \frac{r_+ - r_-}{2r_\pm^2} = \frac{1}{2}|f'(r_\pm)|$ are the (absolute values of) the surface gravities of the event horizon and of the Cauchy horizon, respectively. In particular:

$$r - r_- = Ce^{-2(v+u)K_-} \qquad \text{as } r \to r_-. \tag{1.3}$$

Now, the energy-momentum tensor of ϕ is

$$T_{\mu\nu}[\phi] = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi.$$

The energy current with respect to the Killing vector field $K = \partial_u + \partial_v$ is

$$J^K_\mu[\phi] = T_{\mu\nu}[\phi]K^\nu = |\partial_u \phi|^2 \delta^u_\mu + |\partial_v \phi|^2 \delta^v_\mu.$$

For every fixed $u \in (-\infty, +\infty)$, we have (see (1.1), (1.2) and (1.3)):

$$g(J, X) = |\partial_u \phi|^2 X^u + |\partial_v \phi|^2 X^v \sim c|\partial_u \phi|^2 + e^{2K_- v} |\partial_v \phi|^2, \tag{1.4}$$

as $v \to +\infty$, for a constant $c > 0$. The above is the energy density of ϕ as measured by the family of observers crossing the (right) Cauchy horizon and reveals the exponential growth related to the blueshift instability. We stress that the intensity of the instability depends on the surface gravity of the Cauchy horizon. On the other hand, the above energy density is bounded if

$$|\partial_v \phi|(u, v) \leq Ce^{-K_- v} \tag{1.5}$$

near the Cauchy horizon. In other words, the blueshift mechanism does not produce any instability in the energy if the decay of the solution ϕ is sufficiently rapid.

Thus, in order to understand the consequences of blueshift, it is necessary to investigate the dispersive properties of waves in the black hole spacetime (including the exterior region). These depend on the sign of the cosmological constant. In fact, solutions to the wave equation decay at an exponential rate when $\Lambda > 0$ (see already chapter 2). Such an exponential rate is faster than $e^{-K_- v}$ (see (1.5)) in the case of extremal black holes (for which $K_- = 0$). Similarly, it is expected that (1.5) holds in the near-extremal regime.

In this thesis we prove rigorous results for a non-linear toy model describing perturbations of Reissner-Nordström-de Sitter spacetimes. Such results show that, roughly speaking, the above intuition on the failure of blueshift as an instability mechanism is correct.

The motivation behind this work is to understand the limits of general relativity as a mathematical theory that describes our universe and is expected to adhere to the principles of classical determinism. Even though our results suggest that, in some contexts, determinism may fail, we stress that further investigations are necessary to settle the problem definitely. In particular:

- The results that we obtain are expected to hold for near-extremal charged black holes, which are considered somehow exotic in the astrophysical literature,

- These results assume sufficiently regular (smooth) initial data. By considering rough initial data, numerical evidence and rigorous works at the linear level (see also chapter 2) show that instability is retrieved at the level of the energy of solutions. Notice that working in a lower regularity setting is sometimes motivated by quantum theories of gravity.

1.4 Singularities and the global existence problem

From the point of view of the initial value problem, strong cosmic censorship is an assertion of uniqueness for metric solutions to the Einstein equations. A closely related problem is that of global existence. This is measured in terms of the completeness of future null infinity. When such a hypersurface is incomplete, it is said that the spacetime contains a *naked singularity.* The *weak cosmic censorship conjecture* (WCCC) posits that spacetimes with naked singularities should be unstable, in a suitable sense [13]. Historically [85], the word *strong* was assigned to those versions of cosmic censorship that were phrased without any reference to the observers at infinity. However, despite the nomenclature, there is no causal relation between the PDE formulations of the SCCC and those of the WCCC: there exist spacetimes for which the SCCC fails but the WCCC holds and vice versa.

Both conjectures deal with the term *singularity.* In a seminal paper [88], the Schwarzschild solution was described to have a *singularity* in correspondence of the spheres of radius $r = 2M$, M being the black hole mass and (t, r, θ, ϕ) being the Schwarzschild chart, even though the degeneracy of quantities at $r = 2M$ depends on the coordinate choice. In modern conventions, the term singularity is used to denote coordinate-independent "degenerate behaviours" occurring in black hole regions. This is, on purpose, a loose definition due to the wide variety of phenomena that solutions to the Einstein equations can describe [50, 41]. The singularity theorems of Hawking and Penrose [55], for instance, reserve this name to describe geodesic incompleteness and guarantee the existence of such a degeneracy in terms of open conditions.

Chapter 2

The interior of spherically symmetric charged black holes with $\Lambda > 0$

2.1 An overview of the model

The present work concerns the system of equations describing a spherically symmetric Einstein-Maxwell-charged-Klein-Gordon model with positive cosmological constant Λ, namely:

$$\begin{cases} \mathrm{Ric}(g) - \frac{1}{2}\mathrm{R}(g)\, g + \Lambda g = 2\big(T^{\mathrm{EM}} + T^{\phi}\big), & \text{(Einstein's equations)} \\ dF = 0, \quad d \star F = \star J, & \text{(Maxwell's equations)} \\ \big(D_\mu D^\mu - m^2\big)\phi = 0, & \text{(Klein-Gordon equation)} \end{cases} \tag{2.1}$$

where

$$T^{\mathrm{EM}}_{\mu\nu} = g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$$

is the energy-momentum tensor of the electromagnetic field,

$$T^{\phi}_{\mu\nu} = \mathrm{Re}\big(D_\mu\phi\overline{D_\nu\phi}\big) - \frac{1}{2}g_{\mu\nu}\big(D_\alpha\phi\overline{D^\alpha\phi} + m^2|\phi|^2\big)$$

is the energy-momentum tensor of the charged scalar field and

$$J_\mu = -\frac{iq}{2}\big(\phi\overline{D_\mu\phi} - \overline{\phi}D_\mu\phi\big)$$

is the current appearing in Maxwell's equations. The above system will be presented in full detail in chapter 3.

The main objective of our work is to study the evolution problem associated to the partial differential equations (**PDE**s) (2.1), with respect to initial data prescribed on two transversal characteristic hypersurfaces. Along one of these hypersurfaces, the initial data are specified to mimic the expected behaviour of a charged, spherically symmetric black hole, approaching[1] a sub-extremal Reissner-Nordström-de Sitter solution and interacting with a charged and (possibly) massive scalar field. This scenario is essentially encoded in the requirement of an exponential decay for such a scalar field along the event horizon in terms of an Eddington-Finkelstein type of coordinate, i.e. an *exponential Price law upper bound.* The evolved data then describe the interior of a dynamical black hole. A key point of our study regards the existence of a Cauchy horizon and of metric extensions beyond it, in different classes of regularity and most notably in H^1_{loc}. An overview of our results is given in section 2.2.

This problem arises naturally as a consequence of a decades-long debate on the mathematical treatment of black hole spacetimes. Already in the case $\Lambda = 0$, it was observed that the celebrated Kerr and Reissner-Nordström solutions to the Einstein equations share a daunting property: the future domain of dependence[2] of complete, regular, asymptotically flat, spacelike hypersurfaces is future-extendible in a smooth way. This translates into a failure of classical determinism[3] which is in sharp contrast with the Schwarzschild case, where analogous extensions are forbidden even in the continuous class [93]. On the other hand, observers crossing a Cauchy horizon[4] are generally expected to experience a *blueshift instability* [84, 97], leading to the blow-up of dynamical quantities and thus rescuing the deterministic principle. The instability is related to an unbounded amount of energy associated to test waves propagating in such a universe, as measured along the Cauchy horizon.[5]

This led to the first formulation of the strong cosmic censorship conjecture (**SCCC**), which, roughly speaking, forbids the existence of metric extensions beyond the Cauchy horizon of black hole spacetimes, at least in some regularity class.

In fact, the regularity of such extensions measures to which extent the SCCC fails. Although the Einstein equations are PDEs in the second derivative of the metric tensor, the H^1_{loc} (Christodoulou-Chruściel) regularity is actually the minimal requirement to make sense of the extensions as weak solutions to the Einstein equations (see also [22, 34, 14, 15]), even though local well–posedness is not known to hold in this case [68]. Weaker formulations can also be considered [89].

Being related to an exponential blow-up of the **energy**, the linear instability associated to blueshift concerns the first derivatives of scalar perturbations. Such effect becomes weaker in the near-extremal limit, for which the surface gravity of the Cauchy horizon approaches zero. This instability is absent in the extremal case [47, 48, 49].

Moreover, dispersive effects in the black hole exterior also impact the picture described by the blueshift instability. Scalar perturbations are expected to decay along the event horizon of black holes at a rate which, essentially, is exponential in the case $\Lambda > 0$ [40, 32, 44, 45, 61, 75], inverse polynomial in the case $\Lambda = 0$ (see [87, 53] for heuristics and numerical arguments, see [31, 4, 5, 39, 38, 99, 77, 59] for rigorous

[1] In an appropriate sense, which will be specified rigorously during our study.

[2] We define the future domain of dependence of a spacelike hypersurface S as the set of points p such that every past-directed, past-inextendible causal curve starting at p intersects S, see e.g. [82].

[3] The lack of global uniqueness can occur without any loss of regularity of the solution metric, even though the Einstein equations are hyperbolic, up to their diffeomorphism invariance. See [25] for a discussion of this phenomenon.

[4] That is the boundary of the future domain of dependence of a complete Cauchy hypersurface in the extended spacetime manifold. We refer the reader to the introduction of [22] for a discussion on the topic and to [16] for constructions of specific non-isometric extensions.

[5] At the level of geometric optics, [84] asserts: "*As [an observer crossing the Cauchy horizon $\mathcal{CH}^+$] looks out at the universe that he is 'leaving behind', he sees, in one final flash, as he crosses $\mathcal{CH}^+$, the entire later history of the rest of his 'old universe'* ". See also Fig. 2.1.

results) and logarithmic for $\Lambda < 0$ [63, 64]. This behaviour is known as **Price's law**.

The case $\Lambda > 0$ suggests a scenario where the exponential blow-up of the energy due to blueshift (in the sub-extremal case) can be counterbalanced by an exponentially fast decay in the black hole exterior (at least near extremality), leading to extensions of H^1 regularity at the Cauchy horizon. While this effect has been described rigorously in [18, 60] for the linear wave equation on a Reissner-Nordström-de Sitter background, the present work proves the analogous result in a non-linear setting with charged scalar field (see already section 2.2 and see [22] for the non-linear and uncharged case).

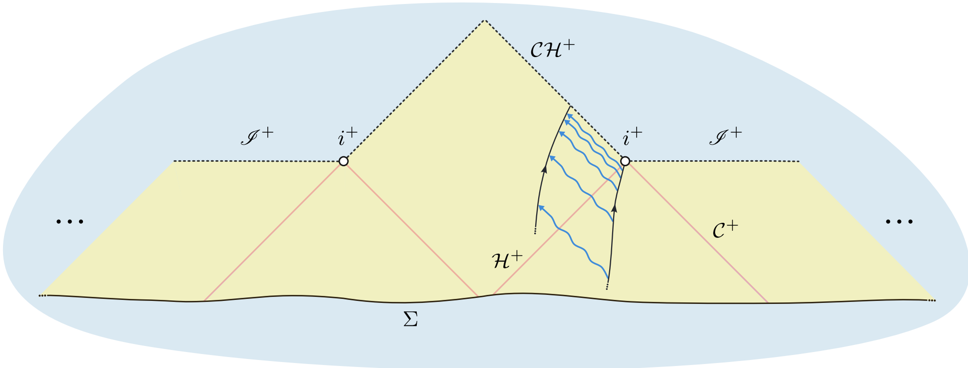


Figure 2.1: A schematic depiction of the blueshift instability in the case, for instance, of a sub-extremal Reissner-Nordström-de Sitter black hole (although we stress that this phenomenon is not specific to the case $\Lambda > 0$). Assume that radiation pulses at constant time intervals, as measured in proper time, from a timelike curve in the exterior of the black hole. The wordline in the exterior of the black hole has infinite affine length. On the other hand, observers along the timelike worldline in the black hole interior can reach the Cauchy horizon $\mathcal{CH}^+$ in finite proper time, thus receiving an infinite amount of pulses at the moment of crossing $\mathcal{CH}^+$.

The current understanding of non-linear effects is influenced by the celebrated work by Israel and Poisson on the Einstein-Maxwell-null dust system [86]. A subsequent turning point was the adoption of the Einstein-Maxwell-(real) scalar field model to study black hole interiors [25]. In particular, the addition of a scalar field not only renders the problem non-trivial (already under spherical symmetry) but is also motivated by the hyperbolic properties expected from the Einstein equations. A review of this construction can be found in the introduction of [30]. In the latter, the analysis of the non-linear problem in the absence of spherical symmetry was initiated. We will review relevant past work in section 2.4.

The present thesis is strictly related to the following formulations of the SCCC for the non-linear system (2.1).

Conjecture 2.1 (C^0 formulation of the SCCC for the Einstein-Maxwell-charged-Klein-Gordon system with $\Lambda > 0$, under spherical symmetry)
Let Σ be a regular, compact (without boundary),[6] spacelike hypersurface. Let us prescribe generic (in the spherically symmetric class) initial data on Σ, for the Einstein-Maxwell-charged-Klein-Gordon system with $\Lambda > 0$. Then, the future maximal globally hyperbolic development[7] of such initial data cannot be

[6]The case of S^3 is often considered in the literature. Non-compact, "asymptotically de Sitter" hypersurfaces can also be examined. See also Fig. 2.3 for an overview on potential global structures.

[7]Which is, roughly speaking, the largest Lorentzian manifold determined by the evolution of Cauchy data via the Einstein equations. See [92] for a precise definition.

locally extended as a Lorentzian manifold endowed with a continuous metric.

Conjecture 2.2 (H^1_{loc} formulation of the SCCC for the Einstein-Maxwell-charged-Klein-Gordon system with $\Lambda > 0$, under spherical symmetry)
Let Σ be a regular, compact (without boundary), spacelike hypersurface. Let us prescribe generic (in the spherically symmetric class) initial data on Σ, for the Einstein-Maxwell-charged-Klein-Gordon system with $\Lambda > 0$. Then, the future maximal globally hyperbolic development of such initial data cannot be locally extended as a Lorentzian manifold endowed with a continuous metric and Christoffel symbols in H^1_{loc}.

In particular, this work contributes to fill a gap in the literature by suggesting a negative resolution to conjecture 2.1 and, more significantly, a negative resolution, for a large subset of the parameter space, to conjecture 2.2. Such resolutions become definitive once the initial conditions of our system are shown to arise from suitable Cauchy data (although we do not deal with this problem in the current work, we explain in section 2.3 why we expect such results to hold in the black hole exterior).

Being a natural generalization of [22] (but notice that the coupled scalar field is now *charged* and possibly *massive*) and [79] (in our case, however, $\Lambda > 0$, thus the Price law is exponential rather than inverse polynomial), we take these works as starting points for our analysis. In section 2.4 we review previous relevant work and compare the techniques we employed with those of [22] and [79].

2.2 Summary of the main results

The main conclusions of our work can be outlined as follows. We first study the characteristic IVP for the first-order system related to (2.1) (this corresponds to equations (3.20)–(3.33)) and establish its local well-posedness, modulo diffeomorphism invariance (see theorem 4.2 and proposition 4.4). We then provide a novel proof of the extension criterion for the first-order system (theorem 5.1), demanding less regularity on the initial data as compared to the analogous result in [72, theorem 1.8] for the second-order system.

Subsequently, we show the equivalence between the first-order and the second-order Einstein-Maxwell-charged-Klein-Gordon systems (see remark 4.5) under additional regularity of the initial data, which we assume for the remaining part of our work. Finally, we assess the global uniqueness of the dynamical black hole under investigation, and its consequences for the SCCC, in the following classes of regularity.

Theorem (stability of the Cauchy horizon and existence of continuous extensions. See corollary 6.20, theorem 6.21)
Let Q_+, ϖ_+ and Λ be, respectively, the charge, mass and cosmological constant associated to a fixed sub-extremal Reissner-Nordström-de Sitter black hole. Let us consider, with respect to the first-order system (3.20)–(3.33)*, the future maximal globally hyperbolic development $(\mathcal{M}, g, F, \phi)$ of spherically symmetric initial data prescribed on two transversal null hypersurfaces, where $q \in \mathbb{R} \setminus \{0\}$ is the charge of the scalar field ϕ and $m \geq 0$ its mass.*

In particular, given $\mathcal{Q} = \mathcal{M}/SO(3)$*, let* (u, v) *be a system of null coordinates on* $\mathcal{Q}$*, determined*[8] *by* (3.44) *and* (3.45) *and such that*

$$g = -\Omega^2(u, v)dudv + r^2(u, v)\sigma_{S^2}, \tag{2.2}$$

where σ_{S^2} *is the standard metric on the unit round sphere. For some* $U > 0$ *and* $v_0 > 0$*, we express the two initial hypersurfaces in the above null coordinate system as* $[0, U] \times \{v_0\} \cup \{0\} \times [v_0, +\infty)$*.*

*Assume that the initial data to the characteristic IVP satisfy assumptions **(A)–(G)**, which require, in particular, that the initial data asymptotically approach those of the afore-mentioned sub-extremal black-hole, and that an exponential Price law upper bound holds: there exists* $s > 0$ *and* $C > 0$ *such that*

$$|\phi|(0, v) + |\partial_v \phi|(0, v) \leq Ce^{-sv}. \tag{2.3}$$

Then, if U *is sufficiently small,*[9] *there exists a unique solution to this characteristic IVP, defined on* $[0, U] \times [v_0, +\infty)$*, given by the metric in* (2.2) *and such that*

$$u \mapsto \lim_{v \to +\infty} r(u, v) \geq \mathfrak{r} > 0$$

is a continuous function from $[0, U]$ *to* $[\mathfrak{r}, r_-]$*, for some* $\mathfrak{r} > 0$*, and*

$$\lim_{u \to 0} \lim_{v \to +\infty} r(u, v) = r_-,$$

where r_- *is the radius of the Cauchy horizon of the reference sub-extremal Reissner-Nordström-de Sitter black hole.*

Moreover, $(\mathcal{M}, g, F, \phi)$ *can be extended up to the Cauchy horizon* $\mathcal{CH}^+ = \{v = +\infty\}$ *with continuous metric and continuous scalar field* ϕ*.*

This **suggests a negative resolution to conjecture 2.1** (see also section 2.3), for every non-zero value of the scalar field charge and for every non-negative value of its mass. This resolution becomes definitive if (2.3) is established for solutions evolved from regular Cauchy data.

Theorem (no-mass-inflation scenario and H^1 extensions. See theorems 7.1 and 7.5)
Under the assumptions of the previous theorem, let s *be the constant in* (2.3) *and consider*

$$\rho \coloneqq \frac{K_-}{K_+} > 1,$$

where K_- *and* K_+ *are the absolute values of the surface gravities of the Cauchy horizon and of the event horizon, respectively, of the reference sub-extremal Reissner-Nordström-de Sitter black hole. We define*

[8] When the initial data of the characteristic IVP coincide with Reissner-Nordström-de Sitter initial data, the coordinate v is equal to the Eddington-Finkelstein coordinate $\frac{r^*+t}{2}$ used near the black hole event horizon. See appendix D for more details.

[9] Compared to the L^∞ norm of the initial data.

the renormalized Hawking mass

$$\varpi = \frac{r}{2}\big(1 - g(dr^\sharp, dr^\sharp)\big) + \frac{Q^2}{2r} - \frac{\Lambda}{6}r^3,$$

where Q is the charge function of the dynamical black hole and Λ is the cosmological constant. If:

$$s > K_- \quad \text{and} \quad \rho < 2, \tag{2.4}$$

then the geometric quantity ϖ is bounded, namely there exists $C > 0$, depending uniquely on the initial data, such that:

$$\|\varpi\|_{L^\infty([0,U]\times[v_0,+\infty))} = C < +\infty.$$

Moreover, for such values of s and ρ, there exists a coordinate system for which g can be extended continuously up to $\mathcal{CH}^+$, with Christoffel symbols in L^2_{loc} and ϕ in H^1_{loc}.

When the decay given by the exponential Price law upper bound is sufficiently fast and the parameters of the reference sub-extremal black hole lie in a certain range,[10] **the above result puts into question the validity of conjecture 2.2** (see also section 2.3).

As shown in Fig. 2.2, this theorem significantly extends the known range of parameters for which H^1 extensions can be obtained, when comparing to the case of a massless and uncharged scalar field [22], in a way which is expected to be sharp in the (s, ρ) parameter space [18, 60]. We stress that the range of parameters allowing for H^1 extensions is not expected to depend on the charge of the scalar field. On the other hand, the techniques of the present work are more general than those previously used in the case of an uncharged scalar field, when $\Lambda > 0$.

Notice that the first condition in (2.4) depends on the choice of the outgoing null coordinate v, see also appendix D.

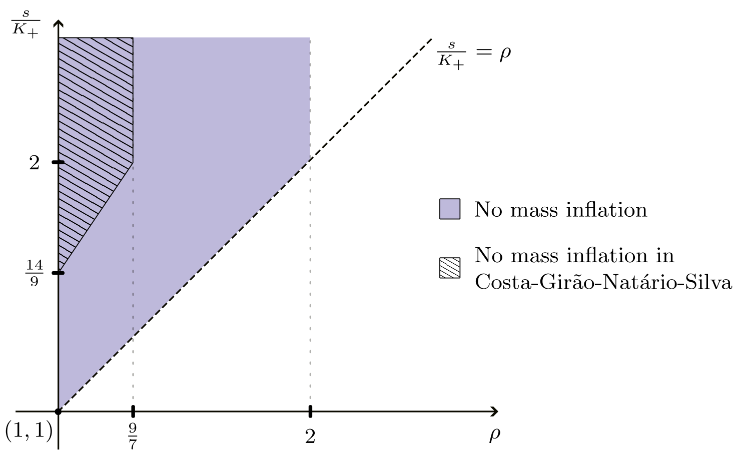


Figure 2.2: Values of s and $\rho = \frac{K_-}{K_+}$ that guarantee the absence of mass inflation for a generic set of initial data, in comparison to the results in [22]. In the extended range that we obtained, H^1_{loc} extensions of the solutions to the IVP are constructed. This extended range includes near-extremal black holes, for which ρ is close to 1. The value of s is expected to depend in a non-trivial way on the parameters of the fixed sub-extremal black hole, see [9, 10, 58, 62, 11].

[10] This includes the case in which the reference black hole is close to being extremal.

2.3 Strong cosmic censorship in full generality

The results of our analysis show that the main conclusion of [22] is still valid when we add a charge (and possibly mass) to the coupled scalar field: **according to the state of the art of the SCCC, the potential scenarios that lead to a violation of determinism in the case $\Lambda > 0$ cannot be ruled out**.

To explain how to reach this conclusion, we first illustrate the connection between our results and conjecture 2.2, and then the relation between the latter and other formulations of the SCCC.

The results presented in section 2.2 stem from the evolution of initial data from two transversal *null* hypersurfaces, one of which modelling the event horizon of a dynamical black hole. This suggests that the black hole we take under consideration has already formed at the beginning of the evolution.[11]

Strictly speaking, however, the SCCC constrains the admissible future developments of *spacelike* hypersurfaces, on which *generic*[12] initial data are specified. Although **we are not committed to any specific global structure**, it is useful to think of the characteristic IVP of section 3.1 as arising from suitable (compact or non-compact) initial data prescribed in the black hole exterior (see also Fig. 2.3). The reason why we believe that our results are relevant to the SCCC is that the initial data we prescribe for our characteristic IVP are compatible with numerical solutions obtained in the exterior of Reissner-Nordström-de Sitter black holes (see the paragraph below). Such results are also expected to hold for perturbations of such black holes, due to the non-linear stability results [57, 61].

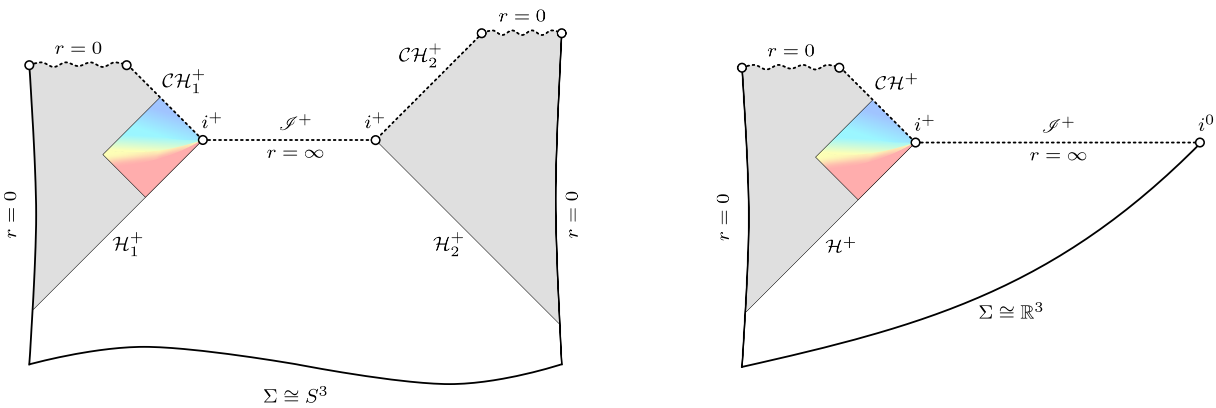


Figure 2.3: Two possible global spacetime structures in which the characteristic IVP of section 3.1 fits in. The coloured region is the one depicted in the forthcoming Fig. 2.4, namely the black hole interior region under inspection in the present work. These global structures require a dynamical formation of trapped surfaces, starting from non-trapped initial data (see also [23, 2]).

In particular, our v coordinate can be compared to half of the outgoing Eddington-Finkelstein coordinate of [10] (see appendix D). This can be used to see that, according to the same article, condition (2.4) can be (numerically) attained at the linear level: $s > K_-$ (i.e. $\beta = \frac{s}{2K_-} > \frac{1}{2}$, in the language of [10]) for linear waves propagating in the exterior of near-extremal (i.e. ρ close to 1) Reissner-Nordström-de Sitter

[11]Notice that the interior and exterior problems can be treated separately, due to domain of dependence arguments, and the exterior region in this context was proved to be stable in [57], with respect to gravitational and electromagnetic perturbations.

[12]By "generic" we mean open and dense in the moduli space of initial data, in a suitable topology (see, e.g. [91] and [73]). This can also be interpreted in terms of a "finite co-dimension condition" (a related discussion can be found in [73] for the non-linear case and in [34] in a linear setting). Genericity of the set of black hole parameters plays a prominent role in [66], for a negative cosmological constant.

black holes. This result is also expected to hold on dynamical black holes sufficiently close to Reissner-Nordström-de Sitter, as the one we analyse. Indeed, notice that on a fixed background s is equal, up to a multiplying factor, to the spectral gap of the Laplace-Beltrami operator $\Box_g$. If we denote by QNM the set of the quasinormal modes of the reference black hole, the spectral gap can also be expressed by

$$\inf_{\substack{\omega \in \mathrm{QNM} \\ \mathrm{Im}(\omega)<0}} (-\mathrm{Im}(\omega)).$$

In particular, s depends on the black hole parameters. At the same time, [57] shows that, under small perturbations, the parameters of the perturbed solution are sufficiently close to the initial ones.

Therefore, once the numerical results of [10] are verified rigorously and extended to the non-linear case, a mathematical proof for the validity of (2.4) would in principle be possible, leading to a negative resolution of conjecture 2.2.

On the other hand, conjectures 2.1 and 2.2 regard initial data which are generic in the spherically symmetric class, but *non-generic* in the much larger *moduli space* of all possible initial configurations. In particular, when we talk about the *SCCC under spherical symmetry*, the validity or failure of such a conjecture does not automatically settle the non-spherically symmetric problem. Nonetheless, due to analogies with the conformal structure of a Kerr-Newman spacetime, it is widely believed [25, 27, 22, 30] (and there is substantial numerical evidence in the case $\Lambda > 0$ [11, 35]) that results obtained in the charged spherically symmetric context can provide vital evidence to either uphold or refute formulations of the SCCC in absence of spherical symmetry.[13]

A larger moduli space can also be achieved by working in a rougher class of initial data. In [34], where global uniqueness is investigated at the linear level, the following is proved: given initial data $(\psi_0, \psi_0') \in H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}$ to the linear wave equation $\Box_g \psi = 0$ on a sub-extremal Reissner-Nordström-de Sitter black hole, with data prescribed on a complete spacelike hypersurface, ψ cannot be extended across the Cauchy horizon in H^1_{loc} (see also the numerical work [37]). This means that, if the numerical results [9] obtained in the exterior of Reissner-Nordström-de Sitter black holes were to be proved rigorously, the initial data leading to a potential failure of the SCCC could be ignored, because they are non-generic in this larger set of rough initial configurations. Whether this Sobolev level of regularity for the initial data is "suitable" may well depend on the problem to be studied and is still a topic of discussion in the literature. We also refer the reader to [33, 76] to compare with instability results in the $\Lambda = 0$ case.

2.4 Previous works and outline of the bootstrap method

The case $\Lambda = 0$: after foundational work by Christodoulou [13], Dafermos [25] first laid the framework for the analysis of the SCCC in spherical symmetry via modern PDE methods. In [27], the C^0 version of the SCCC was proved to be false for the spherically symmetric Einstein-Maxwell-(real) scalar field system, conditionally on the validity of a Price law upper bound, which was later seen to hold in [31].

[13] The case of uncharged black holes, however, needs to be inspected separately. See the numerical study [36] for Kerr-de Sitter. In particular, the H^1 formulation of the SCCC is expected to be violated in Kerr-Newman-de Sitter black holes for parameters sufficiently close to those of a near-extremal Reissner-Nordström-de Sitter solution [35].

The work [73] proved the validity of the C^2 formulation of the SCCC for the same spherically symmetric system.

The first steps towards the analysis of a complex-valued (i.e. *charged*) and possibly massive scalar field were taken in [72]: this is the model that we denoted as the Einstein-Maxwell-charged-Klein-Gordon system. The work [72], in particular, presented a soft analysis of the future boundary of spacetimes undergoing spherical collapse. More recent work on the SCCC for the same system can be found in [79, 81, 80, 67], where continuous spacetime extensions were constructed and several instability results obtained.

The analysis of the non-linear problem without symmetry assumptions and in vacuum, namely the validity of the SCCC for a Kerr spacetime, was initiated in the seminal work [30]. Here, the C^0 formulation of the SCCC was shown to be false, under the assumption of the quantitative stability of the Kerr exterior. The problem is indeed intertwined with stability results of black hole exteriors, object of an intense activity that built on a sequence of remarkable results and notably led, in recent times, to [52, 69, 70] for slowly rotating Kerr solutions. Further celebrated results are the non-linear stability of the Schwarzschild family [29]. For an overview on the problem and recent contributions both in the linear and non-linear realm, we refer the reader to [28, 3, 95, 96, 51, 8] and references therein. See also [6, 5] for results in the extremal case.

The case $\Lambda > 0$: Differently from the asymptotically flat case, Reissner-Nordström-de Sitter black holes present a spacelike future null infinity and an additional Killing horizon $\mathcal{C}^+$ (the cosmological horizon). As a result, scalar perturbations of such black holes decay *exponentially* fast along $\mathcal{H}^+$ [40, 75, 44]. This behaviour competes with the blueshift effect expected near the Cauchy horizon and determining a growth, also exponential, of the main perturbed quantities. While the H^1 version of the SCCC is expected to prevail in the asymptotically flat case (the exponential contribution from blueshift wins over the inverse polynomial tails propagating from the event horizon), the situation drastically changes when $\Lambda > 0$. In the latter case, indeed, knowing the exact asymptotic rates of scalar perturbations along the event horizon, and thus determining which of the two exponential contributions is the leading one, is of vital importance to determine the admissibility of spacetime extensions.

So far, a quantitative description of the exponents for such a decay is available at the linear level via numerical methods. In particular, it has been shown numerically that the H^1 norm (i.e. the energy) of linear waves propagating in sub-extremal Reissner-Nordström-de Sitter black holes is bounded up to the Cauchy horizon, provided that such black hole backgrounds are close to being extremal [9, 10, 78] (see also the rigorous results [58] and [62] in the small mass limit). We refer to [11, 35] for the analogous stability result for scalar perturbations of Kerr-Newman-de Sitter. A different outcome was found in the Kerr-de Sitter case [36], where numerical results showed that the H^1_{loc} version of the SCCC is respected in the linear setting.

On the other hand, the SCCC with $\Lambda > 0$ has been rigorously studied via the Einstein-Maxwell-(real) scalar field model in [19, 20, 21] (compare with [25]). One of the main novelties of this series of papers is a partition of the black hole interior in terms of level sets of the radius function, rather than curves of constant shifts as in [25, 27] (the latter curves fail to be spacelike when $\Lambda \neq 0$). These three works deal with a real scalar field, taken identically zero along the event horizon. Conversely, an exponential Price law (both a lower bound and an upper bound) was prescribed in [22] along the event horizon (compare with [27]), and generic initial data leading to C^0 extensions were constructed. Conditions to have either H^1_{loc} extensions or mass inflation were given in terms of the decay of the scalar field along the event horizon.

The work that we present in the next sections is a natural prosecution of [22] as it replaces the neutral scalar field with a charged one, whose behaviour along the event horizon is dictated by an exponential Price law upper bound. We extend, in a way which is expected to be optimal, the set of parameters s and ρ that allow for H^1_{loc} extensions.

Again, the problem is related with stability results of black hole exteriors: see [57] for the non-linear stability of slowly rotating Kerr-Newman-de Sitter black holes.

The case $\boldsymbol{\Lambda < 0}$: recent progress with the investigation of the SCCC in the case $\Lambda < 0$ can be found, for instance, in [63, 64, 66, 65].

Comparison with [79] and [22]: the present work generalizes the results of [22] to the case of a charged and (possibly) massive scalar field, without requiring any lower bound[14] for it along the initial outgoing characteristic hypersurface. Differently from the neutral case, a charged scalar field is compatible with a more realistic description of gravitational collapse (see Fig. 2.3 for potential global structures). At the same time, the charged setting requires gauge covariant derivatives, brings additional terms in the equations and generally undermines monotonicity properties due to the fact that the scalar field is now complex-valued. With respect to the real case, additional dynamical quantities, such as the electromagnetic potential A_u and the charge function Q, need to be controlled during the evolution. Moreover, the presence of a mass term for the scalar field allows to describe compact objects formed by massive spin-0 bosons.

In particular, we replace many monotonicity arguments exploited in [22] with bootstrap methods. In this sense, our work follows the spirit of [79], where the stability of the Cauchy horizon was proved by bootstrapping the main estimates from the event horizon. In our case, however, we need to propagate exponential terms of the form $e^{-c(s)v}$, with $c(s) > 0$ and v being an Eddington-Finkelstein type of coordinate.

We now summarize the main technical strategies that we adopt in our work (see also Fig. 2.4 for the partition of the black hole interior that we adopt) and how they differ from the techniques of [22] and [79]. Here, the quantities r_- and r_+ represent the radii of the Cauchy horizon and of the event horizon, respectively, of the fixed sub-extremal Reissner-Nordström-de Sitter black hole. We recall that in section 2.2 we presented the coordinates $u = u_{\text{Kru}}$, $v = v_{\text{EF}}$ and the function Ω^2 (see (2.2)). The functions ϖ and μ are defined in (3.15) and (3.16), respectively.

- **Redshift effect**, sections 6.1, 6.3 and 6.5: Along the event horizon, we assume an exponential Price law upper bound e^{-sv}, for a fixed $s \in (0, +\infty)$, for the scalar field. Near the event horizon, the redshift effect competes with the exponential Price law. This is translated into the presence of the

[14] An (integrated or pointwise) lower bound for the scalar field is generally expected in order to prove instability estimates, such as mass inflation. We will not pursue this direction in our study, even though the estimates we obtain in the black hole interior pave the way to a future analysis of instability results.

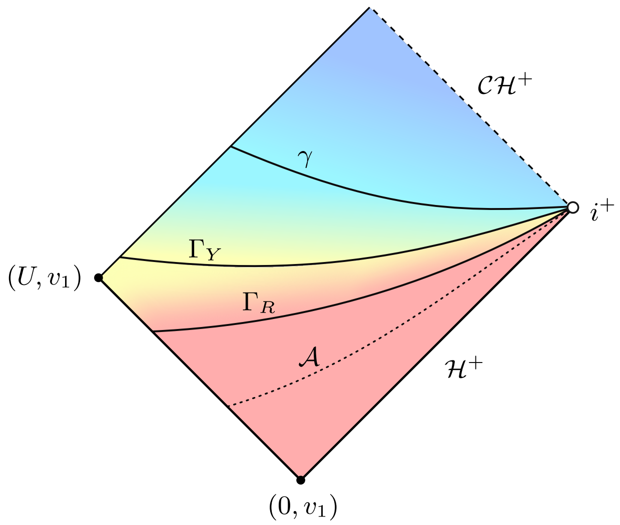


Figure 2.4: A posteriori, these are the spacelike curves that partition the interior of the dynamical black hole for $v_1 \geq v_0$ large. Compare with Fig. 3 in [79] and with the figures in [22, section 4]. For the construction, we consider r_- and r_+, with $0 < r_- < r_+$, respectively the radius of the Cauchy horizon and the radius of the event horizon of the reference sub-extremal black hole. We fix $R < r_+$, close to r_+, and $Y > r_-$ close to r_-. The region to the past of $\Gamma_R = \{r = R\}$ is the redshift region (the curve $\mathcal{A}$ is the apparent horizon). Between Γ_R and $\Gamma_Y = \{r = Y\}$ is the no-shift region. Bounded between Γ_Y and γ (which is not a level set of r) is the early blueshift region. Finally, to the future of γ we have the late blueshift region.

slower[15] decay rate $e^{-c(s)v}$, rather than e^{-sv}. The monotonicity results and Grönwall inequalities adopted in the real-scalar-field case are replaced by a bootstrap argument. The latter is analogous to the procedure in [79], where however in our case we need to take care of the different decay rates in function of s, due to the exponential nature of the estimates.

- **Blueshift effect**, sections 6.5, 6.6, 6.7, 6.8: Near the Cauchy horizon (whose existence is proved at the end of section 6.8) an exponentially growing contribution, arising from blueshift, affects the main estimates. However, this effect is relevant only in a relatively small region in which we can still propagate the previously obtained decay rates, up to a small error in the exponent of the main terms. In [22], this was dealt with by using the monotonicity properties of ϖ (for instance to bound $1 - \mu$ away from zero in the no-shift region and to show the integrability of $\partial_u r$ and $\partial_v r$ near the Cauchy horizon). In our case, however, these monotonicity properties are lacking. In their place, we exploit bootstrap arguments.

  The bootstrap procedure has to differ from that of [79]: there, the estimate $u_{\mathrm{EF}} \sim |\log(2K_+ u_{\mathrm{Kru}})| \sim v_{\mathrm{EF}}$ (see section 3.3 for an overview of the different coordinate systems) is used multiple times. The constant in this estimate leads to a multiplying factor in front of terms of the form v^{-s}. These factors can be dealt with using the smallness parameters of the bootstrap procedure. On the other hand, in our case, such estimates on the null coordinates lead to exponentially large error terms

[15] Here, $c(s)$ is bounded from above by a value depending on the surface gravity of the event horizon of the final black hole, see definition (6.27).

that cannot be removed. To solve this problem, we

$$\text{replace} \quad e^{-c(s)v} \quad \text{with} \quad e^{-\mathcal{C}_s(u,v)} \sim e^{-(c(s)-\tau)v}$$

in the main estimates, for a suitable positive[16] function $\mathcal{C}_s$ and for $\tau > 0$. In particular, the bootstrap estimates need to contain a function that depends both on the Kruskal-type coordinate u and on the Eddington-Finkelstein type of coordinate v.[17]

A key role in the last region is played by the term $\partial_v \log \Omega^2$ (see also the importance of this term in [79]) to replace the monotonicity properties used in [22] and to obtain a finer control on the H^1 norm of the scalar field. Indeed, the product $u\Omega^2$ measures, in some sense, the redshift and blueshift effects.

- ***C^0* and *H^1* extensions**, sections 6.9, 7.1, 7.2: once we propagate the exponential decay rates from $\mathcal{H}^+$, the construction of a C^0 extension of the spacetime metric mainly follows from the results in [27, 22, 73, 79].

  The H^1 extension that we construct is related to the boundedness of the renormalized Hawking mass ϖ (see also [22]). In the expression (7.3) for ϖ (here $\theta = r\partial_v\phi$ and $\lambda = \partial_v r$), all quantities are bounded except possibly for the integral of

$$\frac{|\partial_v \phi|^2}{\partial_v r}. \tag{2.5}$$

  We prove that, for a sufficiently fast decay rate of the scalar field along $\mathcal{H}^+$, i.e. for s sufficiently large, and if the asymptotic black hole is sufficiently near-extremal, $|\partial_v r|$ admits a lower bound that makes (2.5) integrable in v. Thus, ϖ is bounded. The integrability of (2.5) is then used to prove that the Christoffel symbols are locally square integrable, therefore leading to a H^1_{loc} extension of the metric along $\mathcal{CH}^+$.

The **outline of the remaining part of the thesis** is the following: in chapter 3 we delve into the main aspects of the characteristic IVP for the Einstein-Maxwell-charged-Klein-Gordon system under spherical symmetry. Well-posedness of the characteristic IVP is investigated in chapter 4. An extension criterion is given in chapter 5.

In chapter 6 we establish quantitative bounds along $\mathcal{H}^+$ for the main PDE variables of the characteristic IVP, and propagate these bounds to the black hole interior by bootstrap. This allows us to prove the stability of $\mathcal{CH}^+$ and to construct a continuous spacetime extension at the end of the same section.

In chapter 7 we provide a sharp condition on the reference sub-extremal black hole to generically prevent the occurrence of mass inflation. Under such an assumption, we also construct an H^1_{loc} spacetime extension up to and including the Cauchy horizon.

[16] Strictly speaking, $\mathcal{C}_s$ is positive only in a specific region of the black hole interior that has the curve γ as future boundary, see section 6.6. This does not constitute a problem since all estimates to the future of this region can be expressed in terms of $\mathcal{C}_{s|\gamma}$.

[17] We follow the convention of [19, 20, 21, 22], where the null coordinates $(u, v) = (u_{\text{Kru}}, v_{EF})$ are used in the entire black hole interior. See also section 3.3 for a comparison with other conventions.

Chapter 3

The Einstein-Maxwell-charged-Klein-Gordon system under spherical symmetry

The present work revolves around the system of equations describing a spherically symmetric Einstein-Maxwell-charged-Klein-Gordon model with positive cosmological constant Λ:

$$\begin{cases} \mathrm{Ric}_{\mu\nu}(g) - \frac{1}{2}\mathrm{R}(g)g_{\mu\nu} + \Lambda g_{\mu\nu} = 2\big(T^{\mathrm{EM}}_{\mu\nu} + T^{\phi}_{\mu\nu}\big), \\ T^{\mathrm{EM}}_{\mu\nu} = g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}, \\ T^{\phi}_{\mu\nu} = \mathrm{Re}\big(D_\mu\phi\overline{D_\nu\phi}\big) - \frac{1}{2}g_{\mu\nu}\big(D_\alpha\phi\overline{D^\alpha\phi} + m^2|\phi|^2\big), \\ dF = 0, \quad d\star F = \star J, \\ \big(D_\mu D^\mu - m^2\big)\phi = 0. \end{cases}$$

The above system describes a spherically symmetric spacetime $(\mathcal{M}, g)$ interacting with a scalar field ϕ, the latter being endowed with charge $q \in \mathbb{R} \setminus \{0\}$ and mass $m \geq 0$. The units are chosen so that $c = 4\pi G = \epsilon_0 = 1$ and the symbol $\star$ denotes the Hodge-star operator. The charge generates an electromagnetic field which interacts with the spacetime itself and is modelled by the strength field tensor F. The description of the charged scalar field through a complex-valued function and the use of the gauge covariant derivative

$$D_\mu = \nabla_\mu + iqA_\mu \tag{3.1}$$

are standard techniques in the gauge-theoretical framework (see [72] for more details) and in Lagrangian formalism. The current

$$J_\mu = -\frac{iq}{2}\big(\phi\overline{D_\mu\phi} - \overline{\phi}D_\mu\phi\big) \tag{3.2}$$

appearing in Maxwell's equations, in particular, can be seen as the Noether current of the (classical) Lagrangian density $\mathcal{L} = |D_\mu\phi|^2 - m^2|\phi|^2$.

We express the isometric action of $SO(3)$ on our spacetime manifold by assuming

$$g = g_{\mathcal{Q}} + r^2(u,v)\sigma_{S^2}, \tag{3.3}$$

with

$$g_{\mathcal{Q}} \coloneqq -\Omega^2(u,v)dudv,$$

for some real-valued function Ω, where $\sigma_{S^2} = d\theta^2 + \sin^2(\theta)d\varphi^2$ is the standard metric on the unit round 2-sphere, r is the area-radius function and u, v are null coordinates in the $(1+1)$–dimensional manifold $\mathcal{Q} \coloneqq \mathcal{M}/SO(3)$. In particular, we can conformally embed $(\mathcal{Q}, g_{\mathcal{Q}})$ in $(\mathbb{R}^2, \eta)$, where η is the Minkowski metric, and we can restrict ourselves to working with this subset of the two-dimensional space since conformal transformations preserve the causal structure. Moreover, we will make use of the $U(1)$ gauge freedom to assume that ϕ and A (see (3.1)) do not depend on the angular coordinates. We will also require that $\partial_u = \frac{\partial}{\partial u}$ is a future-pointing, ingoing[1] vector field and ∂_v is a future-pointing, outgoing one.

We use the same symbol to denote a spherically symmetric function defined on $\mathcal{M}$, and its push-forward through the projection map $\pi\colon \mathcal{M} \to \mathcal{Q}$.

In order to analyse the behaviour of our dynamical system, we are first naturally led to investigate the maximal globally hyperbolic development (**MGHD**) generated by (spherically symmetric) initial data prescribed on a suitable spacelike hypersurface (see also section 2.3 for possible global structures). After this analysis, the null structure of the equations motivates us to treat the characteristic initial value problem on $\mathcal{Q}$, using the (u, v) coordinate system.

With respect to the afore-mentioned (u, v) null coordinates, we are going to work in the set $[0, U] \times [0, +\infty)$, for some $U > 0$. Therefore, there exists a 1-form A defined on the MGHD of the initial data set, such that $F = dA = F_{uv}(u,v)du \wedge dv$ for some real-valued function F_{uv}. In particular, we can define the real-valued function $Q(u, v)$ such that

$$F = \frac{Q\Omega^2}{2r^2} du \wedge dv. \tag{3.4}$$

Since we can choose the electromagnetic potential A up to an exact form by performing a gauge choice, we can set $A = A_u du$. Thus, the gauge covariant derivative D_v is just ∇_v (see (3.1)) and

$$\partial_v A_u = -\frac{Q\Omega^2}{2r^2}. \tag{3.5}$$

The above expression defines A_u up to a function of u. So, without loss of generality, we will assume that A_u vanishes on the initial ingoing hypersurface. Namely, for $U > 0$ and $v_0 \geq 0$ fixed, we take $A_u(u, v_0) = 0$ for every $u \in [0, U]$.

[1]In the language of [72], a vector field defined on the quotient spacetime manifold is **ingoing** if it points towards the centre of symmetry, i.e. the projection on $\mathcal{M}/SO(3)$ of the set of fixed points of the $SO(3)$ action. See also [72, proposition 2.1] for a related analysis of topologies for the initial data, in the $\Lambda = 0$ case.

3.1 The second-order and first-order systems: the characteristic initial value problem

Using the results of appendix C, we can write down the Einstein equations in the following form:

$$\begin{cases} \partial_u\left(\frac{\partial_u r}{\Omega^2}\right) = -r\frac{|D_u\phi|^2}{\Omega^2}, & \text{(Raychaudhuri equation along the } u \text{ direction)}, & (3.6)\\ \partial_v\left(\frac{\partial_v r}{\Omega^2}\right) = -r\frac{|\partial_v\phi|^2}{\Omega^2}, & \text{(Raychaudhuri equation along the } v \text{ direction)}, & (3.7)\\ \partial_u\partial_v r = \frac{\Omega^2}{4r^3}Q^2 + \frac{\Omega^2 r}{4}\Lambda - \frac{\Omega^2}{4r} - \frac{\partial_v r\partial_u r}{r} + \frac{m^2\Omega^2}{4}r|\phi|^2, & \text{(wave eqn. for } r), & (3.8)\\ \partial_u\partial_v \log\Omega^2 = -2\,\mathrm{Re}\left(D_u\phi\overline{\partial_v\phi}\right) - \frac{\Omega^2Q^2}{r^4} + \frac{\Omega^2}{2r^2} + \frac{2\partial_v r\partial_u r}{r^2}, & \text{(wave eqn. for} \log\Omega^2), & (3.9)\end{cases}$$

Moreover, the equation of motion $\left(D_\mu D^\mu - m^2\right)\phi = 0$ becomes

$$\partial_u\partial_v\phi = -\frac{1}{r}(D_u r\partial_v\phi + \partial_v r D_u\phi) + \frac{iqQ\Omega^2}{4r^2}\phi - \frac{m^2\Omega^2}{4}\phi, \quad \text{(Klein-Gordon equation)} \tag{3.10}$$

where we used that

$$\nabla_\mu\nabla^\mu\phi = -\frac{4}{r\Omega^2}(\partial_u r\partial_v\phi + \partial_v r\partial_u\phi + r\partial_u\partial_v\phi),$$

and

$$\nabla_\mu A^\mu = -\frac{2}{\Omega^2}\partial_v A_u - \frac{4}{\Omega^2 r}A_u\partial_v r = \frac{Q}{r^2} - \frac{4}{\Omega^2 r}A_u\partial_v r.$$

Furthermore, the second Maxwell equation $d\star F = \star J$ implies

$$\begin{cases} \partial_u Q = -qr^2\,\mathrm{Im}\left(\phi\overline{D_u\phi}\right), & (3.11)\\ \partial_v Q = qr^2\,\mathrm{Im}\left(\phi\overline{\partial_v\phi}\right). & (3.12)\end{cases}$$

In order to study the well-posedness of our system of PDEs, it is useful to rewrite the above equations as a first-order system. To do this, we define the following quantities:

$$\nu := \partial_u r, \tag{3.13}$$

$$\lambda := \partial_v r, \tag{3.14}$$

which were first introduced by Christodoulou, and

$$\varpi := \frac{Q^2}{2r} + \frac{r}{2} - \frac{\Lambda}{6}r^3 + \frac{2r}{\Omega^2}\nu\lambda, \tag{3.15}$$

$$\mu := \frac{2\varpi}{r} - \frac{Q^2}{r^2} + \frac{\Lambda}{3}r^2, \tag{3.16}$$

$$\theta := r\partial_v\phi, \tag{3.17}$$

$$\zeta := rD_u\phi, \tag{3.18}$$

$$\kappa := -\frac{\Omega^2}{4\nu}, \tag{3.19}$$

whenever r, Ω^2 and ν are non-zero, an assumption that we will make on the initial hypersurfaces of our PDE system. All the newly defined quantities are real-valued, except for θ and ζ, since the scalar field ϕ is complex-valued. From the above we can obtain the useful relation[2]

$$\lambda = \kappa(1-\mu).$$

We also notice that ϖ is a geometric quantity (often called the **renormalized Hawking mass**), due to the fact that $1-\mu = g(dr^\sharp, dr^\sharp)$. Using the new definitions, some lines of computations show that the Einstein, Klein-Gordon and Maxwell equations imply the following first-order system:[3]

$$\partial_u r = \nu, \tag{3.20}$$

$$\partial_v r = \lambda, \tag{3.21}$$

$$D_u\phi = (\partial_u + iqA_u)\phi = \frac{\zeta}{r}, \tag{3.22}$$

$$D_v\phi = \partial_v\phi = \frac{\theta}{r}, \tag{3.23}$$

$$\partial_v\nu = \partial_u\lambda = -2\nu\kappa\frac{1}{r^2}\left(\frac{Q^2}{r} + \frac{m^2r^3|\phi|^2}{2} + \frac{\Lambda}{3}r^3 - \varpi\right), \tag{3.24}$$

$$\partial_u\varpi = \frac{\nu}{2}m^2r^2|\phi|^2 - Qq\,\mathrm{Im}\left(\phi\bar{\zeta}\right) + \frac{1}{2}\left(\frac{|\zeta|}{\nu}\right)^2\nu\left(1 - \frac{2\varpi}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right), \tag{3.25}$$

$$\partial_v\varpi = \frac{\lambda}{2}m^2r^2|\phi|^2 + Qq\,\mathrm{Im}\left(\phi\bar{\theta}\right) + \frac{|\theta|^2}{2\kappa}, \tag{3.26}$$

$$D_u\theta = (\partial_u + iqA_u)\theta = -\frac{\zeta}{r}\lambda + \frac{\nu\kappa\phi}{r}\left(m^2r^2 - iqQ\right), \tag{3.27}$$

$$D_v\zeta = \partial_v\zeta = -\frac{\theta}{r}\nu + \frac{\nu\kappa\phi}{r}\left(m^2r^2 + iqQ\right), \tag{3.28}$$

$$\partial_u\kappa = \frac{\kappa\nu}{r}\left(\frac{|\zeta|}{\nu}\right)^2, \tag{3.29}$$

$$\partial_u Q = -qr\,\mathrm{Im}(\phi\bar{\zeta}), \tag{3.30}$$

$$\partial_v Q = qr\,\mathrm{Im}(\phi\bar{\theta}), \tag{3.31}$$

$$\partial_v A_u = 2\frac{Q\nu\kappa}{r^2}, \tag{3.32}$$

with the algebraic constraint

$$\lambda = \kappa(1-\mu). \tag{3.33}$$

It is also convenient to define the quantity

$$K \coloneqq \frac{1}{2}\partial_r(1-\mu)(r,\varpi,Q) = \frac{\varpi}{r^2} - \frac{Q^2}{r^3} - \frac{\Lambda}{3}r, \tag{3.34}$$

[2]The variable κ was originally introduced in order to avoid problematic terms showing up in the case $\lambda = 0$.

[3]Strictly speaking, the equality $\partial_v\nu = \partial_u\lambda$ and the equation for $\partial_v\zeta$ require an additional step to be obtained. Indeed, they can be seen to hold for continuous solutions such that all partial derivatives present in the PDE system are continuous, after proving that $\partial_u\partial_v r = \partial_v\partial_u r$ (which follows from the second equality in (3.24)) and that $\partial_u\partial_v\phi = \partial_v\partial_u\phi$ (which follows from (3.22)). See also the proof of theorem 4.2.

which, as we will see, is a generalization of the surface gravity of the Killing horizons of a Reissner-Nordström black hole with cosmological constant Λ. The above quantities then satisfy

$$\partial_v \nu = \partial_u \lambda = \nu\kappa\big(2K - m^2 r|\phi|^2\big), \tag{3.35}$$

by (3.24). By (3.15) and (3.19), we also have:

$$\kappa\varpi = \kappa\left(\frac{Q^2}{2r} + \frac{r}{2} - \frac{\Lambda}{6}r^3\right) - \frac{r}{2}\lambda. \tag{3.36}$$

Therefore, we can recast (3.24) as:

$$\partial_u(r\lambda) = \partial_v(r\nu) = -\nu\kappa\left(\frac{Q^2}{r^2} + m^2 r^2|\phi|^2 - 1 + \Lambda r^2\right). \tag{3.37}$$

Using (3.19), (3.22), (3.23) and (3.33), the equations of the first-order and second-order systems can be expressed in several forms. For our purposes, it will be useful to write the equation for $\partial_u \varpi$ as

$$\partial_u \varpi = \frac{\nu}{2} m^2 r^2 |\phi|^2 - Qqr\,\mathrm{Im}\,(\phi\overline{D_u\phi}) - \frac{2\lambda}{\Omega^2} r^2 |D_u \phi|^2, \tag{3.38}$$

and the Klein-Gordon equation (3.10) as one of the following expressions:

$$D_u(r\partial_v\phi) = -\lambda D_u\phi + m^2\nu\kappa r\phi - i\frac{qQ\nu\kappa}{r}\phi, \tag{3.39}$$

$$\partial_v(rD_u\phi) = -\nu\partial_v\phi + m^2\nu\kappa r\phi + i\frac{qQ\nu\kappa}{r}\phi. \tag{3.40}$$

As noticed in [79] (and the result still holds for non-zero values of Λ), the wave equation (3.9) for $\log\Omega^2$ can be cast as

$$\partial_u\partial_v \log\Omega^2 = \kappa\partial_u(2K) - 2\,\mathrm{Re}\big(D_u\phi\overline{\partial_v\phi}\big) - \frac{2\kappa}{r^2}\left(\partial_u\varpi - \frac{2Q\partial_u Q}{r}\right). \tag{3.41}$$

The proof of this result only requires some algebra and the definitions (3.34) and (3.15), in this exact order.

In the next sections, we are going to discuss the local well-posedness of a characteristic IVP for the first-order system (3.20)–(3.33). Solutions to such a system are elements (λ, ϖ, θ, κ, ϕ, Q, r, ν, ζ, A_u) in an appropriate Cartesian product of function spaces. The system is overdetermined, since we are solving 13 partial differential equations and an algebraic constraint for 10 unknowns. This is why we will solve for equations (3.21)-(3.22), (3.24)-(3.25), (3.27)-(3.30), (3.32) and we will consider equations (3.20), (3.23), (3.26), (3.31), (3.33) as constraints. In particular, for some $U > 0$, $v_0 \geq 0$, we will prescribe initial data on the set $([0,U] \times \{v_0\}) \cup (\{0\} \times [v_0, +\infty)) \subset [0,U] \times [0,+\infty)$, seen as a subset of the conformal embedding of $(\mathcal{Q}, g_{\mathcal{Q}})$ in $(\mathbb{R}^2, \eta)$. In fact, we will specify:

$$\begin{cases} r(u, v_0), \\ \zeta(u, v_0), \\ A_u(u, v_0), \end{cases} \qquad \text{for } u \in [0, U], \tag{3.42}$$

and

$$\begin{cases} \lambda(0, v), \\ \varpi(0, v), \\ \theta(0, v), \\ \kappa(0, v), \\ \phi(0, v), \\ Q(0, v), \end{cases} \qquad \text{for } v \in [v_0, +\infty). \tag{3.43}$$

The constraint equation (3.20) will be imposed on $[0, U] \times \{v_0\}$ so that the function $u \mapsto \nu(u, v_0)$ can be obtained from $u \mapsto r(u, v_0)$. In the following, we will denote the prescribed initial data by the zero subscript, e.g. $r_0(u) \coloneqq r(u, v_0)$ and $A_{u,0}(u) = A_u(u, v_0)$. The coordinates (u, v) have not been fixed yet: we only required them to be null and to stem from the above conformal embedding. We are going to fix a specific gauge in the following.

3.2 Assumptions

Let us make some general assumptions so that our spherically symmetric model describes a dynamical black hole asymptotically approaching (in some sense) a sub-extremal Reissner-Nordström solution with cosmological constant Λ (we refer to [24, 18, 20] for an overview of this spacetime). In this section, we fix the values v_0 and U so that $v_0 \geq 0$ and $0 < U < +\infty$.

A) Gauge-fixing for the null coordinates: exploiting the diffeomorphism invariance of the Einstein-Maxwell-charged-Klein-Gordon system, we fix the u coordinate by setting

$$\nu_0(u) = -1, \quad \forall\, u \in [0, U] \tag{3.44}$$

and we specify the v coordinate by choosing

$$\kappa_0(v) = 1, \quad \forall\, v \in [v_0, +\infty). \tag{3.45}$$

The negativity of ν_0 can be related to the absence of anti-trapped surfaces. We also require the positivity of the area-radius:

$$r_0(u) > 0 \quad \forall\, u \in [0, U] \quad \text{and} \quad r(0, v) > 0 \quad \forall\, v \in [v_0, +\infty). \tag{3.46}$$

Using the above gauge, we define

$$\mathcal{H}^+ := \{(u,v)\colon\ u=0, v \geq v_0\}$$

as the **event horizon** of the dynamical black hole.

B) Regularity assumptions: we suppose that

$$\nu_0, \zeta_0, \lambda_0, \theta_0, \kappa_0 \text{ are } C^0 \text{ functions in their respective domains,}$$

and that

$$r_0, \varpi_0, \phi_0, A_{u,0}, Q_0 \text{ are } C^1 \text{ functions in their respective domains.}$$

We emphasize that θ_0, ζ_0 and ϕ_0 are complex-valued functions. We will also make use of the function $v \mapsto r(0,v)$, whose regularity depends on the regularity of λ_0. These assumptions are enough to prove well-posedness of the first-order system. In order to demonstrate that such a system implies the Einstein-Maxwell equations, it is sufficient to additionally require that ν_0, λ_0 and κ_0 are C^1 functions in their respective domains (see also the forthcoming remark 4.5).

C) Compatibility conditions: we require the following constraints to be satisfied:

$$\begin{cases} r_0'(u) = \nu_0(u), \\ \phi_0'(v) = \frac{\theta_0(v)}{r(0,v)}, \\ \varpi_0'(v) = \frac{\lambda_0(v)}{2} m^2 r(0,v)^2 |\phi_0(v)|^2 + Q_0(v) q \operatorname{Im}(\phi_0(v)\overline{\theta_0}(v)) + \frac{|\theta_0(v)|^2}{2\kappa_0(v)}, \\ Q_0'(v) = q\, r(0,v) \operatorname{Im}(\phi_0(v)\overline{\theta_0}(v)) \\ \lambda_0(v) = \kappa_0(v)\Big(1 - \frac{2\varpi_0(v)}{r(0,v)} + \frac{Q_0(v)^2}{r(0,v)^2} - \frac{\Lambda}{3} r(0,v)^2\Big), \end{cases} \tag{3.47}$$

for every $u \in [0,U]$, every $v \in [v_0, +\infty)$.

D) Absence of anti-trapped surfaces: we require

$$\nu(0,v) < 0, \quad \forall\, v \in [v_0, +\infty). \tag{3.48}$$

Once we construct a solution in a set $\mathcal{D} = [0, U') \times [v_0, +\infty)$ for $0 < U' < +\infty$, the Raychaudhuri equation (3.6) implies that ν remains negative in $\mathcal{D}$.

Starting from chapter 6, we also assume the following.

E) Exponential Price law upper bound: we require that $\Lambda > 0$ and that, on the event horizon of the dynamical black hole, ϕ decays as

$$|\phi|(0,v) + |\partial_v \phi|(0,v) \leq C e^{-sv}, \quad \forall\, v \in [v_0, +\infty) \tag{3.49}$$

for some $C > 0$, $s > 0$. This assumption is one of the main consequences of the presence of a positive cosmological constant Λ. Here, the coordinate v is related to an Eddington-Finkelstein coordinate adopted near the event horizon of a Reissner-Nordström-de Sitter black hole (see appendix D).

F) Asymptotically approaching a sub-extremal black hole: consider a fixed sub-extremal Reissner-Nordström-de Sitter spacetime. Let r_+ and K_+ be, respectively, the radius and the surface gravity of its event horizon, and let Q_+ be its charge and ϖ_+ its mass. Notice that several constraints apply on these four values, due to the sub-extremality condition.[4] Then, on the event horizon, we require:[5]

$$\lim_{v\to+\infty} r(0,v) = r_+, \tag{3.50}$$

$$\lim_{v\to+\infty} Q(0,v) = Q_+, \tag{3.51}$$

$$\lim_{v\to+\infty} \varpi(0,v) = \varpi_+, \tag{3.52}$$

$$\lim_{v\to+\infty} K(0,v) = K_+. \tag{3.53}$$

G) Hawking's area theorem: we assume[6]

$$\lambda(0,v) > 0, \quad \forall\, v \geq v_0. \tag{3.54}$$

3.3 Notations and conventions

Given two non-negative functions f and g, we use the notation $f \lesssim g$ to denote the existence of a positive constant C such that $f \leq Cg$. The relation $f \gtrsim g$ is defined analogously, and we write $f \sim g$ whenever both $f \lesssim g$ and $f \gtrsim g$ hold. Different constants may be denoted by the same symbols if the value of such constant is not relevant in the computations. Whenever the symbols $\lesssim$ and $\gtrsim$ are used, we mean that the respective constants depend on the initial data only, except for the following cases:

- In the redshift region (section 6.3), constants depend on the initial data and possibly on the constant η (see the statement of proposition 6.7),
- In the no-shift region (section 6.5), early blueshift region (sections 6.6 and 6.7) and late blueshift region (section 6.8), constants depend on the initial data, and possibly on η, R and Y (see propositions 6.7 and 6.12).

[4]Given $K_+ := \frac{1}{2}\partial_r(1-\mu)(r_+,\varpi_+,Q_+)$, the values of ϖ_+, Q_+ and r_+ are constrained by the sub-extremality condition $K_+ > 0$ (see also [57, section 3]). Moreover, $(1-\mu)(r_+,\varpi_+,Q_+) = 0$. As in [20], in the course of our work we assume that $(1-\mu)(\cdot,\varpi_+,Q_+)$ admits the three distinct and positive roots r_-, r_+ and r_C, with $r_- < r_+ < r_C$, corresponding to the radii of the Cauchy horizon, event horizon and cosmological horizon, respectively, of a Reissner-Nordström-de Sitter black hole.

[5]More generally, we could assign a condition on the lim sup of r, Q, ϖ, K. This requires minimal changes in the proof of proposition 6.2 and yields the same results.

[6]The case $\lambda_{|\mathcal{H}^+} \equiv 0$ was studied in [20, 21], since Reissner-Nordström-de Sitter data were prescribed along $\mathcal{H}^+$. The case in which $\lambda(0,\cdot)$ is identically zero for large values of the v coordinate, but not for small ones, falls into that analysis after a coordinate shift. Moreover, due to our construction, the initial data along the event horizon are approaching the data of a Reissner-Nordström-de Sitter black hole in a non-trivial way. So, the apparent horizon does not coincide with the event horizon. By Hawking's area theorem [22] and lemma 6.9, then, we are left to examine the case $\lambda_{|\mathcal{H}^+} > 0$.

Coordinate systems: We follow the conventions of [19, 20, 21, 22], where the $(u, v) = (u_{\mathrm{Kru}}, v_{\mathrm{EF}})$ coordinate system (that we specified in section 3.2) is used in the entirety of the black hole interior. Notice that additional coordinate systems have been used in the literature in the case $\Lambda = 0$. For instance, in [79] and [73], our coordinate system is only used in the redshift region, whereas the Eddington-Finkelstein coordinate u_{EF}, defined as

$$u_{\mathrm{EF}} := \frac{1}{2K_+} \log(2K_+ u_{\mathrm{Kru}})$$

is used in the remaining region of the black hole interior. Moreover, the quantity $\nu = \nu_{\mathrm{Kru}}$ (see also $\Omega^2 = \Omega^2_{\mathrm{Kru}}$) that we use throughout the present work corresponds to the quantity ν_{H} (see also Ω^2_H) in [79]. It is therefore different from the quantity

$$\nu_{\mathrm{EF}} = 2K_+ u_{\mathrm{Kru}} \nu_{\mathrm{Kru}}$$

used in [79] to the future of the redshift region.

Chapter 4

Well-posedness of the initial value problem

Following [19], we are going to discuss local existence, uniqueness and continuous dependence with respect to the initial data for the solutions to the first-order system (3.20)-(3.33). In this chapter, we consider the constants $U \geq 0$ and $V \geq 0$, while we take $v_0 = 0$ for the sake of convenience. The initial data κ_0, ν_0 and $A_{u,0}$ are taken to be constant, according to assumption **(A)** and (3.5). The case of more general functions ν_0, κ_0 and $A_{u,0}$ follows in a straightforward way.

Definition 4.1 (solution to the PDE system)
We define a solution to the PDE system (3.20)–(3.32)*, with initial conditions* (3.42), (3.43) *and satisfying the regularity assumptions* ***(B)*** *and constraint* (3.33)*, to be a vector* $(\lambda,\ \varpi,\ \theta,\ \kappa,\ \phi,\ Q,\ r,\ \nu,\ \zeta,\ A_u)$ *of continuous functions defined on* $[0,U]\times[0,V]$*, such that all partial derivatives appearing in such a system are continuous.*

Theorem 4.2 (local existence and uniqueness)
Under the assumptions ***(A)****,* ***(B)****,* ***(C)*** *and* ***(D)****, let us prescribe initial data on the characteristic initial set* $[0,U]\times\{0\}\cup\{0\}\times[0,V]$ *for some* $0<U<+\infty$ *and* $0<V<+\infty$*. We define the quantity*

$$\begin{aligned} N_{i.d.} :=& (\|\lambda_0\| + \|\varpi_0\| + \|\theta_0\| + \|\kappa_0\| + \|\phi_0\| + \|Q_0\|)_{L^\infty([0,V])} \\ &+ (\|r_0\| + \|\nu_0\| + \|\zeta_0\| + \|A_{u,0}\|)_{L^\infty([0,U])}. \end{aligned}$$

Then, there exists a ***time of existence*** $0<\epsilon=\epsilon(N_{i.d.})\leq U$ *for which the characteristic IVP for the first-order system* (3.20)–(3.33) *admits a unique solution in* $\mathcal{D}_\epsilon := [0,\epsilon]\times[0,V]$*, the solution being such that the functions* r,ν *and* κ *are bounded away from zero in* $\mathcal{D}_\epsilon$*.*

Similarly, there exists a ***time of existence*** $0<\epsilon=\epsilon(N_{i.d.})\leq V$ *such that the characteristic IVP admits a unique solution in* $\mathcal{D}^\epsilon := [0,U]\times[0,\epsilon]$ *and such that the functions* r,ν *and* κ *are bounded away from zero in* $\mathcal{D}^\epsilon$*.*

Proof. Local existence can be proved via a fixed-point argument. Given V, where $0<V<+\infty$, let us choose $\epsilon > 0$ small enough, whose value will ultimately depend on $N_{i.d.}$. Let us then consider the metric space

$$
\begin{aligned}
S_\epsilon \coloneqq & B_{R_\lambda}(\lambda_0) \times B_{R_\varpi}(\varpi_0) \times B_{R_\theta}(\theta_0) \times B_{R_\kappa}(1) \times B_{R_\phi}(\phi_0) \times \\
& B_{R_Q}(Q_0) \times I(\alpha, \beta, r_0) \times B_{R_\nu}(-1) \times B_{R_\zeta}(\zeta_0) \times B_{R_A}(0),
\end{aligned}
$$

endowed with the L^∞ norm, where

$$
B_{R_\lambda}(\lambda_0) \coloneqq \{\lambda \in C^0(\mathcal{D}_\epsilon; \mathbb{R})\colon\ \|\lambda(u,v) - \lambda_0(v)\|_{L^\infty(\mathcal{D}_\epsilon)} \leq R_\lambda\}
$$

for $R_\lambda > 0$ and

$$
I(\alpha, \beta, r_0) \coloneqq \{r \in C^0(\mathcal{D}_\epsilon; \mathbb{R})\colon\ \alpha \leq r(u,v) - r_0(u) \leq \beta\},
$$

for some constants $\alpha, \beta \in \mathbb{R}$. Similar definitions hold for the remaining balls.[1]

We notice that, from equations (3.20)–(3.32), we formally have:

$$
\lambda(u,v) = \lambda_0(v) - \int_0^u \frac{2\nu\kappa}{r^2}\left(\frac{Q^2}{r} + \frac{m^2 r^3}{2}|\phi|^2 + \frac{\Lambda}{3}r^3 - \varpi\right)(u',v)du', \tag{4.1}
$$

$$
\begin{aligned}
\varpi(u,v) = {} & \varpi_0(v) e^{-\int_0^u \frac{|\zeta|^2}{r\nu}(u',v)du'} \\
& + \int_0^u e^{-\int_s^u \frac{|\zeta|^2}{r\nu}(u',v)du'} \left(\frac{|\zeta|^2}{2\nu}\left(1 + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right) - qQ\,\mathrm{Im}\big(\bar{\zeta}\phi\big) + \frac{\nu}{2}m^2 r^2 |\phi|^2\right)(s,v)ds,
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
\theta(u,v) = {} & \theta_0(v) e^{-iq\int_0^u A_u(u',v)du'} \\
& + \int_0^u e^{-iq\int_s^u A_u(u',v)du'}\left(-\frac{\zeta\lambda}{r} + \frac{\nu\kappa\phi}{r}\big(m^2 r^2 - iqQ\big)\right)(s,v)ds,
\end{aligned} \tag{4.3}
$$

$$
\kappa(u,v) = e^{\int_0^u \frac{|\zeta|^2}{r\nu}(u',v)du'}, \tag{4.4}
$$

$$
\phi(u,v) = \phi_0(v) e^{-iq\int_0^u A_u(u',v)du'} + \int_0^u \frac{\zeta}{r}(s,v) e^{-iq\int_s^u A_u(u',v)du'} ds, \tag{4.5}
$$

$$
Q(u,v) = Q_0(v) - q\int_0^u r(u',v)\,\mathrm{Im}(\phi\bar{\zeta})(u',v)du', \tag{4.6}
$$

$$
r(u,v) = r_0(u) + \int_0^v \lambda(u,v')dv', \tag{4.7}
$$

$$
\nu(u,v) = -\exp\left(-\int_0^v \frac{2\kappa}{r^2}\left(\frac{Q^2}{r} + \frac{m^2 r^3}{2}|\phi|^2 + \frac{\Lambda}{3}r^3 - \varpi\right)(u,v')dv'\right) \tag{4.8}
$$

$$
\zeta(u,v) = \zeta_0(u) - \int_0^v \left(\frac{\theta}{r}\nu - \frac{\nu\kappa\phi}{r}\big(m^2 r^2 + iqQ\big)\right)(u,v')dv', \tag{4.9}
$$

$$
A_u(u,v) = 2\int_0^v \frac{Q\nu\kappa}{r^2}(u,v')dv'. \tag{4.10}
$$

In the above formulas, we exploited the gauge choice $A_{u,0} \equiv 0$ and the fact that $\nu_0 \equiv -1$ and $\kappa_0 \equiv 1$. Now, we define the operator $T\colon S_\epsilon \to S_\epsilon$ such that, to each element $f \coloneqq (\lambda,\ \varpi,\ \theta,\ \kappa,\ \phi,\ Q,\ r,\ \nu,\ \zeta,\ A_u)$ in S_ϵ, it associates the vector

$$
Tf = (T_\lambda f, T_\varpi f, T_\theta f, T_\kappa f, T_\phi f, T_Q f, T_r f, T_\nu f, T_\zeta f, T_A f),
$$

[1] Notice, however, that ϕ_0, θ_0 and ζ_0 are **complex-valued** continuous functions, so that, e.g., $\|(\mathrm{Re}\,\phi_0, \mathrm{Im}\,\phi_0)\|_{L^\infty} = \|\mathrm{Re}\,\phi_0\|_{L^\infty} + \|\mathrm{Im}\,\phi_0\|_{L^\infty}$.

where:

$$T_\lambda f(u,v) = \lambda_0(v) - \int_0^u \frac{2\nu\kappa}{r^2}\left(\frac{Q^2}{r} + \frac{m^2r^3}{2}|\phi|^2 + \frac{\Lambda}{3}r^3 - \varpi\right)(u',v)du', \quad \forall\,(u,v)\in\mathcal{D}_\epsilon,$$

and analogous definitions hold for $T_\varpi f$, $T_\theta f$, $T_\kappa f$, $T_\phi f$, $T_Q f$, i.e. we use the expressions at the right hand side of equations (4.1)–(4.6). For the remaining components of the vector Tf, we use the expressions at the right hand side of (4.7)–(4.10), but, under the integral signs, we replace r by $T_r f$, λ by $T_\lambda f$ and so on for the other terms. For instance:

$$T_r f(u,v) = r_0(u) + \int_0^v T_\lambda f(u,v')dv', \quad \forall\,(u,v)\in\mathcal{D}_\epsilon,$$

and similar definitions are used for $T_\nu f$, $T_\zeta f$, $T_A f$. Let us now fix the constants R_ϖ, R_θ, R_κ, R_ϕ, R_Q freely, while the remaining constants will be specified in the next lines. We divide the rest of the proof into three separate steps.

1. The operator T is well-defined: indeed, let us assume that f is in S_ϵ. By further reducing ϵ, if needed,[2] we have that $T_\lambda f$, $T_\varpi f$, $T_\theta f$, $T_\kappa f$, $T_\phi f$ and $T_Q f$ are sufficiently close to the respective initial data (see equations (4.1)–(4.6)). This follows indeed by continuity and due to the fact that, since r and ν are bounded away from zero in $\{0\}\times[0,V]$ (see (3.46) and (3.48)), they are also non-zero in $\mathcal{D}_\epsilon$ for ϵ small. So, for instance:

$$|T_\lambda f(u,v) - \lambda_0(v)| \le C_\epsilon, \quad \forall\,(u,v)\in\mathcal{D}_\epsilon,$$

for some $C_\epsilon > 0$, which goes to zero as $\epsilon\to 0$. We focus on $T_r f$, $T_\nu f$, $T_\zeta f$ and $T_A f$ separately. In order to prove that $T_r f$ belongs to $I(\alpha,\beta,r_0)$, we use the definition of $T_r f$ and the previous estimates to write

$$T_r f(u,v) - r_0(u) = \int_0^v T_\lambda f(u,v')dv' \le C_\epsilon V + \int_0^v \lambda(0,v')dv' = C_\epsilon V + r(0,v) - r_0(0),$$

and similarly for a lower bound. Thus $T_r f \in I(\alpha,\beta,r_0)$ if we choose

$$\alpha = -C_\epsilon V + \min_{v\in[0,V]} r(0,v) - r_0(0) \quad \text{and} \quad \beta = C_\epsilon V + \max_{v\in[0,V]} r(0,v) - r_0(0).$$

By picking ϵ sufficiently small, we can additionally ensure that

$$T_r f(u,v) \ge r_0(u) - C_\epsilon V + \min_{v\in[0,V]} r(0,v) - r_0(0) \ge \frac{1}{2}\min_{v\in[0,V]} r(0,v) > 0, \quad \forall\,(u,v)\in\mathcal{D}_\epsilon,$$

by continuity of r.

We can bound $T_\nu f$ in $B_{R_\nu}(-1)$ by choosing $R_\nu = R_\nu(V,\, R_\kappa,\, R_\phi,\, R_Q,\, R_\varpi,\, \alpha,\, \beta)$ large enough (see (4.8)). By the same expression, it also follows that $T_\nu f$ is bounded away from zero and, consequently, $T_\kappa f$ is (see (4.4)). Similarly, we have that $(T_\zeta f, T_A f) \in B_{R_\zeta}(\zeta_0)\times B_{R_A}(0)$ by picking both $R_\zeta = R_\zeta(V,$ R_θ, R_ν, R_κ, R_ϕ, R_Q, α, $\beta)$ and $R_A = R_A(V,\, R_Q,\, R_\nu,\, R_\kappa,\, \alpha,\, \beta)$ large. Therefore, Tf belongs to S_ϵ.

[2]Let ϵ_1,ϵ_2 be such that $0<\epsilon_1<\epsilon_2$. It is straightforward to see that $S_{\epsilon_2}\subsetneq S_{\epsilon_1}$. On the other hand, the additional functions that we need to consider when reducing ϵ are uniformly bounded in terms of the constants R_λ, R_ϖ, R_θ, R_κ, R_ϕ, R_Q, α, β, R_ν, R_ζ and R_A.

2. The operator T is a contraction: let $f_i = (\lambda_i,\, \varpi_i,\, \theta_i,\, \kappa_i,\, \phi_i,\, Q_i,\, r_i,\, \nu_i,\, \zeta_i,\, A_{u,i}) \in S_\epsilon$ for $i = 1, 2$. We now prove that, for every $f_1, f_2 \in S_\epsilon$:

$$\|Tf_1 - Tf_2\|_{L^\infty(S_\epsilon)} \leq p\|f_1 - f_2\|_{L^\infty(S_\epsilon)}, \tag{4.11}$$

for some p, $0 \leq p < 1$. Since $T_\lambda f_i$, $T_\varpi f_i$, $T_\theta f_i$, $T_\kappa f_i$, $T_\phi f_i$ and $T_Q f_i$, for $i = 1, 2$, contain only integrals taken with respect to the variable u, we have, for every $q \in \{\lambda, \varpi, \theta, \kappa, \phi, Q\}$:

$$\|T_q f_1 - T_q f_2\|_{L^\infty(S_\epsilon)} \leq \tilde{C}_\epsilon \|f_1 - f_2\|_{L^\infty(S_\epsilon)},$$

for some constant $\tilde{C}_\epsilon$, where $\tilde{C}_\epsilon$ goes to zero as $\epsilon \to 0$. We used, again, that r and ν are bounded away from zero in $\mathcal{D}_\epsilon$, for ϵ small. We employ these results to estimate the remaining components of Tf, for instance:

$$|T_r f_1 - T_r f_2|\,(u, v) = \left| \int_0^v (T_\lambda f_1 - T_\lambda f_2)(u, v')dv' \right| \leq \tilde{C}_\epsilon V \|f_1 - f_2\|_{L^\infty(S_\epsilon)}, \quad \forall\,(u, v) \in \mathcal{D}_\epsilon.$$

A similar reasoning holds true for $T_\nu f_i$, $T_\zeta f_i$ and $T_A f_i$, $i = 1, 2$, in this order, since they can be expressed in terms of already bounded quantities, and since $T_r f_i$ and $T_\nu f_i$ are bounded away from zero, as previously showed. So, for a sufficiently small choice of ϵ, we have that T is a contraction. Then, it follows from Banach's fixed-point theorem that there exists a unique solution $f \in S_\epsilon$ to our system that satisfies expressions (4.1)-(4.10) in a rigorous way.

Even though we will not elaborate on it, a similar proof follows if we exchange the role of the u coordinate with the one of the v coordinate. In this case we fix $0 < U < +\infty$ and choose $0 < \epsilon(N_{\text{i.d.}}) \leq V$ small enough. We can define the operator T in the same spirit of the previous case, analogously to the procedure in [19].

3. The constraints propagate along the evolution: first, we know that constraint (3.20) holds at $v = 0$. To prove its validity in $\mathcal{D}_\epsilon$, it is enough to show that

$$\partial_v(\partial_u r - \nu) \equiv 0 \quad \text{ in } \mathcal{D}_\epsilon. \tag{4.12}$$

Indeed, we notice that $\lambda = \partial_v r$ is continuous in $\mathcal{D}_\varepsilon$ by definition of solution. On the other hand, (4.7) shows that

$$(u, v) \mapsto \partial_u r(u, v) = r_0'(u) + \int_0^v (\partial_u \lambda)(u, v')dv',$$

is continuous as well (notice that $\partial_u \lambda = \partial_u \partial_v r$ is continuous by (3.24)). Therefore, $r \in C^1(\mathcal{D}_\epsilon)$. Moreover, the second-order mixed partial derivatives of r are also continuous, by the previous two computations. Schwarz's theorem then gives $\partial_v \partial_u r = \partial_u \partial_v r = \partial_u \lambda$. Equality (4.12) finally follows from (3.24).

In order to verify that the algebraic constraint (3.33) is propagated, we notice that

$$\partial_u(\kappa(1 - \mu)) = \partial_u \lambda \quad \text{in } \mathcal{D}_\epsilon,$$

by (3.16), (3.20) (which can now be used in $\mathcal{D}_\epsilon$, by the previous lines), (3.25), (3.29) and (3.30).

The propagation of constraint (3.23) follows after showing that

$$\partial_u\left(\partial_v\phi - \frac{\theta}{r}\right) = -iqA_u\left(\partial_v\phi - \frac{\theta}{r}\right).$$

This is a consequence of (3.22), (3.27), (3.20) (which we can use in $\mathcal{D}_\epsilon$ now, see above), (3.21), (3.28), (3.32), and the fact that $\partial_u\partial_v\phi = \partial_v\partial_u\phi$. The latter can be proved as follows. By (3.22):

$$\phi(u,v) = \phi_0(v) + \int_0^u \left(-iqA_u\phi + \frac{\zeta}{r}\right)(u',v)du',$$

which implies that $\partial_u\partial_v\phi \in C^0(\mathcal{D}_\epsilon)$. Differentiation can be taken under the integral signs by uniform continuity. Moreover, (3.22) implies that $\partial_v\partial_u\phi \in C^0(\mathcal{D}_\epsilon)$ as well.

We now consider constraint (3.31). We differentiate the explicit expression (4.6), use that the constraint is satified on $\{u=0\}$, (3.20) and that

$$qr(0,v)\operatorname{Im}\big(\phi_0(v)\overline{\theta_0}(v)\big) = q\left[r\operatorname{Im}\big(\phi\bar\theta\big)\right](u,v) - \int_0^u \partial_u\big(qr\operatorname{Im}\big(\phi\bar\theta\big)\big)(u',v)du' \quad \forall\,(u,v)\in\mathcal{D}_\epsilon,$$

to write

$$\begin{aligned}\partial_v Q(u,v) =& q\left[r\operatorname{Im}\big(\phi\bar\theta\big)\right](u,v)+\\ &-q\int_0^u\left[\lambda\operatorname{Im}\big(\phi\bar\zeta\big) + r\operatorname{Im}\big(\partial_v\big(\phi\bar\zeta\big)\big) + \nu\operatorname{Im}\big(\phi\bar\theta\big) + r\operatorname{Im}\big(\partial_u\big(\phi\bar\theta\big)\big)\right](u',v)du'.\end{aligned} \tag{4.13}$$

The conclusion follows after noticing (as we will see in the next lines) that the quantity in the square brackets inside the last integral vanishes identically. In fact, (3.28) and (3.22) yield

$$r\operatorname{Im}\big(\partial_v\big(\phi\bar\zeta\big)\big) = \operatorname{Im}\big(\theta\bar\zeta\big) - \nu\operatorname{Im}\big(\phi\bar\theta\big) - qQ\nu\kappa|\phi|^2. \tag{4.14}$$

On the other hand, (3.27) and (3.23) (which holds in $\mathcal{D}_\epsilon$ as we saw above) give

$$r\operatorname{Im}\big(\partial_u\big(\phi\bar\theta\big)\big) = -\operatorname{Im}\big(\theta\bar\zeta\big) - \lambda\operatorname{Im}\big(\phi\bar\zeta\big) + qQ\nu\kappa|\phi|^2. \tag{4.15}$$

Moreover, (4.13) and (3.27) imply that $\partial_u\partial_v Q$ is continuous in $\mathcal{D}_\epsilon$. The same conclusion can be reached for $\partial_v\partial_u Q$, by (3.30) and (3.28). Thus, $\partial_u\partial_v Q = \partial_v\partial_u Q$ due to Schwarz's theorem.

We are now left to deal with constraint (3.26). By the explicit expression (4.2) and by (3.25), both $\partial_u\partial_v\varpi$ and $\partial_v\partial_u\varpi$ are continuous and, in particular, $\partial_u\partial_v\varpi = \partial_v\partial_u\varpi$. Then, we can prove that the constraint is propagated by noticing that

$$\begin{aligned}&\partial_u f(u,v) = -\frac{|\zeta|^2}{r\nu}f(u,v),\\ &\text{with}\quad f := \partial_v\varpi - \frac{\lambda}{2}m^2r^2|\phi|^2 - Qq\operatorname{Im}\big(\phi\bar\theta\big) - \frac{|\theta|^2}{2\kappa}.\end{aligned} \tag{4.16}$$

The above relation is a result of the following steps. First, we exploit (3.16), (3.24), (3.31) and (3.33) (the last two relations hold in $\mathcal{D}_\epsilon$, as seen above) to notice that

$$\partial_v\left(\frac{\lambda}{\nu\kappa}\right) = -\frac{2}{r\nu}\partial_v\varpi + 2\frac{qQ\,\mathrm{Im}\big(\phi\bar{\theta}\big)}{\nu r} + \frac{\lambda}{\nu}m^2 r|\phi|^2. \tag{4.17}$$

Then, we differentiate f and use that $\partial_u\partial_v\varpi = \partial_v\partial_u\varpi$, $\partial_u\partial_v r = \partial_v\partial_u r$, $\partial_u\partial_v Q = \partial_v\partial_u Q$ together with equations (3.30), (3.25), (3.29), (3.24), (3.31), (4.17), (4.14), (4.15) to see that (4.16) holds, up to the additive term

$$\lambda\frac{qQ\,\mathrm{Im}\big(\phi\bar{\zeta}\big)}{r} + \nu\frac{qQ\,\mathrm{Im}\big(\phi\bar{\theta}\big)}{r} + \frac{\nu}{2}m^2r^2\partial_v(|\phi|^2) + \\ -\frac{\lambda}{2}m^2r^2\partial_u(|\phi|^2) - \frac{1}{2\kappa}\partial_u(|\theta|^2) + \frac{\lambda}{2\nu\kappa}\partial_v(|\zeta|^2).$$

The latter can be shown to be zero by using basic properties of complex numbers, such as $\partial_u\big(|\theta|^2\big) = 2\,\mathrm{Re}\big(\bar{\theta}\partial_u\theta\big)$ and $\partial_v\big(|\zeta|^2\big) = 2\,\mathrm{Re}\big(\bar{\zeta}\partial_v\zeta\big)$, jointly with (3.27) and (3.28).

Final remarks: The fixed-point theorem only shows uniqueness of the solution in S_ϵ. To prove uniqueness in the general setting, we consider two solutions f_1 and f_2 having same initial data. Let $\mathcal{D}$ be the intersection of their domains of existence. If we take a point (u, v) in $\mathcal{D}$, we can consider the initial value problem with data prescribed on $[0, u] \times \{0\} \cup \{0\} \times [0, v]$. By the local existence result in S_ϵ, there exists a unique solution f_3 in $\mathcal{E} \coloneqq [0, \epsilon_1] \times [0, v]$ for some $\epsilon_1 > 0$ small, with ϵ_1 depending on the norm of the initial data $N_{\text{i.d.}}$. Since the solution is unique, it must be that $f_1 \equiv f_3 \equiv f_2$ on $\mathcal{E}$.

We can repeat the procedure by considering the initial value problem with initial data in $[\epsilon_1, 2\epsilon_1] \times \{0\} \cup \{\epsilon_1\} \times [0, v]$. These initial data are again controlled by $N_{\text{i.d.}}$, since the L^∞ norm of $f_1 \equiv f_2 \equiv f_3$ in $[0, \epsilon_1] \times [0, v]$ is itself controlled by $N_{\text{i.d.}}$. So, we obtain a further stripe of size ϵ_1 on which $f_1 \equiv f_2$. We can iterate this procedure by subdividing the entire set $\mathcal{D}$ into stripes of side ϵ_1. □

Remark 4.3 (maximal past sets)
A set $\mathcal{P} \subset [0, U] \times [v_0, +\infty)$ is a ***past set*** *if $[0, u] \times [v_0, v] \subset \mathcal{P}$ for every $(u, v) \in \mathcal{P}$. As noticed in [19, Theorem 4.4], every solution to the characteristic IVP with initial data prescribed on $[0, U] \times [v_0, +\infty)$, for some $v_0 \geq 0$, and satisfying assumptions* ***(A)***, ***(B)***, ***(C)*** *and* ***(D)***, *can be extended to a maximal past set $\mathcal{P} \supset [0, U] \times \{v_0\} \cup \{0\} \times [v_0, +\infty)$.*

Proposition 4.4 (continuous dependence with respect to the initial data)
Let us consider two solutions f and $\tilde{f}$ of the first-order system, with initial data (r_0, ν_0, λ_0, ϖ_0, θ_0, ζ_0, κ_0, ϕ_0, Q_0, $A_{u,0}$) and ($\tilde{r}_0$, $\tilde{\nu}_0$, $\tilde{\lambda}_0$, $\tilde{\varpi}_0$, $\tilde{\theta}_0$, $\tilde{\zeta}_0$, $\tilde{\kappa}_0$, $\tilde{\phi}_0$, $\tilde{Q}_0$, $\tilde{A}_{u,0}$), respectively. Assume that the two solutions are defined in $\mathcal{D} \coloneqq [0, U] \times [0, V]$ for some $U, V \in \mathbb{R}_0^+$ and define the two quantities

$$d(U, V) \coloneqq \|f - \tilde{f}\|_{L^\infty(\mathcal{D})},$$

and

$$d_0(U, V) \coloneqq \Big(\|r_0 - \tilde{r}_0\| + \|\nu_0 - \tilde{\nu}_0\| + \|\zeta_0 - \tilde{\zeta}_0\| + \|A_{u,0} - \tilde{A}_{u,0}\|\Big)_{L^\infty([0,U])}$$

$$+ \Big(\|\lambda_0 - \tilde{\lambda}_0\| + \|\varpi_0 - \tilde{\varpi}_0\| + \|\theta_0 - \tilde{\theta}_0\| + \|\kappa_0 - \tilde{\kappa}_0\| + \|\phi_0 - \tilde{\phi}_0\| + \|Q_0 - \tilde{Q}_0\| \Big)_{L^\infty([0,V])}.$$

Then, if $d_0(U,V)$ is sufficiently small, we have:

$$d(U,V) \leq C d_0(U,V),$$

where C is a positive constant depending on U, V and on

$$N(\mathcal{D}) := \|f\|_{L^\infty(\mathcal{D})} + \left\| \frac{1}{r} \right\|_{L^\infty(\mathcal{D})} + \left\| \frac{1}{\nu} \right\|_{L^\infty(\mathcal{D})}.$$

Proof. The proof is closely related to that of lemma 4.6 in [19]. Using the expressions (4.1)-(4.10) for the solutions, we get, for instance:

$$\|\varpi - \tilde{\varpi}\|_{L^\infty(\mathcal{D})} \lesssim \|\varpi_0 - \tilde{\varpi}_0\|_{L^\infty([0,V])} + \Big(\|r - \tilde{r}\| + \|\nu - \tilde{\nu}\| + \|\zeta - \tilde{\zeta}\| + \|Q - \tilde{Q}\| + \|\phi - \tilde{\phi}\| \Big)_{L^\infty(\mathcal{D})},$$

where the multiplying constants may depend on $U, V, N(\mathcal{D})$ and $\|\tilde{f}\|_{L^\infty(\mathcal{D})}$. Similar relations hold for the remaining unknowns of the first-order system. In particular, we get

$$d(U,V) \leq C \left(d_0(U,V) + \int_0^V d(U,V')dV' \right),$$

where C depends on $U, V, N(\mathcal{D})$ and, a priori, also depends on $\|\tilde{f}\|_{L^\infty(\mathcal{D})}$. However, the proof of theorem 4.2 shows that $\tilde{f}$ is controlled by the L^∞ norm of its initial data. Thus, we can conclude that C only depends on U, V and $N(\mathcal{D})$. The claim then follows from Grönwall's inequality. □

Remark 4.5 (On the equivalence between the first-order and second-order system)
*Under assumptions **(A)**, **(B)**, **(C)**, **(D)**, the second-order system* (3.6)–(3.10), (3.30)–(3.31) *implies the first-order system* (3.20)–(3.33).

On the other hand, if (r, Ω^2, ϕ, Q) solves the second-order system (namely r, Ω^2, ϕ, Q satisfy the system, they are continuous functions and all derivatives in the second-order system are continuous), then the Raychaudhuri equations and the wave equation for r imply that $r \in C^2$. Such a property is guaranteed for a solution to the first-order system if we additionally require that

(B2) *ν_0, λ_0 and κ_0 are C^1 functions in their respective domains.*

*A straightforward adaptation of [19, Section 6] reveals that assumption **(B2)** is in fact sufficient to show that a solution to the first-order system solves the second-order system, in the above-mentioned sense.*

*Therefore, we say that the first-order system and the second-order system are equivalent if assumptions **(A)**, **(B)**, **(B2)**, **(C)** and **(D)** are satisfied.*

Remark 4.6 (general properties of solutions)
As a consequence of (4.1)-(4.10), *we have that, for a solution $f = (\lambda,\ \varpi,\ \theta,\ \kappa,\ \phi,\ Q,\ r,\ \nu,\ \zeta,\ A_u)$ of the*

first order system in a past set $\mathcal{P}$:

- *κ is positive,*
- *ν is negative.*

*We notice that the renormalized Hawking mass ϖ is generally not a monotonic function, differently from the case of the Einstein-Maxwell-(**real**) scalar field system. However, we have*

$$\partial_u\left(\varpi - \frac{Q^2}{2r}\right) < 0 \quad \textit{and} \quad \partial_v\left(\varpi - \frac{Q^2}{2r}\right) \geq 0$$

in the region $\{(u,v)\colon\ \lambda(u,v) \geq 0\}$.

Chapter 5

The extension criterion

We are now interested in conditions that let us extend the domain of existence of a solution to the characteristic IVP. Results of this sort were obtained in [26], [19] and [72]. In [26, 72], in particular, the lack of extensions was used to characterize the *first singularities* possibly present in the considered spacetimes, therefore addressing that problem of spacetime predictability which, in more general settings, is captured by the weak cosmic censorship conjecture. In particular, the following result was proved in [72] for the Einstein-Maxwell-(charged) scalar field system with $\Lambda = 0$: given initial data prescribed on $[0, U] \times \{0\} \cup \{0\} \times [0, V]$ for some $U, V \in \mathbb{R}_0^+$, where the initial data satisfy the assumptions of the local existence theorem 4.2, a solution f defined in

$$\mathcal{D} := [0, U') \times [0, V'),$$

with $0 < U' < U$ and $0 < V' < V$, can be extended along both the future ingoing and outgoing directions if we can control the L^∞ norm of the solution and of the inverse of r and Ω^2. More precisely, the result is formulated in a global sense in terms of Cauchy data. In [72], it was also shown that for this to happen, two conditions are sufficient (provided that singularities emanating from spacetime endpoints are avoided): 1) that the area-radius function can be estimated in $\mathcal{D}$ from below and above by positive constants, and 2) that the spacetime volume of $\mathcal{D}$, i.e.

$$\int_{\mathcal{D}} \Omega^2 dudv$$

is bounded from above. The results of [72] apply to our setting, where the cosmological constant is possibly non-zero, since the presence of Λ in (3.8) does not require any substantial change in the proofs of the above statements.

The main part of the proof of the extension criterion in [72] consists of bounding the quantity

$$\iint_{\mathcal{D}} r^2 T_{uv} \sim \int_0^{V'} \int_0^{U'} |\nu|\kappa \left(m^2 r^2 |\phi|^2 + \frac{Q^2}{r^2} \right) dudv \tag{5.1}$$

in order to obtain estimates on the spacetime integrals of several PDE variables. This allows to get a bound on ϕ when integrating the Klein-Gordon equation (3.10):

$$\partial_u\partial_v\phi = -\frac{1}{r}(D_u r\partial_v\phi + \partial_v r D_u\phi) + \frac{iqQ\Omega^2}{4r^2}\phi - \frac{m^2\Omega^2}{4}\phi.$$

As a final step, the bound on ϕ is applied to obtain the remaining uniform bounds. This proof requires a bootstrap argument, due to the need to bound ϕ also at the right hand side of (3.10). Such an argument can be closed after partitioning $\mathcal{D}$ in ϵ-small rectangles.

On the other hand, the expressions (4.1)–(4.10) satisfied by the solution to the characteristic IVP suggest that it might be sufficient to uniformly bound the iterated integrals[1] in (5.1) to estimate the remaining PDE variables. We present here a proof that uses this insight. In particular, we demonstrate that an extension result **for the first-order system** follows from the requirement of an upper and lower bound for the function r,[2] without adopting any bootstrap method and essentially only using the positivity of T_{uv}. We stress that, with respect to the extension criterion **for the second-order system** in [72], slightly less regularity is required: the initial data ν_0, λ_0 and κ_0 only need to be C^0 in their respective domains (see remark 4.5).

Theorem 5.1 (extension criterion)
Let $f_0 = (r_0,\ \nu_0,\ \lambda_0,\ \varpi_0,\ \theta_0,\ \zeta_0,\ \kappa_0,\ \phi_0,\ Q_0,\ A_{u,0})$ be the initial data prescribed on

$$\mathcal{D}_0 \coloneqq [0,U]\times\{0\}\cup\{0\}\times[0,V]$$

*for some $U, V \in \mathbb{R}_0^+$, under the assumptions **(A)**, **(B)**, **(C)** and **(D)**. Let us consider the unique solution $f = (r,\ \nu,\ \lambda,\ \varpi,\ \theta,\ \zeta,\ \kappa,\ \phi,\ Q,\ A_u)$ to the characteristic IVP in the rectangle*

$$\mathcal{D} \coloneqq [0,U')\times[0,V')$$

for some $0 < U' < U$ and $0 < V' < V$. Assume that the area-radius function is bounded away from zero and bounded from above, i.e. there exist L and R such that:

$$0 < L \le r(u,v) \le R < +\infty, \quad \forall\,(u,v)\in\mathcal{D}.$$

Define

$$N_0 \coloneqq \|f_0\|_{L^\infty(\mathcal{D}_0)}$$

and

$$N(\mathcal{D}) \coloneqq \left\|\frac{1}{\kappa}\right\|_{L^\infty(\mathcal{D})} + \left\|\frac{1}{\nu}\right\|_{L^\infty(\mathcal{D})} + \|f\|_{L^\infty(\mathcal{D})}.$$

Then:

$$N(\mathcal{D}) \le C = C(U', V', L, R, N_0) < +\infty$$

[1] As we will see, it is enough to bound just the iterated integral taken with respect to the v variable, using the good signs of the charge function and scalar field terms at the right hand side of (4.8).

[2] This was also the case for the real, massless scalar field system analysed in [27] ($\Lambda = 0$) and in [19] (for any value of Λ), where monotonicity properties were exploited. In our case, the same properties do not generally hold.

and there exists a $\delta > 0$ and a solution $\tilde{f}$ to the same characteristic IVP such that $\tilde{f}$ is defined on $\mathcal{D}^\delta := [0, U' + \delta] \times [0, V' + \delta]$.

Proof. We split the proof into two separate parts. First, we prove the uniform bound on $N(\mathcal{D})$ and, then, construct an explicit extension.

1. Bounded area-radius $\Longrightarrow$ finite norms.

Preamble: in the following, we will use the fact that $\nu < 0$ in $\mathcal{D}$ (see remark 4.6) multiple times, and we will denote by C any positive constant depending on one (or more) value(s) in $\{U', V', L, R, N_0\}$.

Uniform bounds on $\boldsymbol{\kappa}$: due to (4.4), it follows that

$$0 < \kappa \leq 1 \tag{5.2}$$

in $\mathcal{D}$.[3]

Uniform bound on the double integral of $\boldsymbol{Q^2}$ and $\boldsymbol{|\phi|^2}$: for every $(u, v) \in \mathcal{D}$, we can use (3.20), (3.21), (3.37), (5.2) and the boundedness of r to get:

$$\begin{aligned} 0 &\leq -\int_0^v \int_0^u \nu\kappa\Big(\frac{Q^2}{r^2} + m^2 r^2 |\phi|^2\Big)(u', v')du'dv' \\ &= \int_0^v \int_0^u \big[\partial_u(r\lambda) + \nu\kappa\big(\Lambda r^2 - 1\big)\big](u', v')du'dv' \\ &\leq R^2 + \frac{|\Lambda|}{3} R^3 V' + RV' \leq C. \end{aligned} \tag{5.3}$$

Preliminary integral bound on $|\boldsymbol{\lambda}|$: we notice that, after integrating (3.37) in u, the bounds on κ and r give

$$L \cdot \sup_{u \in [0, U')} |\lambda|(u, v) \leq C + \int_0^{U'} |\nu|\kappa\Big(\frac{Q^2}{r^2} + m^2 r^2 |\phi|^2\Big)(u', v)du', \quad \forall\, v \in [0, V').$$

A further integration in the v variable, together with (5.3), gives

$$\int_0^v \sup_{u \in [0, U')} |\lambda|(u, v')dv' \leq C, \quad \forall\, v \in [0, V'). \tag{5.4}$$

Uniform (upper) bound on $|\boldsymbol{\nu}|$: by applying (3.36) to (4.8), and by using the bounds on κ and the positivity of the terms containing Q and ϕ, we can write:

$$|\nu|(u, v) = \exp\Big(-\int_0^v \Big[\kappa\Big(\frac{Q^2}{r^3} - \frac{1}{r} + \Lambda r + m^2 r |\phi|^2\Big) + \frac{\lambda}{r}\Big](u, v')dv'\Big) \leq \frac{R}{L} e^{\left(|\Lambda| R + \frac{1}{L}\right)V'}, \tag{5.5}$$

in $\mathcal{D}$. This gives a uniform bound on $|\nu|$ for every $(u, v) \in \mathcal{D}$.

Uniform bound on an iterated integral: the integral $\int_0^{V'} \nu\kappa\Big(\frac{Q^2}{r^2} + m^2 r^2 |\phi|^2\Big)dv'$, for which Fubini-Tonelli's theorem gives a pointwise estimate, can be uniformly bounded. Indeed, after integrating (3.37),

[3] This estimate can be used, together with the assumptions on the function r, to get a uniform bound on $\iint_{\mathcal{D}} \Omega^2 = -4\int_0^{V'}\Big(\int_0^{U'}(\partial_u r)\kappa du'\Big)dv'$. The extension principle of [72] can then be applied to our system. Our intention, however, is to provide an alternative proof and at the same time to verify explicitly that the presence of Λ does not constitute an obstruction to spacetime extensions.

we have

$$(r\nu)(u,v) - (r\nu)(u,0) - \int_0^v \nu\kappa\big(1-\Lambda r^2\big)(u,v')dv' = -\int_0^v \nu\kappa\bigg(\frac{Q^2}{r^2}+m^2r^2|\phi|^2\bigg)(u,v')dv'$$

for every $(u,v)\in\mathcal{D}$. The left hand side of last expression is bounded uniformly due to (5.2), (5.5) and to the boundedness assumption on r. Hence, we have

$$\int_0^v |\nu|\kappa\bigg(\frac{Q^2}{r^2}+m^2r^2|\phi|^2\bigg)(u,v')dv' \leq C, \quad \forall\,(u,v)\in\mathcal{D}. \tag{5.6}$$

Uniform bound on $\boldsymbol{A_u}$: by applying the Cauchy–Schwarz inequality in (4.10), we have:

$$|A_u|(u,v) \leq 2\bigg(-\int_0^v \nu\kappa\frac{Q^2}{r^2}(u,v')dv'\bigg)^{\frac{1}{2}}\bigg(-\int_0^v \frac{\nu\kappa}{r^2}(u,v')dv'\bigg)^{\frac{1}{2}} \leq C \tag{5.7}$$

in $\mathcal{D}$.

Preliminary bound on $|\boldsymbol{\theta}|$: we observe that (4.3) and bound (5.7) yield:

$$|\theta|(u,v) \leq \sup_{v\in[0,V')}|\theta_0|(v) + \int_0^u \frac{|\zeta\lambda|}{r}(u',v)du' + \int_0^u \frac{|\nu|\kappa}{r}|\phi|\big(m^2r^2+|qQ|\big)(u',v)du', \tag{5.8}$$

in $\mathcal{D}$.

Uniform bound on $|\boldsymbol{\zeta}|$: expression (4.9), together with the uniform bounds on r and ν, lets us write, for every $(u,v)\in\mathcal{D}$:

$$|\zeta|(u,v) \leq \sup_{u\in[0,U')}|\zeta_0|(u) + C\int_0^v |\theta|(u,v')dv' + \int_0^v \frac{|\nu|\kappa}{r}|\phi|\big(m^2r^2+|qQ|\big)(u,v')dv'. \tag{5.9}$$

The last integral term can be bounded again by (5.6) and by the Cauchy-Schwarz inequality, for instance:

$$\int_0^v \frac{|\nu|\kappa}{r}|\phi qQ|(u,v')dv' \leq \frac{1}{mL}\bigg(\int_0^v \frac{|\nu|\kappa}{r^2}q^2Q^2\bigg)^{\frac{1}{2}}\bigg(\int_0^v |\nu|\kappa m^2r^2|\phi|^2\bigg)^{\frac{1}{2}} \leq C. \tag{5.10}$$

By plugging (5.8) in (5.9):

$$|\zeta|(u,v) \leq C + C\int_0^u\int_0^v \frac{|\zeta\lambda|}{r}(u',v')dv'du' + C\int_0^u\int_0^v \frac{|\nu|\kappa}{r}|\phi|\big(m^2r^2+|qQ|\big)(u',v')dv'du',$$

where the order of integration could be reversed due to Fubini-Tonelli's theorem. The last integral can be bounded using (5.6) and (5.10). Moreover, we define

$$\zeta_\epsilon(u) \coloneqq \max_{v\in[0,V'-\epsilon]}|\zeta|(u,v),$$

for every $u\in[0,U')$ and for some fixed $0<\epsilon<V'$. Therefore, the previous lines give

$$\zeta_\epsilon(u) \leq C + C\int_0^u \zeta_\epsilon(u')\bigg(\int_0^{V'-\epsilon}|\lambda|(u',v')dv'\bigg)du'. \tag{5.11}$$

We now use (5.4) in (5.11) to obtain

$$\zeta_\epsilon(u) \leq C\left(1 + \int_0^u \zeta_\epsilon(u')du'\right).$$

By applying Grönwall's inequality, and then taking the limit $\epsilon \to 0$, we finally get a uniform bound on ζ.

Uniform bound on $|\phi|$: equation (4.5), together with the uniform bounds on A_u, ζ and r, gives

$$|\phi|(u,v) \leq \sup_{v\in[0,V')} |\phi_0|(v) + \int_0^u \frac{|\zeta|}{r}(u',v)du' \leq C,$$

where C does not depend on the value of $u \in [0, U')$.

Uniform bound on $|Q|$: equation (4.6), together with the uniform bounds on r, ϕ and ζ, gives

$$|Q|(u,v) \leq \sup_{v\in[0,V')} |Q_0|(v) + |q| \int_0^u (r|\phi||\zeta|)(u',v)du' \leq C,$$

for every v in $[0, V')$.

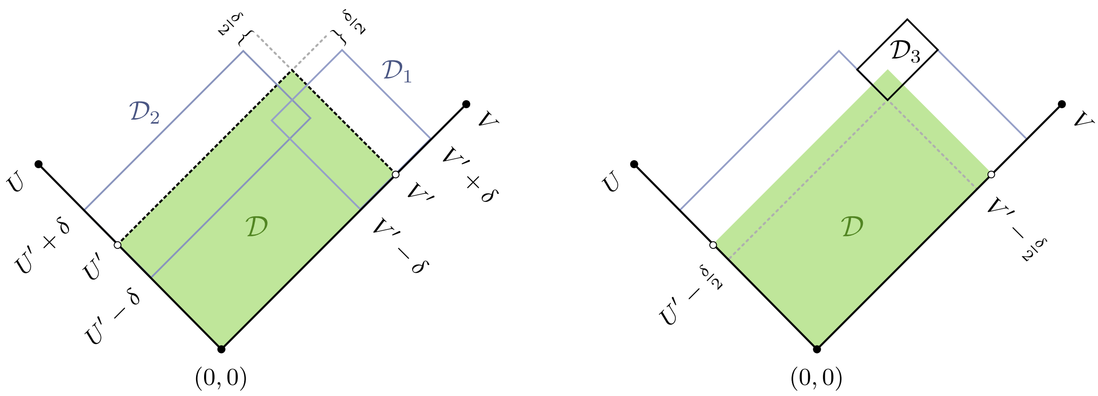
Uniform (lower) bound on $|\nu|$: the above uniform bounds on ϕ and Q are sufficient to prove that ν is bounded away from zero in $\mathcal{D}$, as can be seen from the finiteness of the integrals in expression (5.5).

Uniform (lower) bound on κ: the estimates on ζ, ν and r can be applied to (4.4) to bound κ away from zero.

Remaining uniform bounds: a bound on λ follows after integrating (3.37) in u and noticing that every term at the right hand side of the equation is uniformly bounded. The previous estimate on λ can be applied to (5.8) to get a uniform bound on θ, since we have estimated each function appearing in the integrand terms of (5.8) in the $\sup_{\mathcal{D}}$-norm. Finally, we can divide both sides of (3.36) by κ to bound ϖ uniformly, due to the estimates from above that we have for Q, r, λ, and to the estimates from below that we have for κ and r.

2. Construction of the extension: let $\delta, \epsilon > 0$ be small and consider the initial value problem with initial data prescribed on

$$[0, U' - \epsilon] \times \{V' - \delta\} \cup \{0\} \times [V' - \delta, V' + \delta].$$



By the local existence theorem 4.2, and due to the uniform bound on $N(\mathcal{D})$ proved in the previous steps, there exists a minimum time of existence $\tilde{\delta} > 0$, whose size depends **only** on N_0, such that we can construct a solution to our PDE system in $\mathcal{D}_1 \coloneqq [0, U' - \epsilon] \times [V' - \delta, V' - \delta + \tilde{\delta}]$. Since we chose the minimum time of existence, this result holds whenever we translate the initial null segments along the v–direction, provided that the ingoing segment stays in $\mathcal{D}$ and that the outgoing segment is contained in $\{0\} \times [0, V]$. Without loss of generality, we therefore consider the new shifted rectangle to be $\mathcal{D}_1 \coloneqq [0, U' - \epsilon] \times [V' - \delta, V' + \delta]$, where the value of δ is possibly different from the original one, and take $\epsilon = \frac{\delta}{2}$.

The same construction can be followed with respect to the initial data prescribed on

$$[U' - \delta, U' + \delta] \times \{0\} \cup \{U' - \delta\} \times \left[0, V' - \frac{\delta}{2}\right].$$

In particular, we define $\mathcal{D}_2 \coloneqq [U' - \delta, U' + \delta] \times [0, V' - \frac{\delta}{2}]$, for the same value of δ (if a smaller value of δ is required by the local existence theorem, we repeat the construction of $\mathcal{D}_1$ and $\mathcal{D}_2$ from scratch, starting from the ingoing null segment).

We now consider the point $p = \left(U' - \frac{\delta}{2}, V' - \frac{\delta}{2}\right)$. By performing a coordinate shift, we can set $p = (0, 0)$. We then construct a solution from the initial data prescribed on $\left[0, \frac{3}{2}\delta\right] \times \{0\} \cup \{0\} \times \left[0, \frac{3}{2}\delta\right]$. By the proof of theorem 4.2, the L^∞ norm of the (continuous) solution in the compact sets $\mathcal{D}_1$ and $\mathcal{D}_2$ is controlled by N_0. Therefore, for the initial value problem prescribed on these null segments, the minimum time of existence of solutions is again given by the previously obtained δ. Thus, we obtain a continuous solution in

$$\mathcal{D}_3 \coloneqq [0, \delta] \times \left[0, \frac{3}{2}\delta\right],$$

which is enough to conclude the proof. □

Remark 5.2 (A counter-example)

*Let us consider a solution to the first-order system defined, modulo the $SO(3)$ orbit space, in a half-open rectangle in the Reissner-Nordström-de Sitter spacetime. Assume that the rectangle lies in the outer domain of communications, where one of the corners of the closure of the rectangle coincides with i^+. Although the area-radius function is bounded in such a rectangle, no extensions are possible due to the singularity originating at i^+. The reason for this is that assumption **(D)** does not hold and so theorem 5.1 does not apply: ν vanishes along the cosmological horizon and thus the L^∞ norm of ν^{-1} is unbounded along the initial outgoing null segment. In particular, we do not have a lower bound on the time of existence when constructing local solutions. Notice that, in particular, such a rectangle has infinite spacetime volume.*

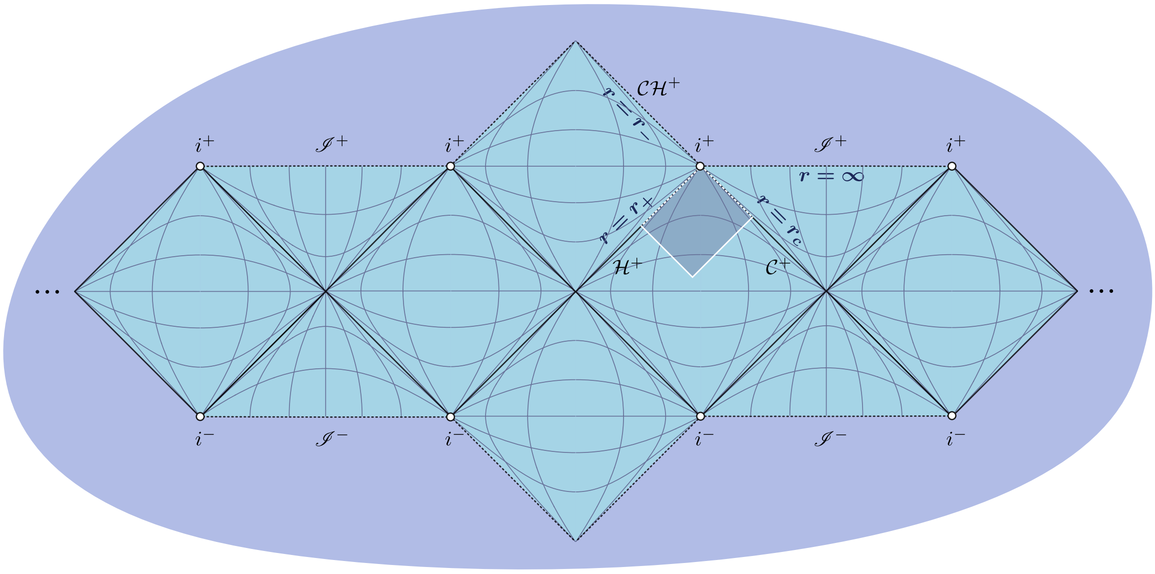


Figure 5.1: The Reissner-Nordström-de Sitter spacetime.

Chapter 6

Existence and stability of the Cauchy horizon

We recall that a set $\mathcal{P} \subset [0, U] \times [v_0, +\infty)$ is called a **past set** if $J^-(u, v) \coloneqq [0, u] \times [v_0, v]$ is contained in $\mathcal{P}$ for every (u, v) in $\mathcal{P}$. Given a set $S \subset [0, U] \times [v_0, +\infty)$, we define

$$J^-(S) \coloneqq \bigcup_{(u,v) \in S} J^-(u, v),$$

and analogous definitions hold for $J^+(S)$, $I^-(S)$ and $I^+(S)$ (where $I^-(u, v) \coloneqq [0, u) \times [v_0, v)$).

In the following, we work with a solution to the characteristic IVP of section 3.1, defined in the maximal past set $\mathcal{P}$ containing

$$\mathcal{D}_0 \coloneqq [0, U] \times \{v_0\} \cup \{0\} \times [v_0, +\infty), \qquad \text{for some } 0 < U < +\infty,\ v_0 \geq 0.$$

The existence of $\mathcal{P}$ is guaranteed by the well-posedness results proved in chapter 4. In turn, its maximality is to be interpreted in the sense that the solution cannot be defined on any larger past set (see also theorem 5.1). From now on, we suppose that assumptions **(A)**, **(B)**, **(B2)** (see remark 4.5), **(C)**, **(D)**, **(E)**, **(F)** and **(G)** hold. In particular, we are going to use the equations from both the first-order and second-order systems, depending on the most convenient choice.

In this context, $\mathcal{P}$ corresponds to a region containing the event horizon and extending inside the dynamical black hole under investigation. In the course of the following proofs, we actually work with solutions defined in

$$\mathcal{P} \cap \{v \geq v_1\},$$

for some $v_1 \geq v_0$. We require (finitely many times) that the values of U and v_1 are, respectively, sufficiently small and sufficiently large. A posteriori, this implies that our results hold in a region adjacent to both the event horizon and the Cauchy horizon of the dynamical black hole under investigation. We will highlight the steps where we constrain the values of U and v_1. This occurs finitely many times and depends only on the L^∞ norm of the initial data and possibly on η, R (see proposition 6.7), Y (see proposition 6.12), and ε (see proposition 6.13). These quantities can ultimately be chosen in terms of the initial data.

In this chapter, we are also going to prove that the **apparent horizon**

$$\mathcal{A} \coloneqq \{(u,v) \in [0,U] \times [v_0,+\infty) \colon \ \lambda(u,v) = 0\}$$

is a non-empty C^1 curve for large values of the v coordinate. It is also convenient to define the **regular region**

$$\mathcal{R} \coloneqq \{(u,v) \in [0,U] \times [v_0,+\infty) \colon \ \lambda(u,v) > 0\},$$

which, as we will see, extends from the event horizon to part of the redshift region. The remaining part of the black hole interior is occupied by the **trapped region**

$$\mathcal{T} \coloneqq \{(u,v) \in [0,U] \times [v_0,+\infty) \colon \ \lambda(u,v) < 0\}. \tag{6.1}$$

For the system under analysis, the regular region is a priori non-empty, differently from the case of the Reissner-Nordström-de Sitter solution. In the latter case, indeed, every 2-sphere in the interior of the black hole region is a trapped surface and, furthermore, the apparent horizon coincides with the event horizon.

Remark 6.1 (on the main constants)
Throughout the bootstrap procedure, we use several auxiliary, positive quantities:

- *Redshift region (proposition 6.7):* η, $\delta = \delta(\eta)$, $R = R(\delta)$,
- *No-shift region (proposition 6.12):* $\varepsilon = \varepsilon(\beta)$, Δ, $Y = Y(\Delta)$,
- *Early blueshift region (see* (6.101)*):* β.

We stress that, in proposition 6.13, we choose Δ *in function of* ε*. However, as explained in remark 6.14, an analogous proof holds even if we choose* Δ *independently from* ε*. All above constants can be ultimately defined in terms of the initial data.*

In particular, we have:

- $\delta \to 0$ *as* $\eta \to 0$ *and* $R \to r_+$ *as* $\delta \to 0$ *(see proposition 6.7),*
- $Y \to r_-$ *as* $\Delta \to 0$ *(see proposition 6.13),*
- $\varepsilon \to 0$ *as* $\beta \to 0$ *(see lemma 6.16).*

6.1 Event horizon

Recall that ϖ_+, Q_+ and Λ are the parameters of the final sub-extremal Reissner-Nordström-de Sitter black hole (see assumption **(F)**), while r_+ and K_+ denote the radius and the surface gravity, respectively, associated to its event horizon.

Proposition 6.2 (bounds along the event horizon)
For every $v \geq v_0$, we have:

$$0 < r_+ - r(0,v) \leq C_{\mathcal{H}} e^{-2sv}, \tag{6.2}$$

$$0 < \lambda(0,v) \leq C_{\mathcal{H}} e^{-2sv}, \tag{6.3}$$

$$\frac{e^{2K_+ v}}{C_{\mathcal{H}}} \leq |\nu|(0,v) \leq C_{\mathcal{H}} e^{2K_+ v}, \tag{6.4}$$

$$|\varpi(0,v) - \varpi_+| \leq C_{\mathcal{H}} e^{-2sv}, \tag{6.5}$$

$$|Q(0,v) - Q_+| \leq C_{\mathcal{H}} e^{-2sv}, \tag{6.6}$$

$$|K(0,v) - K_+| \leq C_{\mathcal{H}} e^{-2sv}, \tag{6.7}$$

$$|\partial_v \log \Omega^2(0,v) - 2K(0,v)| \leq C_{\mathcal{H}} e^{-2sv}, \tag{6.8}$$

$$|D_u \phi|(0,v) \leq \begin{cases} C_{\mathcal{H}} |\nu|(0,v) e^{-sv}, & \text{if } s < 2K_+, \\ C_{\mathcal{H}}(v - v_0), & \text{if } s = 2K_+, \\ C_{\mathcal{H}}, & \text{if } s > 2K_+, \end{cases} \tag{6.9}$$

$$|\phi|(0,v) + |\partial_v \phi|(0,v) \leq C_{\mathcal{H}} e^{-sv}. \tag{6.10}$$

where $C_{\mathcal{H}}$ is a positive constant depending only on the initial data.

Proof. The proof exploits the decay due to the redshift effect, similarly to the proof of [79, proposition 4.4]. In the current case, however, the competition between the redshift effect and the exponential Price law is evident in the estimate for $|D_u \phi|$ (see already remark 6.3).

In the following, the letter C will denote a positive constant depending uniquely on the initial data. Moreover, we will exploit the assumptions (3.45) and (3.48) on κ and ν, respectively, multiple times in the next computations.

Preliminary bounds and exponential growth of $|\nu|$: first, we notice that (3.50) and (3.54) imply

$$0 < r(0, v_0) \leq r(0,v) < r_+ < +\infty, \quad \forall\, v \geq v_0. \tag{6.11}$$

It will also be useful to write (3.35), when evaluated on $\mathcal{H}^+$, as

$$\partial_v \log |\nu|(0,v) = 2K(0,v) - m^2 r(0,v) |\phi|^2(0,v). \tag{6.12}$$

Now, given $\epsilon > 0$, expressions (3.49), (3.53) and the boundedness of r provide a sufficiently large $v_1 > v_0$ such that

$$\left| 2K(0,v) - m^2 r(0,v) |\phi|^2(0,v) - 2K_+ \right| < \epsilon, \quad \forall\, v \geq v_1. \tag{6.13}$$

This estimate will be improved during the next steps of the current proof, but for the moment we can use it in (6.12) and require ϵ to be sufficiently small to obtain

$$0 < K_+ < 2K_+ - \epsilon \leq \partial_v \log |\nu|(0,v) \leq 2K_+ + \epsilon, \quad \forall\, v \geq v_1, \tag{6.14}$$

and so, by integrating and due to (3.44):

$$e^{(2K_+-\epsilon)(v-v_1)} \le |\nu|(0,v) \le e^{(2K_++\epsilon)(v-v_1)}, \quad \forall\, v \ge v_1. \tag{6.15}$$

Note that $\Omega^2(0,v) = -4\nu(0,v)$ for every $v \ge v_0$, by (3.19). So, (3.49), (6.12) and the boundedness of r entail

$$\left|\partial_v \log \Omega^2(0,v) - 2K(0,v)\right| \le Ce^{-2sv}.$$

To obtain (6.3), we first show that $\frac{\lambda}{\nu}$ admits a finite limit as $v \to +\infty$, and that such a limit is zero. Indeed, by assumptions **(A)** and **(F)**, we have that $\lim_{v\to+\infty} \lambda(0,v) = \lim_{v\to+\infty}(1-\mu)(0,v) = 0$. Moreover, due to (6.15) and to the fact that $\lambda_{|\mathcal{H}^+} > 0$:

$$\lim_{v\to+\infty} \frac{\lambda}{\nu}(0,v) = 0. \tag{6.16}$$

Exponential decays: We now focus on the proof of the decay of λ. Using (3.7) and (3.19) we cast one of the Raychaudhuri equations as

$$\partial_v\left(\frac{\lambda}{\nu\kappa}\right) = -r\frac{|\partial_v\phi|^2}{\nu\kappa}.$$

Using the above, assumptions (3.49) and (3.54), and exploiting (6.11) and (6.16), we have:

$$\begin{aligned}
0 < \lambda(0,v) &= \nu(0,v)\int_{+\infty}^{v} \partial_v\left(\frac{\lambda}{\nu}\right)(0,v')dv' \\
&= \nu(0,v)\int_{v}^{+\infty} \frac{r|\partial_v\phi|^2}{\nu}(0,v')dv' \le C\nu(0,v)\int_v^{+\infty}\frac{e^{-2sv'}}{\nu(0,v')}dv'.
\end{aligned}$$

We now multiply and divide by K_+, which is a positive quantity, and use that $K_+ < \partial_v \log|\nu|(0,v')$ (see (6.14)), the fact that $v \mapsto \nu^{-1}(0,v)$ is an increasing function (indeed $\partial_v\nu^{-1} = |\nu|^{-1}\partial_v\log|\nu| > 0$) and (6.15) to write

$$\begin{aligned}
\nu(0,v)\int_v^{+\infty}\frac{e^{-2sv'}}{\nu(0,v')}dv' &\le \frac{|\nu|(0,v)}{K_+}\int_v^{+\infty} e^{-2sv'}\partial_v\left(\frac{1}{\nu}\right)(0,v')dv' \\
&\le C|\nu|(0,v)e^{-2sv}\int_v^{+\infty}\partial_v\left(\frac{1}{\nu}\right)(0,v')dv' = Ce^{-2sv},
\end{aligned} \tag{6.17}$$

which therefore gives $0 < \lambda(0,v) \le Ce^{-2sv}$ for every $v \ge v_0$.

After recalling (3.51), we integrate (3.12) along the event horizon and use (6.11) and the exponential Price law (3.49) to write

$$|Q(0,v) - Q_+| = C\left|\int_v^{+\infty} r^2(0,v')\,\mathrm{Im}(\phi\overline{\partial_v\phi})(0,v')dv'\right| \le Ce^{-2sv}, \quad \forall\, v \ge v_0. \tag{6.18}$$

This gives (6.6).

We can use this after integrating (3.26) (recall that $\theta = r\partial_v\phi$), together with (3.49), (3.52), (6.18), (6.11) and the decay of λ to see that

$$|\varpi(0,v) - \varpi_+| \leq Ce^{-2sv}, \quad \forall\, v \geq v_0, \tag{6.19}$$

therefore giving (6.5).

By integrating (6.17) and using (3.50), it also follows that

$$0 < r_+ - r(0,v) \leq Ce^{-2sv}.$$

Due to definition (3.34) and to the decays proved for Q and ϖ, this is also sufficient to prove

$$|K(0,v) - K_+| \leq Ce^{-2sv}, \quad \forall\, v \geq v_0. \tag{6.20}$$

This result also allows us to improve (6.13) and, following the same steps leading to (6.14) and (6.15), to obtain:

$$|\nu|(0,v) \sim e^{2K_+(v-v_0)}, \quad \forall\, v \geq v_0. \tag{6.21}$$

Bounds on $|D_u\phi|$: we use (3.40), (3.49), (6.18) and (6.11) to write

$$|\partial_v(rD_u\phi)|\,(0,v) = \left|-\nu\partial_v\phi + m^2\nu r\phi + i\frac{qQ\nu}{r}\phi\right|(0,v) \leq C|\nu|(0,v)e^{-sv}. \tag{6.22}$$

Now, let us first assume that $s < 2K_+$ and define the constant $a := K_+ - \frac{s}{2} > 0$. There exists $V \geq v_0$, depending uniquely on the initial data, such that, using (3.35), (3.49) and (6.20), we obtain

$$\partial_v\left(|\nu|e^{-sv}\right)_{|\mathcal{H}^+} = |\nu|(0,v)e^{-sv}\left(2K(0,v) - m^2 r(0,v)|\phi|^2(0,v) - s\right) > a|\nu|(0,v)e^{-sv},$$

for every $v \geq V$. We can use this inequality when we integrate (6.22):

$$\begin{aligned}|rD_u\phi|(0,v) &\leq C\left(1 + \int_{v_0}^{v} |\nu|(0,v')e^{-sv'}dv'\right) < C\left(1 + \frac{1}{a}\int_V^v \partial_v\left(|\nu|(0,v')e^{-sv'}\right)dv'\right)\\ &\leq C\left(1 + \frac{1}{a}|\nu|(0,v)e^{-sv}\right),\end{aligned} \tag{6.23}$$

where we emphasize that C depends uniquely on the initial data. Using (6.21) and the boundedness of r, we conclude that

$$|D_u\phi|(0,v) \leq C|\nu|(0,v)e^{-sv} \lesssim e^{(2K_+-s)v}, \quad \forall\, v \geq v_0,$$

when $s < 2K_+$.

On the other hand, when $s > 2K_+$, (6.21) gives:

$$|rD_u\phi|(0,v) \leq C\left(1 + \int_{v_0}^{v} e^{(2K_+-s)v'}dv'\right) \leq \tilde{C},$$

for some $\tilde{C}$ determined by the initial data and for every $v \geq v_0$. The final estimate follows from (6.11).

When $s = 2K_+$, we can integrate (6.22) and use (6.21) to get

$$|rD_u\phi|(0,v) \leq C\left(1 + \int_{v_0}^{v} |\nu|(0,v')e^{-2K_+v'}dv'\right) \leq \tilde{C}(v - v_0), \quad \forall\, v \geq v_0,$$

for some $\tilde{C} > 0$ depending on the initial data. The final estimate follows again from the boundedness of r.

Finally, (6.10) follows from (3.49). We then choose $C_{\mathcal{H}}$ as the largest of the previous positive constants depending on the initial data. □

Remark 6.3 (Redshift effect)
From the point of view of a family of observers crossing the event horizon $\mathcal{H}^+$, the energy associated to the null geodesics ruling $\mathcal{H}^+$ decays as e^{-2K_+v} due to the redshift effect.

*This is the same rate at which we found the **geometric quantity** $\left|\nu^{-1}D_u\phi\right|$ to decay for $s > 2K_+$ along the event horizon (see* (6.9)*). Here, the constant s is the same as in* (3.49)*. For $s < 2K_+$, the quantity $\left|\nu^{-1}D_u\phi\right|$ decays as e^{-sv} or faster, which is the same rate dictated by the Price law upper bound* (3.49)*. From these two different rates, we deduce the physical relevance of two competing phenomena along the event horizon: the redshift effect and the decay of the scalar field ϕ.*

6.2 Level sets of the area-radius function

The strategy of the next proofs is based on a partition of the past set $\mathcal{P}$. In the two-dimensional quotient space that we are considering, such subsets are mainly separated by curves where the area-radius function is constant. We review the main properties of the sets

$$\Gamma_\varrho := \{(u,v) \in \mathcal{P} : \; r(u,v) = \varrho\} \tag{6.24}$$

for some $\varrho \in (0, r_+)$ (see also [22]).

Lemma 6.4 (properties of Γ_ϱ)
Assume that $\Gamma_\varrho \neq \varnothing$. Then, the following set equality holds:

$$\Gamma_\varrho = \{(u_\varrho(v), v) : \; v \in [v_1, +\infty)\},$$

for some $v_1 \geq v_0$ and for some C^1 function $u_\varrho \colon [v_1, +\infty) \to \mathbb{R}$ such that $r(u_\varrho(\cdot), \cdot) \equiv \varrho$. Moreover, let

$$\underline{v} := \inf\{v \geq v_1 \colon \lambda(u_\varrho(v), v) \leq 0\}. \tag{6.25}$$

Then, the following properties are satisfied:

PΓ1. *Γ_ϱ is a C^1 curve,*

PΓ2. *We have*

$$u_\varrho'(v) = -\frac{\lambda(u_\varrho(v), v)}{\nu(u_\varrho(v), v)}, \tag{6.26}$$

PΓ3. $\lambda(u_\varrho(v), v) \le 0$ *for every* $v \ge \underline{v}$.

Proof. The existence of such a u_ϱ follows from the implicit function theorem, together with the fact that $\nu < 0$ in $\mathcal{P}$ (see remark 4.6). The regularity of Γ_ϱ follows from the regularity of u_ϱ. Furthermore, we obtain (6.26) by differentiating the relation $r(u_\varrho(v), v) = \varrho$.

Property PΓ3 is showed in [22, lemma 4.2], whose results hold in the current case as well, since they are proved just by exploiting the signs of κ and ν (see remark 4.6), the Raychaudhuri equation (3.7), (6.26) and the extension criterion (see theorem 5.1). Lemma 4.2 also shows that the domain of u_ϱ is $[v_1, +\infty)$. ☐

Lemma 6.5 (conditions to bound λ away from zero)
Let $\varrho \in (r_-, r_+)$. *Assume that* $\Gamma_\varrho \neq \emptyset$ *and that for every* $\epsilon > 0$ *there exists* $v_1 \ge v_0$ *and* $c > 0$ *such that*

$$|Q(u_\varrho(v), v) - Q_+| + |\varpi(u_\varrho(v), v) - \varpi_+| < \epsilon, \quad \forall\, v \ge v_1,$$

and

$$0 < c \le \kappa(u_\varrho(v), v) \le 1, \quad \forall\, v \ge v_1.$$

Then, there exist positive constants C_1 *and* C_2 *such that:*

$$-C_2 \le \lambda(u_\varrho(v), v) \le -C_1 < 0, \quad \forall\, v \ge v_1.$$

Proof. Assume that ϵ is sufficiently small. We then use (3.16) to write

$$\begin{aligned}(1-\mu)(u_\varrho(v), v) &= \left(1 - \frac{2}{\varrho}(\varpi - \varpi_+) - \frac{2}{\varrho}\varpi_+ + \frac{Q^2 - Q_+^2}{\varrho^2} + \frac{Q_+^2}{\varrho^2} - \frac{\Lambda}{3}\varrho^2\right)(u_\varrho(v), v) \\ &= (1-\mu)(\varrho, \varpi_+, Q_+) + O(\epsilon),\end{aligned}$$

for every $v \ge v_1$. After inspecting the plot of $(1-\mu)(\cdot, \varpi_+, Q_+)$ (see, e.g. [20, section 3]), using the smallness of ϵ, (3.33) and the bounds on $\kappa(u_\varrho(\cdot), \cdot)$, we have:

$$-C_2 \le \lambda(u_\varrho(v), v) = [\kappa(1-\mu)](u_\varrho(v), v) \le -C_1 < 0,$$

for some positive constants C_1 and C_2. ☐

We use the definitions (6.1) and (6.25) of $\mathcal{T}$ and $\underline{v}$, respectively, for the next result.

Corollary 6.6 (entering the trapped region)
If the assumptions of lemma 6.5 are satisfied, the curve Γ_ϱ *is spacelike for large values of the* v *coordinate. Moreover,* $\underline{v} < +\infty$ *and* $\Gamma_\varrho \cap \{v \ge v_1\} \subset \mathcal{T}$.

Proof. The conclusion follows from the negativity of ν in $\mathcal{P}$ and of λ in $\Gamma_\varrho \cap \{v \ge v_1\}$, for some $v_1 \ge v_0$ (see lemma 6.5). ☐

6.3 Redshift region

For some fixed R, chosen close enough to r_+ (see assumption **(F)**), we now study the region $J^-(\Gamma_R)$, after proving that it is non-empty.[1] We denote this set as the **redshift region** (see also [25, 73, 79]). Most of the estimates showed in proposition 6.2 propagate throughout the redshift region, and the interplay between the decay of the scalar field and the redshift effect that we observed along the event horizon $\mathcal{H}^+$ (see remark 6.3) is still present.

To obtain quantitative bounds in this region, it is sufficient to bootstrap the estimates on $|\phi|$, $|\partial_v\phi|$, $|D_u\phi|$, κ and $|\nu|$ from $\mathcal{H}^+$, and exploit these to achieve the remaining bounds. In particular, we employ the results of proposition 6.2 and use the smallness of $r_+ - r$ to close the bootstrap inequalities. The proof is an adaptation of the one in [79, proposition 4.5] to the case of exponential estimates. In particular, this requires a careful treatment of the Grönwall argument used to close the bootstrap for $|D_u\phi|$. The estimates that we obtain depend on the integrable quantity $u\Omega^2(0,v) \sim ue^{2K_+v}$ (see proposition 6.2).

Proposition 6.7 (propagation of estimates by redshift)
Given a fixed $\eta \in (0, K_+)$, let $\delta \in (0, \eta)$ be small compared to the initial data and to η. Moreover, let R be a fixed constant such that

$$0 < r_+ - R < \delta$$

and consider the function

$$c(s) := \begin{cases} s, & \text{if } 0 < s \leq 2K_+ - \eta, \\ 2K_+ - \eta, & \text{if } s \geq 2K_+ - \eta. \end{cases} \tag{6.27}$$

Then, given $v_1 \geq v_0$ large, whose size depends on the initial data, we have $J^-(\Gamma_R) \cap \{v \geq v_1\} \neq \varnothing$. Moreover, for every $(u,v) \in J^-(\Gamma_R) \cap \{v \geq v_1\}$, the following inequalities hold:

$$2(r(0,v) - r(u,v)) \leq u\Omega^2(0,v) \leq C\delta, \tag{6.28}$$

$$0 < r_+ - r(u,v) \leq Ce^{-2sv} + u\Omega^2(0,v), \tag{6.29}$$

$$|\lambda|(u,v) \leq Ce^{-2sv} + u\Omega^2(0,v), \tag{6.30}$$

$$|\nu|(u,v) \sim \Omega^2(0,v), \tag{6.31}$$

$$|\varpi(u,v) - \varpi_+| \leq Ce^{-2c(s)v}, \tag{6.32}$$

$$|Q(u,v) - Q_+| \leq Ce^{-2c(s)v}, \tag{6.33}$$

$$|K(u,v) - K_+| \leq C\Big(e^{-2c(s)v} + u\Omega^2(0,v)\Big), \tag{6.34}$$

$$|\partial_v \log\Omega^2(u,v) - 2K(u,v)| \leq Ce^{-2c(s)v}, \tag{6.35}$$

$$|D_u\phi|(u,v) \leq C|\nu|(u,v)e^{-c(s)v}, \tag{6.36}$$

$$|\phi|(u,v) + |\partial_v\phi|(u,v) \leq Ce^{-c(s)v}, \tag{6.37}$$

[1] By definition (6.24), $\Gamma_R = \{r = R\}$ is a (possibly empty) subset of the maximal past set $\mathcal{P}$. Since $\nu < 0$ in $\mathcal{P}$ (see remark 4.6) and due to the extension criterion (theorem 5.1), the curve being empty means that R is smaller than any value of the area-radius function in $[0,U] \times [v_0, +\infty)$. We will see that this is never the case, i.e. for every choice of $v_1 \geq v_0$ and U, we can find a curve in $[0,U] \times [v_1, +\infty)$ such that $r = R$ along it.

for a positive constant C that depends only on the initial data and on η. Additionally, we have:

$$\partial_u \lambda(u,v) < 0, \quad \forall\,(u,v) \in J^-(\Gamma_R) \cap \{v \geq v_1\}. \tag{6.38}$$

Proof. In the following, $C_{\mathcal{H}}$ denotes the constant appearing in the statement of proposition 6.2. We will use the notation

$$C = C(N_{\text{i.d.}}, \eta)$$

to denote a positive constant depending on (a suitable norm of) the initial data of our IVP and on η. We will use the same letter C to denote possibly different constants, when the exact expression of such constants is not important for the bootstrap. To close the bootstrap argument, we will require δ to be sufficiently small with respect to $N_{\text{i.d.}}$ and η. Moreover, we use that $0 < \kappa \leq 1$ and $\nu < 0$ in $\mathcal{P}$ (see (4.4) and remark 4.6) and that $\lambda_{|\mathcal{H}^+} > 0$ (see assumption **(G)**) multiple times.

Setting up the bootstrap procedure: we recall that $\mathcal{P}$ is the maximal past set where we can define a solution to the characteristic IVP of section 3.1, with initial data prescribed on $[0,U] \times \{v_0\} \cup \{0\} \times [v_0, +\infty)$. In the following, we consider $v_1 \geq v_0$ to be sufficiently large with respect to the initial data and to η, and define the set

$$\mathcal{P}_\delta \coloneqq \mathcal{P} \cap \{(u,v) \in [0,U] \times [v_1, +\infty) \colon\ 0 < r_+ - r(u,v) \leq \delta\}. \tag{6.39}$$

The latter is non-empty, since the area-radius function provides an increasing parametrization on $\mathcal{H}^+$, and $r(0,v) \to r_+$ as $v \to +\infty$ (see (3.50) and (3.54)).

We now **define E** as the set of points q in $\mathcal{P}_\delta$ such that the following four inequalities hold for every (u,v) in $J^-(q) \cap \mathcal{P}_\delta$:

$$|\phi|(u,v) + |\partial_v \phi|(u,v) \leq 4C_{\mathcal{H}} e^{-c(s)v}, \tag{6.40}$$

$$|D_u \phi|(u,v) \leq M|\nu|(u,v) e^{-c(s)v}, \tag{6.41}$$

$$\kappa(u,v) \geq \frac{1}{2}, \tag{6.42}$$

$$\frac{1}{8}\Omega^2(0,v) \leq |\nu|(u,v) \leq \Omega^2(0,v), \tag{6.43}$$

where $M > 0$ is a constant depending uniquely on the initial data.

The set E is non-empty due to the estimates we obtained along the event horizon. Moreover, we notice that, since E is a past set by construction, inequalities (6.40)–(6.43) can be integrated along causal curves ending in (u,v) and starting from the event horizon or from the null segment $[0,U] \times \{v_1\}$.

Outline of the proof: in the following, we show that estimates (6.28)–(6.38) are valid in E and, at the same time, use this result to *close the bootstrap* and show that $E = \mathcal{P}_\delta$. The conclusion then follows after proving that $\varnothing \neq J^-(\Gamma_R) \cap \{v \geq v_1\} \subset \mathcal{P}_\delta$.

Closing the bootstrap: let us now fix $(u,v) \in E$. First, we stress that we have the following bounds on the area-radius function:

$$\begin{cases} 0 < r(0,v) - r(u,v) \leq r_+ - r(u,v) \leq \delta, \\ r(u,v) > \frac{r_+}{2}, \\ r(u,v) < r_+, \end{cases} \tag{6.44}$$

due to (6.39), assumption **(G)** and by taking $\delta < \frac{r_+}{2}$.

We notice that, by (6.44) and (6.43):

$$\delta \geq r(0,v) - r(u,v) = \int_0^u |\nu|(u',v)du' \geq Cu\Omega^2(0,v). \tag{6.45}$$

In particular:

$$u\Omega^2(0,v) \leq C\delta. \tag{6.46}$$

We stress that in [79] this relation was taken as the definition of the redshift region and, on the other hand, the bounds for r were derived. A similar computation and (6.2) yield

$$r_+ - r(u,v) = r_+ - r(0,v) + r(0,v) - r(u,v) \leq Ce^{-2sv} + u\Omega^2(0,v). \tag{6.47}$$

Now, we use (3.11) and bootstrap inequalities (6.40) and (6.41) to get

$$|\partial_u Q|(u,v) \leq CMC_{\mathcal{H}} r^2(u,v)|\nu|(u,v)e^{-2c(s)v}.$$

After integrating in u and using (6.44):

$$|Q(u,v) - Q(0,v)| \leq CMC_{\mathcal{H}}\delta e^{-2c(s)v}. \tag{6.48}$$

Then, by proposition 6.2 and by (6.27):

$$|Q(u,v) - Q_+| \leq |Q(u,v) - Q(0,v)| + |Q(0,v) - Q_+| \leq C(N_{\text{i.d.}}, M)e^{-2c(s)v}. \tag{6.49}$$

Furthermore, by integrating (3.37) and using (6.40), (6.44) and the above bound on Q, we obtain

$$|r\lambda|(u,v) \leq r_+|\lambda|(0,v) + \left(C(N_{\text{i.d.}}, \eta) + O\left(e^{-2c(s)v}\right)\right)\int_0^u |\nu|(u',v)du'.$$

The decay of $\lambda_{|\mathcal{H}^+}$ proved in proposition 6.2 and (6.43), yield

$$|\lambda|(u,v) \leq Ce^{-2sv} + u\Omega^2(0,v), \tag{6.50}$$

for a positive constant $C = C(N_{\text{i.d.}}, \eta)$.

We can apply the latter bound, together with bootstrap inequalities (6.40)–(6.42) and with (3.19), to estimate ϖ. Indeed, (3.38) gives

$$|\partial_u \varpi|(u,v) \leq C|\nu|(u,v)e^{-2c(s)v}.$$

Thus, (6.44) and the bounds along the event horizon let us write

$$|\varpi(u,v) - \varpi_+| \leq |\varpi(u,v) - \varpi(0,v)| + |\varpi(0,v) - \varpi_+| \leq Ce^{-2c(s)v}. \tag{6.51}$$

The definition of K (see (3.34)), together with bounds (6.44), (6.47), (6.49) and (6.51), can be used to obtain

$$\begin{aligned}|K(u,v) - K_+| &\leq \frac{1}{r_+^2}|\varpi(u,v) - \varpi_+| + \frac{1}{r_+^3}\left|Q^2(u,v) - Q_+^2\right| + \frac{|\Lambda|}{3}|r(u,v) - r_+| \\ &\quad + |\varpi|(u,v)\left|\frac{1}{r^2(u,v)} - \frac{1}{r_+^2}\right| + Q^2(u,v)\left|\frac{1}{r^3(u,v)} - \frac{1}{r_+^3}\right| \\ &\leq C\big(e^{-2sv} + u\Omega^2(0,v)\big) + O(e^{-2c(s)v}) \leq C\Big(e^{-2c(s)v} + u\Omega^2(0,v)\Big),\end{aligned} \tag{6.52}$$

for v_1 sufficiently large.

In order to close the bootstrap for κ, we use equation (3.29), the bootstrap inequalities (6.41), (6.43) and recall that $\zeta = rD_u\phi$ to get

$$|\partial_u \log\kappa|(u,v) = \frac{r|D_u\phi|^2}{|\nu|}(u,v) \leq CM^2|\nu|(u,v)e^{-2c(s)v} \leq CM^2\Omega^2(0,v)e^{-2c(s)v}.$$

So, after integrating from $\mathcal{H}^+$, we use (3.45) and (6.44) to get:

$$\exp(-CM^2u\Omega^2(0,v)e^{-2c(s)v}) \leq \kappa(u,v) \leq 1. \tag{6.53}$$

Since $u\Omega^2(0,v)$ is bounded (see (6.46)), the above closes the bootstrap if we choose v_1 large enough.

Notice that (3.35), the positivity of K (see (6.52)) and the smallness of $|\phi|$ imply that

$$\partial_v\nu = \partial_u\lambda < 0 \tag{6.54}$$

holds in E.

We can now improve estimate (6.41) by following the procedure in [79] (see also the Grönwall inequalities exploited in [22, section 5]). Let us consider a constant $a > 0$, which will be chosen in the next lines, and define

$$f(u,v) := r(u,v)\frac{D_u\phi}{\nu}(u,v). \tag{6.55}$$

Then, (3.35) and (3.40) imply

$$\partial_v(e^{av}f(u,v)) = e^{av}\left(-\partial_v\phi + m^2r\kappa\phi + \frac{iqQ\kappa}{r}\phi\right) + e^{av}f(u,v)(a - \kappa(2K - m^2r|\phi|^2)). \tag{6.56}$$

Before continuing, we fix a in such a way that the following two conditions are satisfied: we require

$$a - \kappa\big(2K - m^2 r|\phi|^2\big)(u,v') < 0, \quad \forall\, v' \geq v_1$$

and

$$a - c(s) > \frac{\eta}{2} > 0.$$

Notice that such a choice of a is admissible due to (6.53), (6.27), (6.52) and (6.40), provided that δ is chosen sufficiently small.

This choice of a is then exploited when we integrate (6.56) and use (6.40), (6.44) and (6.49):

$$\begin{aligned} e^{av}|f|(u,v) &\leq e^{av_0}|f|(u,v_0)e^{\int_{v_0}^{v}\left(a-\kappa(2K-m^2r|\phi|^2)\right)(u,v')dv'} \\ &\quad + \int_{v_0}^{v} e^{av'}\left|-\partial_v\phi + m^2 r\kappa\phi + i\frac{qQ\kappa}{r}\phi\right|(u,v')e^{\int_{v'}^{v}\left(a-\kappa\left(2K-m^2r|\phi|^2\right)\right)(u,s)ds}dv' \\ &\leq C(N_{\text{i.d.}},v_1)(e^{av_0}|f|(u,v_0)+1) + C(N_{\text{i.d.}},\eta)\int_{v_1}^{v} e^{(a-c(s))v'}dv', \end{aligned}$$

where we used

$$\int_{v_0}^{v_1} e^{av'}\left|-\partial_v\phi + m^2 r\kappa\phi + i\frac{qQ\kappa}{r}\phi\right|(u,v')e^{\int_{v'}^{v_1}\left(a-\kappa(2K-m^2r|\phi|^2)\right)(u,s)ds}dv' \leq C(N_{\text{i.d.}},v_1),$$

with $C(N_{\text{i.d.}},v_1)$ depending on the initial data, due to theorem 4.2, and on v_1. Notice that, so far, we increased the size of v_1 based on the L^∞ norm of the initial data and on η. Thus, we have $C(N_{\text{i.d.}},v_1) = C(N_{\text{i.d.}},\eta)$.

Finally, we obtain:

$$\left|\frac{D_u\phi}{\nu}\right|(u,v) \leq C(N_{\text{i.d.}},\eta)\Big(e^{-av} + e^{-c(s)v}\Big),$$

where we emphasize that the above constant does not depend on M. The bootstrap inequality (6.41) then closes after choosing $M > 0$ sufficiently large with respect to the initial data. From now on, the constant M will be absorbed in $C(N_{\text{i.d.}},\eta)$.

Now, if we integrate (3.9) and use (6.40), (6.41), (6.49), (6.50), the fact that $\Omega^2 = 4|\nu|\kappa$, the boundedness of r and κ and, finally, (6.43):

$$\left|\partial_v \log\frac{\Omega^2(u,v)}{\Omega^2(0,v)}\right| \leq C\int_0^u |\nu|(u',v)du' \leq Cu\Omega^2(0,v). \tag{6.57}$$

Now, notice that, by assumption **(A)**:

$$\frac{\Omega^2(u,v_0)}{\Omega^2(0,v_0)} = \kappa(u,v_0) = 1 + o(1),$$

as u goes to zero.[2] So, if we integrate (6.57) along the v direction and use (6.4) and (6.43), we obtain:

$$\left|\log\frac{\Omega^2(u,v)}{\Omega^2(0,v)} + o(1)\right| \leq Cu\int_{v_0}^{v}\Omega^2(0,v')dv' \leq Cue^{2K_+v} \leq Cu\Omega^2(0,v),$$

[2] In the following, we assume that v_1 is sufficiently large. By (6.45) and (6.4), this implies that u is small.

for some $C = C(N_{\text{i.d.}}, \eta)$ that may differ from term to term. Therefore, using (6.46):

$$e^{-C\delta+o(1)} \leq e^{-Cu\Omega^2(0,v)+o(1)} \leq \frac{\Omega^2(u,v)}{\Omega^2(0,v)} \leq e^{Cu\Omega^2(0,v)+o(1)} \leq e^{C\delta+o(1)}, \tag{6.58}$$

where the $o(1)$ notation refers to the limit $v_1 \to +\infty$. Since $\Omega^2 = 4|\nu|\kappa$, the above and (6.42) are enough to close the bootstrap inequality (6.43), provided that δ is chosen small and v_1 is sufficiently large with respect to the initial data and η.

Equations (3.29), (3.41) and the fact that $\zeta = rD_u\phi$ give

$$\begin{aligned}\partial_u\big(\partial_v \log\Omega^2 - 2K\big) &= (\kappa-1)\partial_u(2K) - 2\operatorname{Re}\big(D_u\phi\overline{\partial_v\phi}\big) - \frac{2\kappa}{r^2}\left(\partial_u\varpi - \frac{2Q\partial_u Q}{r}\right), \\ &= \partial_u\left[2K(\kappa-1)\right] - 2K\frac{\kappa r|D_u\phi|^2}{\nu} - 2\operatorname{Re}\big(D_u\phi\overline{\partial_v\phi}\big) - \frac{2\kappa}{r^2}\left(\partial_u\varpi - \frac{2Q\partial_u Q}{r}\right).\end{aligned} \tag{6.59}$$

By integrating the above from the event horizon and using (3.45), (6.8), (6.40), (6.41), (6.44), (6.52), (6.53), the previous bounds on Q, $\partial_u Q$, ϖ and $\partial_u\varpi$:

$$\begin{aligned}|\partial_v \log\Omega^2(u,v) - 2K(u,v)| &\lesssim e^{-2sv} + 2K(u,v)(1-\kappa(u,v)) + e^{-2c(s)v}\int_0^u |\nu|(u',v)du' \\ &\lesssim e^{-2sv} + 1 - e^{-Ce^{-2c(s)v}} + \delta e^{-2c(s)v}.\end{aligned}$$

Now, notice that, given a fixed $C > 0$, the function $h(x) \coloneqq 1 - e^{-Ce^{-x}} - Ce^{-x}$ is non-positive. Indeed, $h(0) < 0$, $h'(x) > 0$ for every x in $\mathbb{R}$ and $\lim_{x\to+\infty} h(x) = 0$. Thus, recalling (6.27):

$$|\partial_v \log\Omega^2(u,v) - 2K(u,v)| \leq Ce^{-2c(s)v}. \tag{6.60}$$

In the next steps, we will bound ϕ and $\partial_v\phi$. By integrating $\partial_u\phi = D_u\phi - iqA_u\phi$ and using (6.10), (6.41), (6.44), we have:

$$\begin{aligned}|\phi|(u,v) &= \left|\phi(0,v)e^{-iq\int_0^u A_u(u',v)du'} + \int_0^u (D_u\phi)(u',v)e^{-iq\int_{u'}^u A_u(s,v)ds}du'\right| \\ &\leq C_{\mathcal{H}}e^{-sv} + C(N_{\text{i.d.}},\eta)e^{-c(s)v}\int_0^u |\nu|(u',v)du' \leq (C_{\mathcal{H}} + C(N_{\text{i.d.}},\eta)\delta)e^{-c(s)v}.\end{aligned} \tag{6.61}$$

By taking δ small:

$$|\phi|(u,v) < \frac{3}{2}C_{\mathcal{H}}e^{-c(s)v}. \tag{6.62}$$

Now, the wave equation (3.39) for ϕ can be integrated to give

$$\begin{aligned}(r\partial_v\phi)(u,v) = (r\partial_v\phi)(0,v)e^{-iq\int_0^u A_u(u',v)du'} + \\ + \int_0^u e^{-iq\int_{u'}^u A_u(s,v)ds}\left(-\lambda D_u\phi + \nu\kappa m^2 r\phi - i\frac{qQ\nu\kappa\phi}{r}\right)(u',v)du'.\end{aligned} \tag{6.63}$$

Then, expressions (6.10), (6.40), (6.41), (6.44), (6.49) and (6.50) yield

$$|r\partial_v\phi|(u,v) \leq r_+C_{\mathcal{H}}e^{-sv} + C(N_{\text{i.d.}},\eta)e^{-c(s)v}\int_0^u |\nu|(u',v)du' \leq r_+C_{\mathcal{H}}e^{-sv} + C(N_{\text{i.d.}},\eta)\delta e^{-c(s)v},$$

where $C(N_{\text{i.d.}}, \eta)$ may possibly differ from term to term and the quantity $O(e^{-2c(s)v})$ stemming from the estimate in Q was reabsorbed in the costants, assuming that v_1 is sufficiently large. The fact that $r(u,v) \geq \frac{r_+}{2}$ (see (6.44)), together with a suitably small choice of δ, implies

$$|\partial_v \phi|(u,v) < \frac{5}{2} C_{\mathcal{H}} e^{-c(s)v}.$$

Combined with (6.62), this result closes the bootstrap inequality (6.40). Due to the connectedness of $\mathcal{P}_\delta$, the previous steps reveal the set equality $E = \mathcal{P}_\delta$.

Localization of $J^-(\Gamma_R)$: To conclude the proof, it is enough to show that $J^-(\Gamma_R) \cap \{v \geq v_1\} \neq \varnothing$ and that $J^-(\Gamma_R) \cap \{v \geq v_1\} \subset \mathcal{P}_\delta$. First, we choose v_1 sufficiently large so that $u_R(v_1) \leq U$: this choice of v_1 is possible by the exponential decay of the u coordinate in this region (see (6.46) and recall that $\Omega^2(0,v) \sim e^{2K_+ v}$) and entails that $\Gamma_R = \{r = R\} \neq \varnothing$, due to the extension criterion (see theorem 5.1). In particular, $J^-(\Gamma_R) \cap \{v \geq v_1\} \neq \varnothing$.

Moreover, Γ_R is a spacelike curve for v_1 large (see corollary 6.6), and thus $u \leq u_R(v) \leq U$ for every (u,v) in $J^-(\Gamma_R) \cap \{v \geq v_1\}$. So, using again that $\nu < 0$ in $\mathcal{P}$, we have, for every (u,v) in $J^-(\Gamma_R) \cap \{v \geq v_1\}$:

$$0 < r_+ - r(u,v) \leq r_+ - r(u_R(v), v) = r_+ - R < \delta,$$

and thus $J^-(\Gamma_R) \cap \{v \geq v_1\} \subset \mathcal{P}_\delta$. □

Remark 6.8
By corollary 6.6 and by (6.54)*, there exists a C^1 function $u \mapsto v_R(u)$ defined for $0 \leq u \leq u_R(v_1)$ such that (see also lemma 6.4):*

$$\Gamma_R \cap \{v \geq v_1\} = \{(u_R(v), v) \colon\ v \geq v_1\} = \{(u, v_R(u)) \colon\ 0 \leq u \leq u_R(v_1)\}.$$

We considered the case $v \geq v_1$ because (6.54) *holds for large values of the v coordinate. However, notice that we might have*

$$\Gamma_R \cap \{v \geq v_1\} \subsetneq \Gamma_R,$$

see definition (6.24)*. Since we are going to study solutions to the characteristic IVP restricted to large values of the v coordinate, we will make no distinction between Γ_ϱ and $\Gamma_\varrho \cap \{v \geq v_1\}$, for $\varrho \in (0, r_+)$, whenever the restriction to large values of v is clear from the context.*

Furthermore, by differentiating the relation $r(u, v_R(u)) = R$, we can see that

$$v_R'(u) = -\frac{\nu(u, v_R(u))}{\lambda(u, v_R(u))} < 0. \tag{6.64}$$

6.4 Apparent horizon

Assume that the constants η, R and δ of proposition 6.7 are fixed. This allows us to give a parametrization of the apparent horizon

$$\mathcal{A} = \{(u, v) \in [0, U] \times [v_0, +\infty) \colon \lambda(u, v) = 0\}$$

and to show that, for large values of the v coordinate, $\mathcal{A}$ is a C^1 curve that lies in the causal past of Γ_R. We report here the main results on $\mathcal{A}$, that follow from methods analogous to those used in [22, section 4]. In particular, in the quotient manifold, $\mathcal{A} \cap \Gamma_R$ can only contain isolated points or outgoing null segments of finite length. In the far future, we can be more precise: Γ_R never intersects the apparent horizon for large values of v.

Lemma 6.9 (on the sign of λ along null cones)
Let $\mathcal{P}$ be the maximal past set where a solution to our characteristic IVP is defined. Take $(u, v) \in \mathcal{P}$. If $\lambda(u, v) = 0$ (respectively, $\lambda(u, v) < 0$), then $\lambda(u, v + V) \leq 0$ (respectively, $\lambda(u, v + V) < 0$) for every $V > 0$ for which $(u, v + V) \in \mathcal{P}$.

Proof. By (3.7):

$$\partial_v \left(\frac{\lambda}{\Omega^2} \right) \leq 0.$$

Moreover, by (3.19), (4.4) and (4.8), we know that $\Omega^2 = -4\nu\kappa > 0$ in $\mathcal{P}$. So, let us assume that $\lambda(u, v) = 0$ (the other case follows similarly). Then, for every $V > 0$ such that $(u, v + V) \in \mathcal{P}$:

$$\frac{\lambda}{\Omega^2}(u, v) = 0 \quad \text{implies} \quad \frac{\lambda}{\Omega^2}(u, v + V) \leq 0,$$

and thus $\lambda(u, v + V) \leq 0$. □

Remark 6.10
At the end of chapter 6 we will prove that $[0, U] \times [v_1, +\infty] \subset \mathcal{P}$ for a sufficiently small U and a sufficiently large v_1. Therefore, a posteriori, the statement in lemma 6.9 holds for every $V > 0$.

Corollary 6.11 (characterization of the apparent horizon)
Under the assumptions of proposition 6.7, we have that:

P$\mathcal{A}$1. *For every $\tilde{u} \in [0, U]$, the set $\Gamma_R \cap \mathcal{A} \cap \{u = \tilde{u}\}$, if non-empty, consists only of a single point or only of a null segment of finite length.*

Moreover, for $v \geq v_1$, the set $\mathcal{A}$ is a non-empty C^1 curve and the following properties are satisfied:

P$\mathcal{A}$2. *The inclusion*

$$\mathcal{A} \cap \{v \geq v_1\} \subset I^-(\Gamma_R) \tag{6.65}$$

holds;

P$\mathcal{A}$3. *The apparent horizon can be parametrized as*

$$\mathcal{A} \cap \{v \geq v_1\} = \{(u_{\mathcal{A}}(v), v) :\ v \geq v_1\}$$

for some C^1 function $u_{\mathcal{A}}$ defined on $[v_1, +\infty)$ such that $\lambda(u_{\mathcal{A}}(v), v) = 0$ on such a domain;

P$\mathcal{A}$4. *We have*

$$u'_{\mathcal{A}}(v) = -\frac{\partial_v \lambda(u_{\mathcal{A}}(v), v)}{\partial_u \lambda(u_{\mathcal{A}}(v), v)} \leq 0, \quad \forall\, v \geq v_1. \tag{6.66}$$

Proof. Let us fix $\tilde{u} \in [0, U]$ such that $\mathcal{C} \coloneqq \Gamma_R \cap \mathcal{A} \cap \{u = \tilde{u}\} \neq \varnothing$ (this is the case if $\mathcal{A} \neq \varnothing$, $v_0 \leq \underline{v}$ and we choose $\tilde{u} = u_R(\underline{v})$, where $\underline{v}$ and the function u_R are defined in lemma 6.4 for a generic curve of constant area-radius). Moreover, let us assume that there exists $p = (\tilde{u}, v_p)$, $q = (\tilde{u}, v_q) \in \mathcal{C}$ with $v_p \neq v_q$. Without loss of generality, we require $v_p < v_q$. By definition of $\mathcal{A}$, we have $\lambda(p) = \lambda(q) = 0$. By lemma 6.9, it follows that $\lambda(u, v_t) = 0$ for every $v_p < v_t < v_q$. So, in particular, if two points belong to $\mathcal{C}$, then the outgoing null segment connecting such points is entirely contained in $\mathcal{C}$. We stress that it is excluded that the above null segment has infinite length, due to the decay of the function $v \mapsto u_R(v)$ shown in (6.28) (see also (6.4)).

Notice that, for sufficiently large values of the v coordinate, the set $\mathcal{C}$ must be empty for any choice of $\tilde{u}$, due to the fact that $\lambda(u_R(v), v) < 0$ for $v \geq v_1$ (see lemma 6.5, whose assumptions are satisfied due to proposition 6.7). In the rest of the proof, we assume that $v \geq v_1$.

By (3.54), $\lambda(0, v_1) > 0$. However, $\lambda(u_R(v_1), v_1) < 0$ by lemma 6.5. So, by continuity, there exists $\tilde{u} \in]0, u_R(v_1)[$ such that $\lambda(\tilde{u}, v_1) = 0$, and thus $\mathcal{A}$ is non-empty. In particular, $\tilde{u}$ is unique due to (6.38).

Moreover, the previous reasoning shows that

$$\mathcal{A} \cap \{v \geq v_1\} \cap I^-(\Gamma_R) \neq \varnothing.$$

To prove property **P$\mathcal{A}$2**, it is sufficient to show that $\mathcal{A} \cap \{v \geq v_1\} \cap J^+(\Gamma_R) = \varnothing$. This is enough, since Γ_R is a spacelike curve for $v \geq v_1$ (see corollary 6.6).

To show the above, we notice that, again by lemma 6.5, we have $\lambda(u_R(v), v) < 0$ for $v \geq v_1$. But then, by lemma 6.9 and since Γ_R is spacelike in this region, we necessarily have $\lambda < 0$ in $J^+(\Gamma_R)$.

The existence of the function $u_{\mathcal{A}}$ follows from the implicit function theorem and from (6.38), which holds in the current case due to property **P$\mathcal{A}$2**. The regularity of $u_{\mathcal{A}}$ also implies the regularity of the curve $\mathcal{A}$.

The expression for $u'_{\mathcal{A}}$ can be obtained by differentiating the relation $\lambda(u_{\mathcal{A}}(v), v) = 0$. Moreover, the sign of $u'_{\mathcal{A}}$ in (6.66) is determined by the fact that $\partial_u \lambda < 0$ in $J^-(\Gamma_R)$ and that $\partial_v \lambda_{|\mathcal{A}} \leq 0$ (see lemma 6.9). ☐

6.5 No-shift region

In regions corresponding to smaller values of the area-radius, the redshift effect is counterbalanced by a blueshift effect which is expected to originate in the Cauchy horizon (whose existence and stability will be proved as a consequence of the results of chapter 6). A by-product of this phenomenon is the propagation, throughout this new region, of most of the estimates already obtained in proposition 6.7. We recall that the constants η, δ and R can be fixed according to the statement of proposition 6.7, and that $r_- \in (0, r_+)$ is one of the roots of the function $(1-\mu)(\cdot, \varpi_+, Q_+)$. A more precise formulation of the above heuristics is then the following: let us choose[3] $Y \in (r_-, R)$, close enough to r_-, and work in the set

$$J^+(\Gamma_R) \cap J^-(\Gamma_Y), \tag{6.67}$$

which we denote by the name **no-shift region**, after proving that it is non-empty. The following properties hold in this set:

- the functions $r_+ - r(\cdot,\cdot)$ and $r(\cdot,\cdot) - r_-$ are bounded, but not necessarily δ–small,
- the surface gravity K of the dynamical black hole changes sign: it remains positive near Γ_R and becomes negative near Γ_Y. In this context, we introduce the quantity

$$K_- := \frac{1}{2}|\partial_r(1-\mu)(r_-, \varpi_+, Q_+)|, \tag{6.68}$$

  namely the absolute value of the surface gravity along the Cauchy horizon[4] of the final sub-extremal black hole.

In particular, the proof of proposition 6.7, in which the smallness of $r_+ - r$ is used to close the bootstrap inequalities, cannot be directly applied to this region. Nonetheless, we can subdivide the region (6.67) in finitely many narrow subsets on which we have a finer control on the area-radius function. The validity of the respective estimates is then proved in each subset by induction, where the basis step follows from the estimates on the redshift region, while the inductive step is proved with the aid of bootstrap (this technique was used in [79], see also [27]). A key role will be played by the bound on the geometric quantity $1-\mu$, which lets us relate the quantity ν defined in this region to $\nu_{|\Gamma_R}$. We stress that, in the uncharged case [22], the bound on $1-\mu$ could be obtained directly from the monotonicity of the renormalized Hawking mass. Such a property is not present in the current case.

Proposition 6.12 (propagation of estimates due to coexisting redshift and blueshift effects)
Let $\Delta > 0$ be small, compared to the initial data, and let $Y > 0$ be such that

$$0 < Y - r_- < \Delta.$$

Then, given $v_1 \geq v_0$ large, we have $J^+(\Gamma_R) \cap J^-(\Gamma_Y) \cap \{v \geq v_1\} \neq \varnothing$ and, for every (u,v) in $J^+(\Gamma_R) \cap J^-(\Gamma_Y) \cap \{v \geq v_1\}$, the following relations hold:

$$-C_{\mathcal{N}} \leq \lambda(u,v) \leq -c_{\mathcal{N}} < 0, \tag{6.69}$$

[3] The letter Y, which denotes the constant radius in the region where redshift and blueshift effects balance each other, stands for the yellow color, which lies between the red and blue colors in the visible electromagnetic spectrum.

[4] We adopt the convention of [22, 73], where K_- is a positive quantity. In [79], this quantity is defined with the opposite sign.

$$|\nu|(u,v) \sim |\nu|(u, v_R(u)) \tag{6.70}$$

$$|\varpi(u,v) - \varpi_+| \leq C_{\mathcal{N}} e^{-2c(s)v}, \tag{6.71}$$

$$|Q(u,v) - Q_+| \leq C_{\mathcal{N}} e^{-2c(s)v}, \tag{6.72}$$

$$|K(u,v) - K_+| + |K(u,v) + K_-| \leq C_{\mathcal{N}}, \tag{6.73}$$

$$|\partial_v \log \Omega^2(u,v) - 2K(u,v)| \leq C_{\mathcal{N}} e^{-2c(s)v}, \tag{6.74}$$

$$|D_u\phi|(u,v) \leq C_{\mathcal{N}} |\nu|(u,v) e^{-c(s)v}, \tag{6.75}$$

$$|\phi|(u,v) + |\partial_v \phi|(u,v) \leq C_{\mathcal{N}} e^{-c(s)v}, \tag{6.76}$$

$$u \sim e^{-2K_+ v}, \tag{6.77}$$

$$v - v_R(u) \leq C_{\mathcal{N}}, \tag{6.78}$$

for some positive constants $C_{\mathcal{N}}, c_{\mathcal{N}}$ *depending on the initial data, and possibly on* η*,* R *and* Y*. Here,* v_R *is the function that gives a parametrization of* Γ_R *as*

$$\Gamma_R = \{r = R\} = \{(u, v_R(u)) : \ u \in [0, u_R(v_1)]\},$$

and $c(s)$ *was defined in* (6.27)*.*

Proof. We recall that there exists a value $r_0 \in (r_-, r_+)$ such that[5] $\partial_r(1-\mu)(r_0, \varpi_+, Q_+) = 0$.

Definitions of the constants: For every positive integer l, we define the following constants inductively:

$$\begin{cases} C_l := 4C_{l-1}, \\ C_0 := 4C_{\mathcal{H}}, \end{cases} \tag{6.79}$$

where $C_{\mathcal{H}}$ is the constant given by proposition 6.2, and we consider[6] $N \in \mathbb{N}$, which will be chosen sufficiently large during the proof, based on the initial data, on R and on Y. Moreover, let

$$\epsilon = \epsilon(N) := \frac{R - Y}{N} > 0. \tag{6.80}$$

For $l = 0, \dots, N$, we define the sets

$$\mathcal{N}_l := J^+(\Gamma_R) \cap \{v \geq v_1\} \cap \{(u,v) \in \mathcal{P} : \ R - l\epsilon \leq r(u,v) \leq R - (l-1)\epsilon\}$$

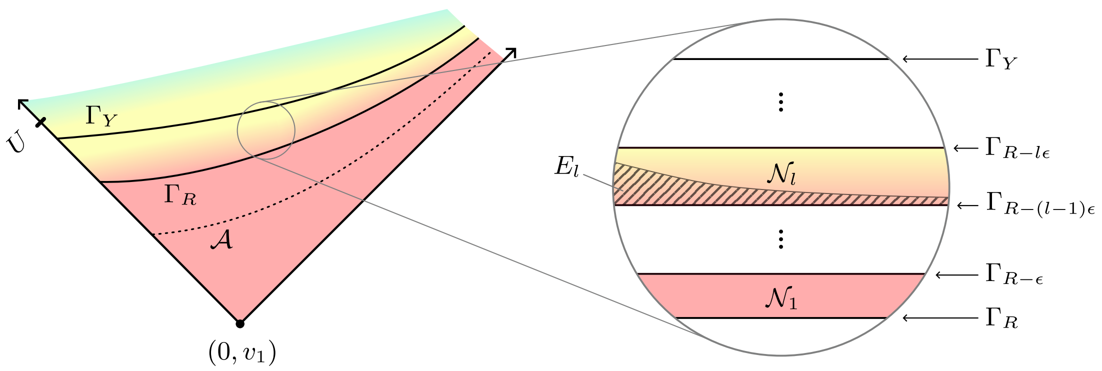
and

$$\mathcal{N} := \bigcup_{l=0}^{N} \mathcal{N}_l.$$

We observe that, for every $l = 0, \dots, N$, we have $\mathcal{N}_l \neq \varnothing$, by the extension criterion (see theorem 5.1). Since $\nu < 0$ in $\mathcal{P}$ and $\lambda < 0$ in $J^+(\Gamma_R)$ (see lemma 6.5, which holds along Γ_R, and lemma 6.9), these newly defined sets are connected. Finally, it is also straightforward to see that $\mathcal{N}_0 = \Gamma_R \cap \{v \geq v_1\}$. In the following, we will repeatedly use that $0 < \kappa \leq 1$ in $\mathcal{P}$ (see (4.4)).

[5] See also section 3 in [20] for a more detailed analysis of the function $r \mapsto (1-\mu)(r, \varpi_+, Q_+)$.

[6] In particular, our choice will imply that $N \to +\infty$ if either $R \to r_+$ or $Y \to r_-$.



1. Induction: We want to prove, by induction, that for every $l \in \{0, \ldots, N\}$ the following holds:

$$|\phi|(u,v) + |\partial_v \phi|(u,v) \le C_l e^{-c(s)v}, \tag{6.81}$$

$$|D_u \phi|(u,v) \le MC_l |\nu|(u,v) e^{-c(s)v}, \tag{6.82}$$

$$\kappa(u,v) \ge \frac{1}{2}, \tag{6.83}$$

$$|\partial_v \log \Omega^2(u,v) - 2K(u,v)| \le Ce^{-2c(s)v}, \tag{6.84}$$

for every $(u,v) \in \mathcal{N}_l$, for some constants $C = C(N_{\text{i.d.}}, \eta, R, Y)$ and $M = M(N_{\text{i.d.}}, \eta, R, Y)$. As we will see, this result implies that inequalities (6.69)–(6.78) are satisfied. The above inequalities hold for $l = 0$ by the results obtained in the redshift region (see, in particular, (6.35), (6.40)–(6.42)) and by choosing M large compared to the initial data.

Let us now consider the inductive step: let $l \in \{1, \ldots, N\}$ be fixed and assume that, for every $i = 0, \ldots, l-1$, inequalities (6.81)–(6.84) are satisfied in $\mathcal{N}_i$. To prove that these inequalities also hold for $i = l$ in $\mathcal{N}_l$, we adapt the bootstrap techniques of [79].

2. Bootstrap: we define E_l as the set of points q in $\mathcal{N}_l$ such that the following inequalities hold for every (u,v) in $J^-(q) \cap \mathcal{N}_l$:

$$|\phi|(u,v) + |\partial_v \phi|(u,v) \le C_l e^{-c(s)v}, \tag{6.85}$$

$$|D_u \phi|(u,v) \le MC_l |\nu|(u,v) e^{-c(s)v}, \tag{6.86}$$

$$\kappa(u,v) \ge \frac{1}{2}. \tag{6.87}$$

The set E_l is non-empty since (6.85)–(6.87) hold in $\{r = R - (l-1)\epsilon\} \subset \partial\mathcal{N}_{l-1} \cap \partial\mathcal{N}_l$, by the induction hypothesis. Now, let $(u,v) \in E_l$ be fixed.

We will repeatedly use the following estimates for r:

$$\begin{cases} r_- < Y \le r \le R < r_+, & \text{in } \mathcal{N}, \\ 0 \le R - r(u,v) \le R - Y, \\ 0 \le R - (l-1)\epsilon - r(u,v) \le \epsilon, \\ 0 \le r(u,v) - (R - l\epsilon) \le \epsilon. \end{cases} \tag{6.88}$$

Now, equation (3.11), together with (6.85), (6.86) and (6.88), gives

$$|\partial_u Q|(u,v) \le CMC_l^2 |\nu|(u,v) e^{-2c(s)v}. \tag{6.89}$$

If we integrate the latter from the curve Γ_R and use the results of proposition 6.7 and (6.79):

$$|Q(u,v) - Q_+| \le |Q(u,v) - Q(u_R(v),v)| + |Q(u_R(v),v) - Q_+| \le C(N_{\text{i.d.}}, \eta, M) e^{-2c(s)v}. \tag{6.90}$$

Notice that the above constant depending on M is coupled with the term $e^{-2c(s)}$, which decays faster than the leading-order term of the next inequalities that we take into account.

We can proceed similarly for ϖ by using (3.19) and integrating equation (3.38) from Γ_R. Indeed, we can use the fact that $|\lambda|$ is bounded from above by a positive constant (this can be proved by following the same steps that led to (6.50) in the redshift region), together with (6.85)–(6.87), (6.88), (6.90) and the bounds obtained in the redshift region, to write

$$|\partial_u \varpi|(u,v) \le C|\nu|(u,v) e^{-2c(s)v} \tag{6.91}$$

and thus

$$|\varpi(u,v) - \varpi_+| \le |\varpi(u,v) - \varpi(u_R(v),v)| + |\varpi(u_R(v),v) - \varpi_+| \le Ce^{-2c(s)v}. \tag{6.92}$$

The inequality

$$-C(N_{\text{i.d.}}) < (1-\mu)(u,v) < -C(N_{\text{i.d.}}, R, Y) < 0, \tag{6.93}$$

follows from (6.90), (6.92), (6.87) and lemma 6.5 (its proof does not require r to be constant: it is sufficient that the area-radius is bounded away from r_- and r_+ as in the current case). In particular, $(1-\mu)(u,v)$ approaches zero when either $R \to r_+$ or $Y \to r_-$. Whence (6.87) entails

$$-C(N_{\text{i.d.}}) < \lambda(u,v) = [\kappa(1-\mu)](u,v) < -\frac{C(N_{\text{i.d.}}, R, Y)}{2} < 0. \tag{6.94}$$

We now show the validity of (6.77). Due to (6.28):

$$2(r(0, v_R(u)) - R)\Omega^{-2}(0, v_R(u)) \le u \le C\delta\Omega^{-2}(0, v_R(u)), \tag{6.95}$$

where v_R was introduced in remark 6.8. Furthermore, the difference between $v_R(u)$ and v is bounded. In fact, if we integrate (6.94) from $v_R(u)$ to v:

$$-C(N_{\text{i.d.}})(v - v_R(u)) < r(u,v) - R < -\frac{C(N_{\text{i.d.}}, R, Y)}{2}(v - v_R(u)).$$

Hence, due to (6.88):

$$0 \le v - v_R(u) \le 2\frac{R - r(u,v)}{C(N_{\text{i.d.}}, R, Y)} \le 2\frac{R - Y}{C(N_{\text{i.d.}}, R, Y)}. \tag{6.96}$$

Since $\Omega^2(0,y) \sim e^{2K_+ y}$ for every y, if we plug (6.96) in (6.95) then (6.77) is satisfied.

Now, using (3.19) and (3.33), the Raychaudhuri equation (3.7) can be cast in the form

$$\partial_v \left(\frac{\nu}{1-\mu} \right) = \frac{\nu}{1-\mu} \frac{r|\partial_v \phi|^2}{\lambda}, \tag{6.97}$$

to get a bound on ν. Indeed, by integrating the above from $v_R(u)$ to v, we have

$$|\nu|(u,v) = |\nu|(u, v_R(u)) \frac{|1-\mu|(u,v)}{|1-\mu|(u, v_R(u))} e^{\int_{v_R(u)}^{v} \frac{r}{\lambda} |\partial_v \phi|^2 (u,v') dv'}. \tag{6.98}$$

By (6.85), (6.88) and (6.94):

$$\left| \int_{v_R(u)}^{v} \frac{r}{\lambda} |\partial_v \phi|^2 (u,v') dv' \right| \leq C(N_{\text{i.d.}}, R, Y) e^{-2c(s) v_R(u)}. \tag{6.99}$$

Thus, (6.98) and (6.93) give

$$|\nu|(u,v) \sim |\nu|(u, v_R(u)). \tag{6.100}$$

We can now close bootstrap inequality (6.86). By inspecting (3.40) and using (6.85), (6.88), (6.90) and (6.100), we have

$$|\partial_v (r D_u \phi)|(u,v) \leq C(N_{\text{i.d.}}) C_l |\nu|(u,v) e^{-c(s)v} \leq C(N_{\text{i.d.}}, R, Y) C_l |\nu|(u, v_R(u)) e^{-c(s)v},$$

for v_1 large. Notice that the above constant does not depend on M. Before integrating the above inequality from Γ_R, we notice that (6.36) gives

$$|r D_u \phi|(u, v_R(u)) \leq C(N_{\text{i.d.}}, \eta) |\nu|(u, v_R(u)) e^{-c(s) v_R(u)}$$

Therefore, the relation $C_l = 4^{l+1} C_{\mathcal{H}}$ and estimates (6.96) and (6.100) allow us to write

$$\begin{aligned} |r D_u \phi|(u,v) &\leq (C(N_{\text{i.d.}}, \eta) + C(N_{\text{i.d.}}, R, Y)) C_l |\nu|(u, v_R(u)) e^{c(s)(v - v_R(u))} e^{-c(s)v} \\ &\leq C(N_{\text{i.d.}}, \eta, R, Y) C_l |\nu|(u,v) e^{-c(s)v}. \end{aligned}$$

Inequality (6.86) is then closed due to (6.88) and by choosing M sufficiently large with respect to the initial data, η, R and Y (this might possibly require a larger value for v_1, as well).

The bootstrap on κ can then be closed by following the same steps that precede (6.53) in the proof of proposition 6.7.

Now, $\Gamma_{R-(l-1)\epsilon}$ is non-empty (see the end of the proof of proposition 6.7). Lemma 6.4 then shows that we have the parametrization $\Gamma_{R-(l-1)\epsilon} = \{(u_{l-1}(v), v) : \ v \geq v_1\}$ for some function u_{l-1}.

In order to close the bootstrap for ϕ and $\partial_v \phi$, we first notice that, by the inductive assumption (i.e. (6.81) for the $(l-1)$-th step) and by (6.79), we have:

$$|\phi|(u_{l-1}(v), v) \leq \frac{C_l}{4} e^{-c(s)v}.$$

Then, by integrating $\partial_u\phi = D_u\phi - iqA_u\phi$ and using (6.86) and (6.88), we obtain

$$|\phi|(u,v) \le \frac{C_l}{4}e^{-c(s)v} + C_l C(N_{\text{i.d.}}, R, Y)e^{-c(s)v}\int_{u_{l-1}(v)}^{u} |\nu|(u',v)du' \le \left(\frac{1}{4} + C(N_{\text{i.d.}}, R, Y)\epsilon\right)C_l e^{-c(s)v},$$

By choosing N sufficiently large (see (6.80)), the last constant multiplying $C_l e^{-c(s)v}$ is strictly smaller than $\frac{3}{8}$.

Similarly, we integrate (3.39) from $u_{l-1}(v)$ to u (see also (6.63)) and use (6.85), (6.86), (6.88), (6.94) and the inductive assumption (see (6.81) for the $(l-1)$-th step) to obtain

$$\begin{aligned}|r\partial_v\phi|(u,v) &\le \frac{C_l}{4}(R-(l-1)\epsilon)e^{-c(s)v} + C_l C(N_{\text{i.d.}}, R, Y)e^{-c(s)v}\int_{u_{l-1}(v)}^{u} |\nu|(u',v)du' \\ &\le C_l e^{-c(s)v}\left(\frac{1}{4}(R-(l-1)\epsilon) + C(N_{\text{i.d.}}, R, Y)\epsilon\right),\end{aligned}$$

assuming that v_1 is sufficiently large. If we divide both sides of the above by $r(u,v) \ge R - l\epsilon$ (see (6.88)), the first term enclosed in brackets can be estimated using that $R - l\epsilon \ge Y$ and that, taking N large:

$$\frac{R-l\epsilon+\epsilon}{R-l\epsilon} < 1 + \frac{\epsilon}{Y} < 2.$$

Thus, for N large:

$$|\partial_v\phi|(u,v) < \frac{5}{8}C_l e^{-c(s)v}.$$

This result and the previous bound on ϕ close the bootstrap inequality (6.85). Hence, we have $E_l = \mathcal{N}_l$.

3. Remaining estimates: We are now left to prove (6.84) in $\mathcal{N}_l$. Similarly to the case of the redshift region, this follows after integrating (6.59) from $\Gamma_{R-(l-1)\epsilon}$, and using (6.84) (in $\mathcal{N}_{l-1}$), (6.85), (6.86), (6.88), (6.89), (6.90), (6.91), (6.92) and the fact that K is bounded. In particular, given $(u,v) \in \mathcal{N}_l$:

$$\begin{aligned}|\partial_v \log\Omega^2(u,v) - 2K(u,v)| &\le |\partial_v \log\Omega^2 - 2K|(u_{l-1}(v),v) + [|2K|(1-\kappa)]\,(u,v) \\ &\quad + [|2K|(1-\kappa)]\,(u_{l-1}(v),v) + Ce^{-2c(s)v}\int_{u_{l-1}(v)}^{u}|\nu|(u',v)du' \\ &\le C\left(1 - e^{-Ce^{-2c(s)v}} + e^{-2c(s)v}\right) \\ &\le Ce^{-2c(s)v},\end{aligned}$$

for v_1 sufficiently large. In particular, we used that $x \mapsto 1 - e^{-Ce^{-x}} - Ce^{-x}$ is a non-positive function, as seen right after (6.59).

It then follows that inequalities (6.69)–(6.78) are satisfied in $\mathcal{N}$. Moreover, we showed in the previous steps that $\Gamma_{R-(l-1)\epsilon} \neq \varnothing$, for $l \in \{1, \dots, N\}$. The last step of the induction yields $\varnothing \neq \Gamma_Y \cap \{v \ge v_1\} \subset \mathcal{N}$, and we further know that Γ_Y is a spacelike curve, for $v \ge v_1$, by corollary 6.6. The latter result, together with the fact that $\nu < 0$ in $\mathcal{P}$, entails that $\varnothing \neq J^+(\Gamma_R) \cap J^-(\Gamma_Y) \cap \{v \ge v_1\} \subset \mathcal{N}$ (see last paragraph of the proof of proposition 6.7). ☐

6.6 Early blueshift region: general behaviour

In the next steps, we inspect the consequences of the blueshift effect in the region $J^+(\Gamma_Y)$. Given a real number

$$\beta > 0, \tag{6.101}$$

sufficiently small compared to the initial data, we define the curve

$$\gamma := \Big\{(u, (1+\beta)v_Y(u)) \colon \; u \in [0, u_Y(v_1)]\Big\}. \tag{6.102}$$

For a suitably large choice of v_1, the curve γ is spacelike, since that is the case for $\Gamma_Y = \{(u, v_Y(u)) \colon u \in [0, u_Y(v_1)]\}$. Indeed, after recalling the expression of g in (3.3), it is immediate to verify that $g(X, X) > 0$, $X = \partial_u + (1+\beta)v_Y' \partial_v$ being the vector field tangent to γ. We also use the notation

$$v_\gamma(u) := (1+\beta)v_Y(u),$$

and we denote by $u_\gamma(v)$ the u-coordinate of the unique point along the curve γ corresponding to the outgoing null coordinate v.

The aim of the curve γ is to probe a spacetime region where the blueshift effect starts to be relevant, but whose influence is weak enough so that the quantities describing the dynamical black hole can be controlled as done in the previously inspected regions. This is why we denote the region

$$J^+(\Gamma_Y) \cap J^-(\gamma)$$

by the name **early blueshift region**.

Here the dynamical solution is still quantitatively close to the final sub-extremal Reissner-Nordström-de Sitter black hole, in the sense that ϖ converges to ϖ_+ and Q converges to Q_+ at an exponential rate. However, due to the non-zero value of β (which distinguishes γ from a curve of constant radius), we lose decay with respect to the no-shift region. Also, although this region is delimited by a curve of constant area-radius only in the past, the function r is still bounded away from zero for v_1 large.

Some technical difficulties, which were absent in the uncharged $\Lambda > 0$ case and in the charged $\Lambda = 0$ case, arise in this region. Differently from [20] and [22], we cannot directly apply BV estimates (originally presented in [27]) on the quantities $D_u\phi$ and $\partial_v\phi$, due to new terms in equations (3.27) and (3.28) that were absent in the uncharged case. We then prove the results by bootstrap, similarly to the procedure followed in the redshift region, where now we close the bootstrap inequalities by exploiting the smallness of $r - r_-$. The procedure differs from the bootstrap technique used in [79], since the approximations of the v coordinate used there give rise to exponential errors that cannot be removed[7] using the smallness of $r - r_-$, see also section 2.4.

A prominent role in the proof is played by $\mathcal{C}_s$, a function in both the u and v coordinates that we

[7]Notice that that the same problem occurred in the no-shift region of section 6.5. However, these error terms could be controlled due to (6.78).

define and that determines the exponential decay of the main quantities. This function is positive in the early blueshift region[8] and replaces the linear function $v \mapsto c(s)v$ of the previous regions. The aim of $\mathcal{C}_s$ is to interpolate between the exponential decay known along the curve Γ_Y (see proposition 6.12) and the one expected along γ (see [22]), so that the function $|\nu| \exp(-\mathcal{C}_s)$, appearing in many integral estimates, is monotonically increasing in v. In particular, $\mathcal{C}_s$ is chosen in such a way that it is monotonic in both coordinates, so that the main integral estimates of this region can be achieved similarly to the ones of the redshift and no-shift regions.

Proposition 6.13 (estimates in the early blueshift region)
Let $\beta > 0$. Then, for every $\varepsilon > 0$ small enough, depending only on the initial data, there exists $\Delta \in (0, \varepsilon)$ and $Y > 0$ such that

$$0 < Y - r_- < \Delta, \tag{6.103}$$

and such that the following inequalities hold for every $(u, v) \in J^+(\Gamma_Y) \cap J^-(\gamma) \cap \{v \geq v_1\}$:

$$r(u, v) \geq r_- - \varepsilon, \tag{6.104}$$

$$|\varpi(u, v) - \varpi_+| \leq Ce^{-2(c(s)-\tau)v}, \tag{6.105}$$

$$|Q(u, v) - Q_+| \leq Ce^{-2(c(s)-\tau)v}, \tag{6.106}$$

$$|K(u, v) + K_-| \leq C\varepsilon, \tag{6.107}$$

$$|\partial_v \log \Omega^2(u, v) - 2K(u, v)| \leq Ce^{-(c(s)-\tau)v}, \tag{6.108}$$

$$|D_u\phi|(u, v) \leq C|\nu|(u, v)e^{-(c(s)-\tau)v}, \tag{6.109}$$

$$|\phi|(u, v) + |\partial_v\phi|(u, v) \leq Ce^{-(c(s)-\tau)v}, \tag{6.110}$$

for some $v_1 \geq v_0$ large, where $\tau > 0$, $\tau = O(\beta)$ and where $C > 0$ depends on the initial data and possibly on η, R and Y. The curve γ is defined in (6.102)*. The positive constant K_- is given by* (6.68) *and $c(s)$ is defined in* (6.27)*.*

Proof. In the following, we will repeatedly use that $0 < \kappa \leq 1$, that $\nu < 0$ in $\mathcal{P}$ and that $\lambda < 0$ in $J^+(\Gamma_Y)$ (this follows from (6.69) and lemma 6.9, together with the fact that Γ_Y is a spacelike curve). In the next lines the constant $C_{\mathcal{N}} = C_{\mathcal{N}}(N_{\text{i.d.}}, \eta, R, Y)$ (defined in the statement of proposition 6.12) will be used.

Given $C > 0$ depending uniquely on the initial data, we define

$$\mathcal{C}_s(u, v) := c(s)v_Y(u) - 2(2K_- + C\varepsilon)(v - v_Y(u)), \tag{6.111}$$

which is positive in $J^+(\Gamma_Y) \cap J^-(\gamma)$. Recall that, in the latter region, $v_Y(u) \leq v \leq (1 + \beta)v_Y(u)$. So, if β is small:

$$\mathcal{C}_s(u, v) = c(s)v_Y(u) - 2(2K_- + C\varepsilon)(v - v_Y(u)) \geq [c(s) - 2(2K_- + C\varepsilon)\beta]v_Y(u) > 0,$$

for every $(u, v) \in J^+(\Gamma_Y) \cap J^-(\gamma) \cap \{v \geq v_1\}$. In particular, $\mathcal{C}_s$ is a **positive function in the early**

[8] It is not necessarily positive to the future of this region, but this will not be a problem, see already section 6.8.

blueshift region. Using the restrictions on the v coordinate again, the above also implies

$$\mathcal{C}_s(u,v) > (c(s)-\tau)v, \tag{6.112}$$

for $\tau > 0$, $\tau = O(\beta)$. Furthermore, the fact that $v_Y'(u) < 0$ for every $u \in [0, u_R(v_1)]$ (see also (6.64)) gives

$$\partial_u \mathcal{C}_s < 0 \quad \text{in } J^+(\Gamma_Y) \cap J^-(\gamma) \cap \{v \geq v_1\}. \tag{6.113}$$

Setting up the bootstrap procedure: let E be the set of points q in $\mathcal{D} := J^+(\Gamma_Y) \cap J^-(\gamma) \cap \{v \geq v_1\}$ such that the following inequalities hold for every (u,v) in $J^-(q) \cap \mathcal{D}$:

$$r(u,v) \geq r_- - 2\varepsilon, \tag{6.114}$$

$$|\phi|(u,v) + |\partial_v \phi|(u,v) \leq 4C_{\mathcal{N}} e^{-\mathcal{C}_s(u,v)}, \tag{6.115}$$

$$|D_u \phi|(u,v) \leq M C_{\mathcal{N}} |\nu|(u,v) e^{-\mathcal{C}_s(u,v)}, \tag{6.116}$$

$$\kappa(u,v) \geq \frac{1}{2}, \tag{6.117}$$

for some suitably large $M = M(N_{\text{i.d.}}) > 0$.

The validity of the bootstrap inequalities along Γ_Y is promptly ensured for v_1 large, since $Y > r_-$, due to (6.76), (6.75), (6.83) and to the fact that (see (6.111)):

$$\mathcal{C}_s(u, v_Y(u)) = c(s) v_Y(u), \quad \forall\, u \in [0, u_Y(v_1)].$$

Closing the bootstrap: let $(u,v) \in E$. First, we notice that, by construction of E:

$$0 < \frac{r_-}{2} < r_- - 2\varepsilon \leq r(u,v) \leq Y < r_+, \tag{6.118}$$

provided that ε is sufficiently small.

Similarly to the previous regions (and now using (6.113)), the inequalities (6.115), (6.116), (6.118) applied to (3.11) and the results of proposition 6.12 yield

$$|Q(u,v) - Q_+| \leq |Q(u,v) - Q(u_Y(v),v)| + |Q(u_Y(v),v) - Q_+| \leq C(N_{\text{i.d.}}, \eta, R, Y) e^{-2\mathcal{C}_s(u,v)}, \tag{6.119}$$

where, in particular, we used $e^{-c(s)v_Y(u)} \leq e^{-\mathcal{C}_s(u,v)}$.

Furthermore, integrating (3.37) from Γ_Y and using (6.115), (6.118) and (6.119), together with the bound (6.94) obtained in the no-shift region:

$$|\lambda|(u,v) \leq C = C(N_{\text{i.d.}}), \tag{6.120}$$

In order to bound ϖ, we consider (3.38) and apply (3.19), the bootstrap inequalities (6.115), (6.116), (6.117), the previous bounds on $|\lambda|$ and Q and finally integrate using (6.118) and (6.113) to get

$$|\varpi(u,v)-\varpi_+| \le |\varpi(u,v)-\varpi(u,v_Y(u))| + |\varpi(u,v_Y(u))-\varpi_+| \lesssim e^{-2\mathcal{C}_s(u,v)}. \tag{6.121}$$

Now, since $J^+(\Gamma_Y)\cap J^-(\gamma)$ is stricly contained in the trapped region $\mathcal{T}$ for v large (see (6.65)), relation (3.33) entails that $(1-\mu)(u,v)<0$. In particular, by (3.16):

$$(1-\mu)(u,v) = 1 - \frac{2}{r}\varpi_+ + \frac{Q_+^2}{r^2} - \frac{\Lambda}{3}r^2 - \frac{2}{r}(\varpi(u,v)-\varpi_+) + \frac{Q(u,v)^2-Q_+^2}{r^2} < 0.$$

Due to (6.119) and (6.121), the latter implies that

$$(1-\mu)(r(u,v),\varpi_+,Q_+) < \frac{2}{r}(\varpi(u,v)-\varpi_+) - \frac{Q(u,v)^2-Q_+^2}{r^2} \lesssim e^{-2\mathcal{C}_s(u,v)}.$$

Thus, by studying the plot of $r \mapsto (1-\mu)(r,\varpi_+,Q_+)$ (see e.g. [20, section 3]), we have

$$r(u,v) > r_- - \varepsilon, \tag{6.122}$$

provided that v_1 is sufficiently large. This closes the bootstrap inequality (6.114) for r.

We now use (3.34) and (6.68) to write

$$\begin{aligned}|K(u,v)+K_-| &= \left|\frac{\varpi}{r^2}(u,v) - \frac{Q^2}{r^3}(u,v) - \frac{\Lambda}{3}r(u,v) - \left(\frac{\varpi_+}{r_-^2} - \frac{Q_+^2}{r_-^3} - \frac{\Lambda}{3}r_-\right)\right| \\ &\le |\varpi|(u,v)\left|\frac{1}{r^2(u,v)} - \frac{1}{r_-^2}\right| + \frac{|\varpi(u,v)-\varpi_+|}{r_-^2} \\ &\quad + Q^2(u,v)\left|\frac{1}{r^3(u,v)} - \frac{1}{r_-^3}\right| + \frac{|Q^2(u,v)-Q_+^2|}{r_-^3} + \frac{|\Lambda|}{3}|r(u,v)-r_-|.\end{aligned}$$

The terms proportional to $|\varpi-\varpi_+|$ and $|Q-Q_+|$ give $O(e^{-2\mathcal{C}_s(u,v)})$, due to the previously obtained estimates. Moreover, we note that the inequality $r(u,v)\le Y$, together with (6.103), (6.122) and the fact that $\Delta<\varepsilon$ gives:

$$|r(u,v)-r_-| < \max\{\Delta,\varepsilon\} = \varepsilon.$$

On the other hand, from (6.118) and the above we obtain:

$$\left|\frac{1}{r^2(u,v)} - \frac{1}{r_-^2}\right| < C(N_{\text{i.d.}})\varepsilon,$$

and similarly for the cubic case. Therefore, we conclude that

$$|2K(u,v)+2K_-| \le C(N_{\text{i.d.}})\varepsilon, \tag{6.123}$$

for v_1 sufficiently large. In particular, K is a negative quantity in the early blueshift region.

The bootstrap condition on κ can be closed as done in the redshift region (see e.g. (6.53)), now using (3.29) (recall that $\zeta = rD_u\phi$), (6.116), (6.118) and the fact that κ is close to 1 in the no-shift region. We emphasize that κ can be taken close to 1 also in the current region, provided that v_1 is sufficiently large. Hence, for a suitable choice of v_1:

$$|\kappa(2K - m^2 r|\phi|^2)| > (1 - o(1))(2K_- - C\varepsilon) > 2K_- - 2C\varepsilon > 0, \tag{6.124}$$

due to the smallness of ε, where $C > 0$ depends uniquely on the initial data.

The estimate for $\partial_v \log \Omega^2$ is obtained analogously to the procedure of the previous regions, i.e. by integrating (6.59) from Γ_Y and using (6.74), (6.113), (6.115), (6.116), (6.118), (6.123), the estimate for $\partial_v \log \Omega^2$ in the no-shift region and the previous bounds on $\partial_u Q$, Q, $\partial_u \varpi$ and ϖ.

To close the bootstrap for ϕ we integrate $\partial_u \phi = -iqA_u\phi + D_u\phi$ from the curve Γ_Y (see also (6.61) for an analogous expression). Thus, we use (6.76), (6.103), (6.113), (6.116), (6.122) and the fact that $e^{-c(s)v_Y(u)} \leq e^{-\mathcal{C}_s(u,v)}$ to get, for a suitably large v_1:

$$\begin{aligned} |\phi|(u,v) &\leq C_{\mathcal{N}} e^{-c(s)v_Y(u)} + MC_{\mathcal{N}} e^{-\mathcal{C}_s(u,v)} \int_{u_Y(v)}^{u} |\nu|(u',v)du' \\ &\leq C_{\mathcal{N}}(1 + (Y - r_- + \varepsilon)M)e^{-\mathcal{C}_s(u,v)} \\ &\leq C_{\mathcal{N}}(1 + 2\varepsilon M)e^{-\mathcal{C}_s(u,v)}. \end{aligned} \tag{6.125}$$

The quantity $1 + 2\varepsilon M(N_{\text{i.d.}})$ is strictly smaller than $\frac{3}{2}$ for ε small.

To bound $|\partial_v \phi|$, we integrate (3.39) from Γ_Y and obtain an expression for $r\partial_v\phi$ analogous to (6.63). Then, by using (6.76), (6.113), (6.118), (6.119), (6.120), (6.122), the bootstrap inequalities for ϕ and $D_u\phi$, and the fact that $e^{-c(s)v_Y(u)} \leq e^{-\mathcal{C}_s(u,v)}$:

$$\begin{aligned} |r\partial_v\phi|(u,v) &\leq C_{\mathcal{N}} \left(Y + C(N_{\text{i.d.}}) \int_{u_Y(v)}^{u} |\nu|(u',v)du' \right) e^{-\mathcal{C}_s(u,v)} \\ &\leq C_{\mathcal{N}}(Y + (Y - r_- + \varepsilon)C(N_{\text{i.d.}}))e^{-\mathcal{C}_s(u,v)}, \end{aligned}$$

for v_1 large. After dividing both sides of the above by $r \geq r_- - \varepsilon$ (see (6.122)) and using (6.103) and (6.118) to notice that

$$\frac{Y}{r_- - \varepsilon} < 2 \qquad \text{and} \qquad \frac{Y - r_- + \varepsilon}{r_- - \varepsilon} < \frac{4\varepsilon}{r_-},$$

for ε small, we conclude that

$$|\partial_v\phi|(u,v) \leq C_{\mathcal{N}} \left(2 + \frac{4\varepsilon}{r_-} C(N_{\text{i.d.}}) \right) e^{-\mathcal{C}_s(u,v)} < \frac{5}{2} C_{\mathcal{N}} e^{-\mathcal{C}_s(u,v)}.$$

This result, together with (6.125), is sufficient to close bootstrap inequality (6.115).

We now improve the bound (6.116) for $D_u\phi$. First, we notice that, by integrating (3.35) from Γ_Y:

$$|\nu|(u,v) = |\nu|(u, v_Y(u)) \exp\left(\int_{v_Y(u)}^{v} \kappa(2K - m^2 r|\phi|^2)(u,v')dv' \right). \tag{6.126}$$

Moreover, by (3.40):

$$|rD_u\phi|(u,v) \leq |rD_u\phi|(u,v_Y(u)) + C(N_{\text{i.d.}})C_{\mathcal{N}}\int_{v_Y(u)}^{v} |\nu|(u,v')e^{-\mathcal{C}_s(u,v')}dv'.$$

Since (3.35), (6.111) and (6.123) give[9]

$$\begin{aligned}\partial_v\Big(|\nu|(u,v)e^{-\mathcal{C}_s(u,v)}\Big) &= |\nu|(u,v)e^{-\mathcal{C}_s(u,v)}\big(\kappa(2K - m^2r|\phi|^2)(u,v) + 4K_- + 2C\varepsilon\big)\\ &\geq |\nu|(u,v)e^{-\mathcal{C}_s(u,v)}(2K_- + C\varepsilon) > 0\end{aligned}$$

in the bootstrap set E, then:

$$\int_{v_Y(u)}^{v} |\nu|(u,v')e^{-\mathcal{C}_s(u,v')}dv' \leq \frac{1}{2K_- + C\varepsilon}\int_{v_Y(u)}^{v} \partial_v\big(|\nu|e^{-\mathcal{C}_s}\big)dv' \leq \frac{|\nu|(u,v)e^{-\mathcal{C}_s(u,v)}}{2K_-}.$$

Expressions (6.75), (6.126), (6.123), (6.115) and the definition of $\mathcal{C}_s$ then yield

$$\begin{aligned}|rD_u\phi|(u,v) &\leq C_{\mathcal{N}}|\nu|(u,v)\Big(r_+e^{-c(s)v_Y(u)+(2K_-+C\varepsilon)(v-v_Y(u))} + C(N_{\text{i.d.}})e^{-\mathcal{C}_s(u,v)}\Big)\\ &\leq C_{\mathcal{N}}C(N_{\text{i.d.}})|\nu|(u,v)e^{-\mathcal{C}_s(u,v)},\end{aligned}$$

for a suitably large v_1. Hence, the bootstrap closes by choosing M large compared to the initial data (this might possibly reduce the size ε even further).

Finally, we can express the above estimates in terms of $c(s)-\tau$, rather than $\mathcal{C}_s(u,v)$, using (6.112). □

Remark 6.14

We observe that:

- *The inequality $\Delta < \varepsilon$ that we assumed in the statement of proposition 6.13 is not needed for the proof. Such a condition can be replaced by an assumption on the smallness of $\Delta + \varepsilon$ (with respect to the initial data).*
- *Proposition 6.13 can be proved in an analogous fashion if we replace* (6.111) *with the following definition of $\mathcal{C}_s$:*

  $$\mathcal{C}_s(u,v) = c(s)v_Y(u) - \alpha(2K_- + C\varepsilon)(v - v_Y(u)),$$

  for $(u,v) \in J^+(\Gamma_Y) \cap J^-(\gamma) \cap \{v \geq v_1\}$ and for any $\alpha > 1$.

6.7 Early blueshift region: bounds along γ

In this section, we improve the estimates of ν and λ along the future boundary of the early blueshift region.

Lemma 6.15 (integral bounds on Ω^2)

The following results hold:

[9] Here and in the following steps the expression of $\mathcal{C}_s$ plays a prominent role. The same reasoning would not work if we replaced $\mathcal{C}_s(u,v)$ with $(c(s)-\tau)v$.

1. *If there exists* $D > 0$ *such that*

$$\partial_v \log \Omega^2(w,z) \le -D, \quad \forall\, (w,z) \in J^+(\Gamma_Y) \cap J^-(\gamma),$$

and if v *is sufficiently large, then:*

$$\int_{u_Y(v)}^{u} \Omega^2(\tilde{u},v)d\tilde{u} \le C(N_{i.d.},R,Y)\frac{\beta}{1+\beta} v\, e^{-D\frac{\beta}{1+\beta}v}, \quad \forall\, (u,v) \in \gamma. \tag{6.127}$$

2. *Let* $E \subseteq J^+(\gamma)$ *be a past set such that* $\gamma \subset E$. *If there exists* $D > 0$ *such that*

$$\partial_v \log \Omega^2(w,z) \le -D, \quad \forall\, (w,z) \in J^+(\Gamma_Y) \cap J^-(E), \tag{6.128}$$

and if v *is sufficiently large, then:*

$$\int_{u_\gamma(v)}^{u} \Omega^2(\tilde{u},v)d\tilde{u} \le C(N_{i.d.},R,Y)(v - v_\gamma(u))e^{-D\frac{\beta}{1+\beta}v}, \quad \forall\, (u,v) \in E.$$

Proof. We will prove the second statement. Inequality (6.127) follows similarly.

First, we notice that estimate (6.93) can be improved when $1-\mu$ is restricted to the curve Γ_Y. Indeed, we have

$$(1-\mu)(u_Y(z),z) = (1-\mu)(Y,\varpi_+,Q_+) + 2\frac{\varpi_+ - \varpi(u_Y(z),z)}{Y} + \frac{Q^2(u_Y(z),z) - Q_+^2}{Y^2},$$

for every $z \ge v_1$. Using (6.72) and (6.71):

$$\Big| |1-\mu|(u_Y(z),z) - |1-\mu|(Y,\varpi_+,Q_+) \Big| \lesssim e^{-2c(s)z}, \tag{6.129}$$

for every z suitably large. An analogous estimate can be proved for $|1-\mu|(w,v_Y(w))$, for every w.

Let us now integrate (6.128) from Γ_Y to $(\tilde{u},\tilde{v}) \in E$ and use the fact that $\tilde{v} \ge (1+\beta)v_Y(\tilde{u})$ to get

$$\log \frac{\Omega^2(\tilde{u},\tilde{v})}{\Omega^2(\tilde{u},v_Y(\tilde{u}))} \le -D\frac{\beta}{1+\beta}\tilde{v},$$

and thus

$$\Omega^2(\tilde{u},\tilde{v}) \le \Omega^2(\tilde{u},v_Y(\tilde{u}))e^{-D\frac{\beta}{1+\beta}\tilde{v}}, \quad \forall\, (\tilde{u},\tilde{v}) \in E. \tag{6.130}$$

Now, if we differentiate the relation $r(\cdot,v_Y(\cdot)) = Y$, we have

$$\nu(\cdot,v_Y(\cdot)) + \lambda(\cdot,v_Y(\cdot))v_Y'(\cdot) \equiv 0 \quad \text{ in } [0,u_Y(v_1)].$$

Given $(u,v) \in \text{int}E$, we can integrate the above in $[u_\gamma(v),u]$, use that $u_\gamma = v_\gamma^{-1}$ and perform a change of variables (see (6.64)):

$$\int_{u_\gamma(v)}^{u} \nu(\tilde{u},v_Y(\tilde{u}))d\tilde{u} = \int_{v_Y(u)}^{v_Y(u_\gamma(v))} \lambda(u_Y(\tilde{v}),\tilde{v})d\tilde{v}. \tag{6.131}$$

On the other hand, if we integrate (6.130) from the curve $\gamma \subset E$ to $(u,v) \in \mathrm{int}E$ and use: (6.130), the facts that $\Omega^2 = 4|\nu|\kappa$ and $0 < \kappa \le 1$ in $\mathcal{P}$ and the relations (6.131), (3.33) and (6.129), in this order:

$$\begin{aligned}\int_{u_\gamma(v)}^{u} \Omega^2(\tilde{u},v)d\tilde{u} &\le e^{-D\frac{\beta}{1+\beta}v}\int_{u_\gamma(v)}^{u} \Omega^2(\tilde{u},v_Y(\tilde{u}))d\tilde{u} \le 4e^{-D\frac{\beta}{1+\beta}v}\int_{u_\gamma(v)}^{u} |\nu|(\tilde{u},v_Y(\tilde{u}))d\tilde{u} \\ &= 4e^{-D\frac{\beta}{1+\beta}v}\int_{v_Y(u)}^{v_Y(u_\gamma(v))} |\lambda|(u_Y(\tilde{v}),\tilde{v})d\tilde{v} \\ &\le C(N_{\mathrm{i.d.}},R,Y)(v_Y(u_\gamma(v)) - v_Y(u))e^{-D\frac{\beta}{1+\beta}v} \\ &\le C(N_{\mathrm{i.d.}},R,Y)(v - v_\gamma(u))e^{-D\frac{\beta}{1+\beta}v},\end{aligned}$$

for v_1 suitably large. □

We recall that, in the following, the notation $\lesssim$ refers to constants depending on the initial data, η, R and Y.

Lemma 6.16 (bounds along γ)
Under the assumptions of proposition 6.13, if we further suppose that U is sufficiently small, the following inequalities hold for every $(u,v) \in \gamma \cap \{v \ge v_1\}$:

$$e^{-(2K_- + c_1\varepsilon)\frac{\beta}{1+\beta}v} \lesssim -\lambda(u,v) \lesssim e^{-(2K_- - c_2\varepsilon)\frac{\beta}{1+\beta}v}, \tag{6.132}$$

$$u^{-1+\frac{K_-}{K_+}\beta+\tilde{\alpha}} \lesssim -\nu(u,v) \lesssim u^{-1+\frac{K_-}{K_+}\beta-\alpha}, \tag{6.133}$$

for some positive constants α, $\tilde{\alpha}$, c_1 and c_2. The constants α and $\tilde{\alpha}$ depend on the initial data, on β and on ε, whereas c_1 and c_2 depend on the initial data and on η, R and Y. Moreover, for every

$$\beta > 0$$

small compared to the initial data, there exists $\varepsilon > 0$ such that

$$\frac{K_-}{K_+}\beta - \alpha(\beta,\varepsilon) > 0.$$

Remark 6.17 (on the estimates along the curve γ)
As β tends to zero (see (6.101)), the curve γ approaches the curve of constant radius Γ_Y. The importance of a non-zero value for β is apparent in the results of lemma 6.16, showing that, along γ:

- *Not only the quantity $|\lambda|$ is bounded (as occurs in the redshift, no-shift and early blueshift regions, see (6.30), (6.69) and (6.120)) but it also decays at a rate which is integrable in $[v_1, +\infty)$; this result will be vital to prove that r is bounded and bounded away from zero at the Cauchy horizon;*

- *The "blow-up" behaviour of $|\nu|$ in the u coordinate (property already present in the previous regions, see (6.4), (6.31), (6.70)) is now integrable in u: this can be used in alternative to the previous point to show the stability of the Cauchy horizon, once we propagate this result to $J^+(\gamma)$.*

We also notice that, in $J^+(\Gamma_Y) \cap J^-(\gamma)$, requiring U to be small is equivalent to requiring suitably large values of v_1, due to the spacelike nature of γ and its location in the black hole interior. Indeed, we showed that every $u \in [0, u_Y(v_1)]$ satisfies (6.77)*:*

$$u \sim e^{-2K_+ v_Y(u)},$$

and a similar relation holds for points along γ, since $v_Y(u) = (1+\beta)^{-1} v_\gamma(u)$. On the other hand, in the region $J^+(\gamma)$ the restrictions on the v coordinate do not constrain the ingoing null coordinate.

proof of lemma 6.16. Let $(u, v) \in \gamma \cap \{v \geq v_1\}$. To prove the bounds for $|\lambda|$, some preliminary estimates will be useful. First, we obtain an upper bound on $\frac{\nu}{1-\mu}$ by exploiting (6.97):

$$0 < \frac{\nu}{1-\mu}(u, v) \leq \frac{\nu}{1-\mu}(u, v_Y(u)). \tag{6.134}$$

Furthermore, by differentiating the relation $r(\cdot, v_Y(\cdot)) = Y$, we note that

$$\nu(\cdot, v_Y(\cdot)) + \lambda(\cdot, v_Y(\cdot)) v_Y'(\cdot) \equiv 0 \quad \text{ in } [0, u_Y(v_1)].$$

If we integrate the latter in $[u_Y(v), u]$, use that $u_Y = v_Y^{-1}$ and perform a change of variables:

$$\int_{u_Y(v)}^{u} \nu(\tilde{u}, v_Y(\tilde{u})) d\tilde{u} = \int_{v_Y(u)}^{v} \lambda(u_Y(\tilde{v}), \tilde{v}) d\tilde{v}. \tag{6.135}$$

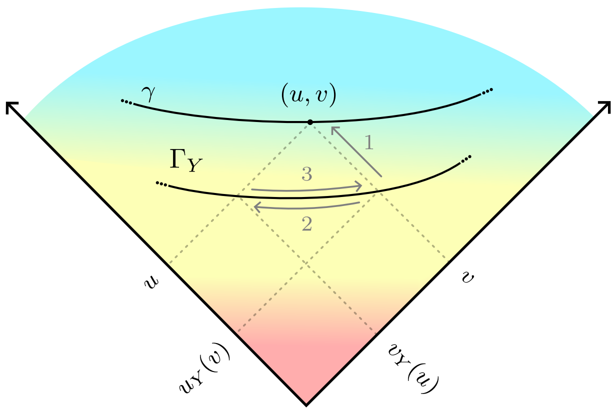


Figure 6.1: The integration procedure for $\nu(1-\mu)^{-1}$.

Now, we obtain integral bounds on $\frac{\nu}{1-\mu}$ as a preliminary step. An upper bound is achieved as follows. We use (6.134), together with (6.129), (6.135), (3.33) and with the definition of the curve γ, in this precise order, to obtain

$$0 < \int_{u_Y(v)}^{u} \frac{\nu}{1-\mu}(u', v) du' \leq \int_{u_Y(v)}^{u} \frac{\nu}{1-\mu}(\tilde{u}, v_Y(\tilde{u})) d\tilde{u}$$
$$\leq \frac{1}{C(N_{\text{i.d.}}, R, Y) + o(1)} \int_{u_Y(v)}^{u} (-\nu)(\tilde{u}, v_Y(\tilde{u})) d\tilde{u} = \frac{1}{C(N_{\text{i.d.}}, R, Y) + o(1)} \int_{v_Y(u)}^{v} (-\lambda)(u_Y(\tilde{v}), \tilde{v}) d\tilde{v}$$

$$\leq \frac{C(N_{\text{i.d.}}, R, Y) + o(1)}{C(N_{\text{i.d.}}, R, Y) + o(1)} \int_{v_Y(u)}^{v} \kappa(u_Y(\tilde{v}), \tilde{v}) d\tilde{v} \leq (1 + o(1)) \frac{\beta}{1+\beta} v, \tag{6.136}$$

where the notation $o(1)$ refers to the limit $u \to 0$. This yields a lower bound on $|\lambda|$ as follows. We use (3.33) to cast (3.35) as

$$\partial_u \lambda = \lambda \frac{\nu}{1-\mu}(2K - m^2 r |\phi|^2). \tag{6.137}$$

If we integrate the latter from Γ_Y and use that $|\lambda|(u_Y(v), v) = [\kappa(1-\mu)](u_Y(v), v)$, together with the boundedness of r, (6.110), (6.123), (6.129), (6.136) and the fact that κ is close to 1 in the early blueshift region:

$$\begin{aligned} |\lambda|(u,v) &= |\lambda|(u_Y(v), v) \exp\left(\int_{u_Y(v)}^{u} \frac{\nu}{1-\mu} \big(2K - m^2 r |\phi|^2\big)(u', v) du' \right) \\ &\geq C(N_{\text{i.d.}}, R, Y) \exp\left(-(2K_- + C(N_{\text{i.d.}})\varepsilon)(1 + o(1)) \frac{\beta}{1+\beta} v \right) \\ &\geq C(N_{\text{i.d.}}, R, Y) e^{-(2K_- + c_1 \varepsilon)\frac{\beta}{1+\beta} v} \end{aligned} \tag{6.138}$$

for U sufficiently small, where the constant c_1 depends uniquely on the initial data.

Now, we imitate the previous steps to get an upper bound on $|\lambda|$. Since $\lambda < 0$ and $\partial_u |\lambda| < 0$ in this region (see (3.35) and (6.107)), we use (6.110), (6.138) and the fact that $v = (1+\beta) v_Y(u)$ to write:

$$\begin{aligned} \int_{v_Y(u)}^{v} \frac{|\partial_v \phi|^2}{|\lambda|}(u, v') dv' &\leq \int_{v_Y(u)}^{v} \frac{|\partial_v \phi|^2 (u, v')}{|\lambda|(u_\gamma(v'), v')} dv' \\ &\lesssim \int_{v_Y(u)}^{v} e^{-2(c(s) - \tau) v' + (2K_- + c_1 \varepsilon) \frac{\beta}{1+\beta} v'} dv' \\ &\lesssim e^{-a v_Y(u)} = e^{-\frac{a}{1+\beta} v}, \end{aligned}$$

for some $a > 0$ and for β small. In particular, provided that v_1 is chosen sufficiently large, a reasoning similar to that of the steps (6.97)–(6.100) leads to

$$\frac{\nu}{1-\mu}(u,v) \geq \frac{1}{(1+o(1))} \cdot \frac{\nu}{1-\mu}(u, v_Y(u)), \tag{6.139}$$

where the notation $o(1)$ refers to the limit $u \to 0$.

Analogously to the steps in (6.136), but now using (6.139) with (6.129), (6.135), (3.33), with the fact that κ is close to 1 in the early blueshift region, and with the definition of the curve γ, in this precise order, we obtain:

$$\int_{u_Y(v)}^{u} \frac{\nu}{1-\mu}(u', v) du' \geq (1 + o(1)) \frac{\beta}{1+\beta} v, \tag{6.140}$$

for $(u,v) \in \gamma \cap \{v \geq v_1\}$. We now integrate (6.137) from Γ_Y and use that $|\lambda|(u_Y(v), v) = [\kappa(1-\mu)](u_Y(v), v)$, together with the boundedness of r, (6.110), (6.123) and (6.140):

$$|\lambda|(u,v) = |\lambda|(u_Y(v), v) \exp\left(\int_{u_Y(v)}^{u} \frac{\nu}{1-\mu} \big(2K - m^2 r |\phi|^2\big)(u', v) du' \right)$$

$$\le C(N_{\text{i.d.}}, R, Y) \exp\left(-(2K_- - C(N_{\text{i.d.}})\varepsilon)(1+o(1))\frac{\beta}{1+\beta} v \right)$$
$$\le C(N_{\text{i.d.}}, R, Y) e^{-(2K_- - c_2 \varepsilon)\frac{\beta}{1+\beta} v}, \tag{6.141}$$

where $c_2 > 0$ depends uniquely on the initial data and can be chosen so that $2K_- - c_2\varepsilon > 0$.

We now proceed to estimate $|\nu|$ along γ. First, by relation (6.77) proved in the no-shift region, we have

$$u \sim e^{-2K_+ v_Y(u)}, \quad \forall\, u \in [0, u_Y(v_1)]. \tag{6.142}$$

Now, for every $0 < \sigma < 2K_-$ and for every $u \le u_Y(v_1)$:

$$e^{-(2K_- \pm \sigma)\beta v_Y(u)} \sim u^{\frac{2K_- \pm \sigma}{2K_+}\beta} = u^{\frac{K_-}{K_+}\beta \pm \alpha}, \tag{6.143}$$

with $0 < \alpha = \frac{\sigma\beta}{2K_+} \lesssim 1$.

We now use, in this order, (6.126), (6.100) (from the no-shift region), (6.124), the definition (6.102) of γ, (6.31), (6.28) and (6.143) to obtain:

$$\begin{aligned}
|\nu|(u,v) &= |\nu|(u, v_Y(u)) \exp\left(\int_{v_Y(u)}^{v} \kappa(2K - m^2 r|\phi|^2)(u,v')dv' \right) \\
&\le C(N_{\text{i.d.}}, R, Y)|\nu|(u, v_R(u)) e^{-(2K_- - 2C\varepsilon)\beta v_Y(u)} \\
&\le C(N_{\text{i.d.}}, R, Y)\frac{1}{u} e^{-(2K_- - 2C\varepsilon)\beta v_Y(u)} \\
&\lesssim u^{-1 + \frac{K_-}{K_+}\beta - \alpha},
\end{aligned}$$

for every $(u,v) \in \gamma \cap \{v \ge v_1\}$, where $0 < \alpha \lesssim \varepsilon\beta$. In particular, for every β, we can assume without loss of generality that ε was chosen sufficiently small so that

$$\frac{K_-}{K_+}\beta - \alpha > 0. \tag{6.144}$$

The lower bound in (6.133) then follows analogously to the previous steps, after recalling that $v_Y(u) - v_R(u)$ is bounded by a constant (see (6.96)). □

6.8 Late blueshift region

In the causal future of the curve γ, also dubbed the **late blueshift region**, the dynamical black hole departs considerably[10] from the fixed sub-extremal solution. After showing that the upper bounds in (6.132) and (6.133) propagate to $J^+(\gamma)$, it will follow that r remains bounded away from zero in this entire region. This will be the key ingredient to show the existence, for the dynamical model, of a Cauchy horizon $\mathcal{CH}^+$ beyond which the metric g can be continuously extended. We stress that the smallness of the region $J^+(\gamma)$ is what allows us to propagate decaying estimates despite the adverse contribution of

[10] By this, we mean that we have less control on the main dynamical quantities, e.g. K and ϖ, with respect to the previous regions that we analysed. A precise notion of 'distance' between the dynamical black-hole and the fixed sub-extremal Reissner-Nordström-de Sitter solution depends on the norm used to control the PDE variables.

the blueshift effect (see already lemma 6.19).

The core idea of the proof is again a bootstrap argument involving the quantity $\partial_v\phi$. The proof, however, differs significantly from the bootstrap procedures of the previous regions, due to the lack of an upper bound[11] on the quantities $|\varpi|$ and $|K|$ and due to the absence of a lower bound on κ. A natural way to close the argument is to control ϕ and $D_u\phi$ by integrating the estimate on $\partial_v\phi$ that we bootstrap from γ. This is also possible due to a bound on $\partial_v \log \Omega^2$, which is related to K via (3.41).

The technical difficulties of the proof can be summarised as follows. Due to the presence of the non-constant charge function Q, we cannot apply the soft analysis used in [20, 22] to propagate the bounds on $|\nu|$ and $|\lambda|$ from the curve γ. A bootstrap argument lets us circumvent this difficulty and obtain additional estimates on $|\phi|$, $|\partial_v\phi|$, $|D_u\phi|$ and $|Q|$. Two core ingredients of the proof are given by a bound on $\partial_v \log \Omega^2$, which also plays a crucial role in [79], and by the sub-extremality condition $\frac{K_-}{K_+} > 1$.

Proposition 6.18 (estimates in the late blueshift region)
Under the assumptions of proposition 6.13, if U and β are sufficiently small and if ε is chosen small so that (see lemma 6.16):

$$\frac{K_-}{K_+}\beta - \alpha(\beta,\varepsilon) > 0,$$

then the following inequalities hold for every $(u,v) \in J^+(\gamma) \cap \{r \geq r_- - \varepsilon,\, v \geq v_1\}$:

$$0 < -\lambda(u,v) \lesssim \frac{\Omega^2(u,v)}{\Omega^2(u,v_\gamma(u))}|\lambda|(u,v_\gamma(u)) + e^{-\min\{2c(s)-4\tau, 2K_- - \tau\}v}, \tag{6.145}$$

$$0 < -\nu(u,v) \lesssim u^{-1+\frac{K_-}{K_+}\beta - \alpha}, \tag{6.146}$$

$$|\partial_v\phi|(u,v) \lesssim e^{-(c(s)-\tau)v}, \tag{6.147}$$

$$|\phi|(u,v) \lesssim e^{-(c(s)-\tau)v_\gamma(u)}, \tag{6.148}$$

$$|D_u\phi|(u,v) \lesssim u^{-1+\frac{K_-}{K_+}\beta-\alpha} e^{-(c(s)-\tau)v_\gamma(u)}, \tag{6.149}$$

$$\partial_v \log \Omega^2(u,v) = -2K_- + O(\varepsilon), \tag{6.150}$$

$$|Q|(u,v) \lesssim |Q|(u,v_\gamma(u)) + e^{-2(c(s)-\tau)v_\gamma(u)}, \tag{6.151}$$

where $\tau > 0$, $\tau = O(\beta)$.

Proof. As for the previous proofs, we will repeatedly use that $0 < \kappa \leq 1$ and $\nu < 0$ in $\mathcal{P}$, and that $\lambda < 0$ in $J^+(\gamma)$ (see lemma 6.9 and lemma 6.16). The value of ε is taken small so that, in particular:

$$\frac{3}{4}r_- < r_- - \varepsilon \leq r \leq Y \quad \text{in } J^+(\Gamma_Y) \cap \{r \geq r_- - \varepsilon,\, v \geq v_1\}. \tag{6.152}$$

Moreover, we will use that

$$c(s) < 2K_-. \tag{6.153}$$

The above follows from (6.27) and from the fact that $\frac{K_-}{K_+} > 1$ (see appendix A in [21]).

[11]The main obstructions to this result stemming from the quadratic term $|D_u\phi|^2$ in (3.38) and the term κ^{-1} in (3.26). The renormalized Hawking mass was shown to blow up in the real and massless case [22] for a class of initial data and by further assuming a lower bound on $|\partial_v\phi|$ along the event horizon.

Setting up the bootstrap procedure: Define E as the set of points q in $\mathcal{D} := J^+(\gamma) \cap \{r \geq r_- - \varepsilon,\, v \geq v_1\}$ such that the following inequalities hold for every (u,v) in $J^-(q) \cap \mathcal{D}$:

$$|r\partial_v \phi|(u,v) \leq M e^{-(c(s)-\tau)v}, \tag{6.154}$$

$$|\lambda|(u,v) \leq C(N_{\text{i.d.}}), \tag{6.155}$$

$$\partial_v \log \Omega^2(u,v) \leq -K_-, \tag{6.156}$$

where $M = M(N_{\text{i.d.}}) > 0$ is a large constant and $\tau > 0$, $\tau = O(\beta)$. Moreover, notice that, whenever (u,v) belongs to γ, we have $(u,v) = (u_\gamma(v), v) = (u, v_\gamma(u))$.

Closing the bootstrap: let us fix $(u,v) \in E$. By integrating the bootstrap inequality (6.154) and using (6.110) and the above bounds on r, we can write:

$$|\phi|(u,v) \leq |\phi|(u, v_\gamma(u)) + C(N_{\text{i.d.}}, M) \int_{v_\gamma(u)}^{v} e^{-(c(s)-\tau)v'} dv' \leq C(N_{\text{i.d.}}, \eta, R, Y, M) e^{-(c(s)-\tau)v_\gamma(u)}. \tag{6.157}$$

Since the constants R, Y and η will be left unchanged in the next steps, we can absorb those in $C = C(N_{\text{i.d.}}, M) > 0$.

In the causal future of γ, the charge function Q remains bounded. Indeed, by integrating (3.12) and using (6.152), (6.154) and (6.157):

$$|Q|(u,v) \leq |Q|(u, v_\gamma(u)) + C(N_{\text{i.d.}}, M) e^{-2(c(s)-\tau)v_\gamma(u)}. \tag{6.158}$$

Now, (3.37), (3.19) and the bounds (6.152), (6.156), (6.157) and (6.158) give

$$|\partial_v(r\nu)|(u,v) \lesssim \Omega^2(u,v) \leq -\frac{\Omega^2(u,v)}{K_-} \partial_v \log \Omega^2(u,v) = -\frac{\partial_v[\Omega^2](u,v)}{K_-}.$$

After integrating the latter from γ and exploiting (6.133), (3.19) and the boundedness of r:

$$|\nu|(u,v) \lesssim |\nu|(u, v_\gamma(u)) + \Omega^2(u, v_\gamma(u)) \leq C(N_{\text{i.d.}}) u^{-1+\frac{K_-}{K_+}\beta - \alpha}.$$

To estimate $|D_u \phi|$, we first notice that bound (6.109), obtained for the early blueshift region, and (6.133) entail

$$|D_u \phi|(u, v_\gamma(u)) \lesssim u^{-1+\frac{K_-}{K_+}\beta-\alpha} e^{-(c(s)-\tau)v_\gamma(u)}.$$

Therefore, the relations (6.152), (6.154), (6.156), (6.157), the fact that $\Omega^2 = -4\nu\kappa$ and the above bounds on ν and Q can be used to integrate (3.40) as follows:

$$\begin{aligned}
|rD_u\phi|(u,v) &\lesssim |rD_u\phi|(u, v_\gamma(u)) + \int_{v_\gamma(u)}^{v} |\nu \partial_v \phi|(u,v') dv' + C(N_{\text{i.d.}}) \int_{v_\gamma(u)}^{v} \Omega^2 |\phi|(u,v') dv' \\
&\lesssim u^{-1+\frac{K_-}{K_+}\beta-\alpha} e^{-(c(s)-\tau)v_\gamma(u)} + C(N_{\text{i.d.}}, M) u^{-1+\frac{K_-}{K_+}\beta-\alpha} e^{-(c(s)-\tau)v_\gamma(u)} \\
&\quad - C(N_{\text{i.d.}}, M) e^{-(c(s)-\tau)v_\gamma(u)} \int_{v_\gamma(u)}^{v} \frac{\partial_v[\Omega^2](u,v')}{K_-} dv',
\end{aligned}$$

where all above constants are positive. So, since $\Omega^2(u,v_\gamma(u)) \lesssim u^{-1+\frac{K_-}{K_+}\beta-\alpha}$:

$$|D_u\phi|(u,v) \leq C(N_{\text{i.d.}},M)u^{-1+\frac{K_-}{K_+}\beta-\alpha}e^{-(c(s)-\tau)v_\gamma(u)}. \tag{6.159}$$

To close bootstrap inequality (6.156), we first note that the results (6.108) and (6.123) obtained in the early blueshift region give

$$\begin{aligned}|\partial_v \log\Omega^2(u_\gamma(v),v)+2K_-| &\leq |\partial_v\log\Omega^2(u_\gamma(v),v)-2K(u_\gamma(v),v)|+|2K(u_\gamma(v),v)+2K_-|\\ &= o(1)+O(\varepsilon),\end{aligned} \tag{6.160}$$

where $o(1)$ refers to the limit $v_1 \to +\infty$. On the other hand, it is useful to notice that (6.154) and the previous bound on $|D_u\phi|$ yield[12]

$$\int_{u_\gamma(v)}^{u} |D_u\phi\,\partial_v\phi|(u',v)du' = o(1), \tag{6.161}$$

as $v_1 \to +\infty$.

Furthermore, (6.156) also holds in $J^+(\Gamma_Y)\cap J^-(E)$ (see (6.107) and (6.108)). By applying lemma 6.15 with $D = K_-$:

$$\int_{u_\gamma(v)}^{u} \Omega^2(u',v)du' \leq C(N_{\text{i.d.}})(v-v_\gamma(u))e^{-K_-\frac{\beta}{1+\beta}v} \leq e^{-K_-\frac{\beta}{2+2\beta}v}, \tag{6.162}$$

for v_1 sufficiently large. Now, if we integrate (3.9) from the curve γ and use (6.152), (6.155), (6.160), (6.161), (6.162) and the previous bound on $|\nu|$:

$$\begin{aligned}\partial_v\log\Omega^2(u,v)+2K_-+O(\varepsilon)+o_{v_1}(1) &\leq \partial_v\log\Omega^2(u,v)-\partial_v\log\Omega^2(u_\gamma(v),v)\\ &= o_{v_1}(1)+o_U(1),\end{aligned}$$

where $o_U(1)$ refers to the limit $U \to 0$. This closes bootstrap inequality (6.156). Notice that a similar reasoning gives

$$\partial_v\log\Omega^2(u,v) = -2K_- + O(\varepsilon)$$

for v_1 large and U small. Hence, after integration:

$$e^{-(2K_-+\tau)(v-v_\gamma(u))} \leq \frac{\Omega^2(u,v)}{\Omega^2(u,v_\gamma(u))} \leq e^{-(2K_--\tau)(v-v_\gamma(u))}, \tag{6.163}$$

where $\tau > 0$, $\tau = O(\beta)$ and we assumed that ε is small compared to β.[13]

[12] We recall that $\frac{K_-}{K_+}\beta-\alpha > 0$ by assumptions.

[13] Recall that for every small choice of β, the value of ε may possibly be taken smaller so that $\frac{K_-}{K_+}\beta-\alpha > 0$. See also the proof of lemma 6.16.

To close the bootstrap on $|\lambda|$: integrate (3.7) and use (6.152) to get

$$\frac{|\lambda|}{\Omega^2}(u,v) \leq \frac{|\lambda|}{\Omega^2}(u, v_\gamma(u)) + C(N_{\text{i.d.}}) \int_{v_\gamma(u)}^{v} \frac{|\partial_v \phi|^2}{\Omega^2}(u,v')dv'.$$

Then, after using (6.154) and (6.163):

$$|\lambda|(u,v) \leq \frac{\Omega^2(u,v)}{\Omega^2(u,v_\gamma(u))}\left(|\lambda|(u,v_\gamma(u)) + C(N_{\text{i.d.}}, M)e^{-(2K_-+\tau)v_\gamma(u)} \int_{v_\gamma(u)}^{v} e^{-(2c(s)-2K_-+\tau)v'}dv'\right). \tag{6.164}$$

We finally obtain (6.145) after distinguishing between the cases $c(s) \leq K_-$ and $c(s) > K_-$, and exploiting (6.163) again. In particular, (6.155) is closed.

For the next step, it is useful to notice that (6.163), (6.157), (6.153) and (6.133) (recall that $\Omega^2 \sim |\nu|$ along γ) give:

$$\begin{aligned}\int_{u_\gamma(v)}^{u} \big(\Omega^2|\phi|\big)(u',v)du' &\leq C(N_{\text{i.d.}}, M)\int_{u_\gamma(v)}^{u} \Omega^2(x, v_\gamma(x))e^{-(2K_- -\tau)(v-v_\gamma(x))}e^{-(c(s)-\tau)v_\gamma(x)}dx \\ &\leq C(N_{\text{i.d.}}, M)u^{\frac{K_-}{K_+}\beta-\alpha}e^{-(c(s)-\tau)v}. \end{aligned} \tag{6.165}$$

To close the bootstrap inequality on $\partial_v\phi$, we integrate (3.39) (see (6.63) for an explicit expression) and use (3.19), (6.110), the boundedness of r, of the charge Q, (6.145), (6.165), (6.159), (6.132) and (6.153):

$$\begin{aligned}|r\partial_v\phi|(u,v) &\lesssim |r\partial_v\phi|(u_\gamma(v),v) + \int_{u_\gamma(v)}^{u}\big(|\lambda D_u\phi| + \Omega^2|\phi|\big)(u',v)du' \\ &\leq C(N_{\text{i.d.}})e^{-(c(s)-\tau)v} + C(N_{\text{i.d.}}, M)e^{-(2K_- -\tau)v}\int_{u_\gamma(v)}^{u} e^{(2K_- -c(s))v_\gamma(x)}x^{-1+\frac{K_-}{K_+}\beta-\alpha}dx \\ &+ C(N_{\text{i.d.}}, M)u^{\frac{K_-}{K_+}\beta-\alpha}e^{-\min\{2c(s)-2\tau, 2K_- -\tau\}v} + C(N_{\text{i.d.}}, M)u^{\frac{K_-}{K_+}\beta-\alpha}e^{-(c(s)-\tau)v} \\ &\leq \left(C(N_{\text{i.d.}}) + C(N_{\text{i.d.}}, M)u^{\frac{K_-}{K_+}\beta-\alpha}\right)e^{-(c(s)-\tau)v},\end{aligned}$$

assuming that β and ε are small compared to the initial data. Inequality (6.154) is then improved, provided that U is sufficiently small. □

Lemma 6.19

For every $\tilde{\varepsilon} > 0$ small, there exists $U_{\tilde{\varepsilon}} > 0$ such that

$$r(u,v) > r_- - 2\tilde{\varepsilon}$$

for every (u,v) in $J^+(\gamma) \cap \{u < U_{\tilde{\varepsilon}}\}$.

Proof. The proof is analogous to the one of lemma 7.2 in [20], where the results on ν and λ exploited in such a proof can be replaced with those of proposition 6.18. □

Corollary 6.20

Let $U_{\tilde{\varepsilon}} > 0$ be the value provided by lemma 6.19 for some $\tilde{\varepsilon} < \frac{r_-}{2}$. Then:

$$[0, U_{\tilde{\varepsilon}}] \times [v_0, +\infty) \subset \mathcal{P},$$

and the estimates of proposition 6.18 hold in $J^+(\gamma) \cap \{u < U_{\tilde{\varepsilon}}\}$. Moreover, the limit

$$\lim_{v \to +\infty} r(u, v)$$

exists and is finite for every $u \in [0, U_{\tilde{\varepsilon}}]$, and

$$\lim_{u \to 0} \lim_{v \to +\infty} r(u, v) = r_-.$$

Proof. The inclusion follows from the extension criterion (see theorem 5.1) and lemma 6.19. The limits are well-defined by the monotonicity of r in $J^+(\gamma)$, namely the fact that $\lambda < 0$ (see (6.69) and lemma 6.9) and $\nu < 0$. The final limit follows from lemma 6.19. □

6.9 A continuous extension

The construction obtained so far allows us to represent the Cauchy horizon $\mathcal{CH}^+$ of the dynamical black hole as the level set of a regular function. The problem of constructing a continuous extension is well understood (see e.g. [27, 22, 73, 79]), so we only provide an overview.

In particular, let (u, V) be a new coordinate system defined as follows. Given a sufficiently regular function $f\colon [v_0, +\infty) \to \mathbb{R}$ such that the limit of its first derivative as $v \to +\infty$ exists and such that

$$\frac{df(v)}{dv} \underset{v \to +\infty}{\longrightarrow} -2K_-, \tag{6.166}$$

we define

$$V(v) \coloneqq 1 - \int_v^{+\infty} e^{f(y)} dy, \quad \forall\, v \in [v_0, +\infty). \tag{6.167}$$

The choice $f(v) = -2K_- v$ was taken into account in [73] to construct a continuous extension in the case $\Lambda = 0$, where the authors studied the evolution of two-ended asymptotically flat initial data. Here, we are going to consider any function f such that

$$\frac{df(v)}{dv} = 2K(u_\gamma(v), v), \quad \forall\, v \in [v_0, +\infty), \tag{6.168}$$

as done in [67]. This gives, specifically,

$$\mathcal{CH}^+ = V^{-1}(\{1\}).$$

The solutions to our IVP expressed in the coordinate system (u, V) will be denoted by a subscript, e.g. $r_V(u, V) = r(u, v)$ and $\lambda_V(u, V) = \partial_V r_V(u, V) = e^{-f(v)} \lambda(u, v)$. We designate the smallest value that V can take by $V_0 = V(v_0)$.

In the following, we restrict the range of the ingoing null coordinate to the set $[0, U]$, where we use again the letter U to denote a value chosen sufficiently small, so that corollary 6.20 holds for some fixed $\tilde{\varepsilon} < \frac{r_-}{2}$.

In this context, we show that the metric g and the scalar field ϕ admit a continuous extension beyond the Cauchy horizon, in the sense of theorem 11.1 in [27]. This is a strong signal for the potential failure of conjecture 2.1, whose definitive resolution however requires a complementary study of the exterior problem.

Theorem 6.21 (A C^0 extension of g and ϕ)
For every $0 < \delta_U < U$, the functions r_V, Ω_V^2 and ϕ_V admit a continuous extension to the set $[\delta_U, U] \times [V_0, 1]$. Moreover, on $[\delta_U, U]$, the functions $r_V(\cdot, 1)$ and $\Omega_V^2(\cdot, 1)$ are positive.

Proof. We prove the existence of an extension for r, Ω^2 and ϕ, in this order.

Extension of r: first, by monotonicity of r in $I^+(\mathcal{A})$, the limit

$$\lim_{v \to +\infty} r(u, v) = \lim_{V \to 1} r_V(u, V) =: r_V(u, 1) \tag{6.169}$$

exists for every $u \in [\delta_U, U]$. Moreover, the above convergence is uniform in u. Indeed, due to the decay of $|\lambda|$ proved in the late blueshift region (see (6.145)):

$$\int_{V(v)}^{1} |\lambda_V|(u, V')dV' = \int_{v}^{+\infty} |\lambda|(u, v')dv' \leq e^{-av},$$

for some $a > 0$ and for every $u \in [\delta_U, U]$. So, by the fundamental theorem of calculus:

$$\sup_{u \in [\delta_U, U]} |r_V(u, V) - r_V(u, 1)| = \sup_{u \in [\delta_U, U]} \left| \int_{V}^{1} \lambda_V(u, V')dV' \right| \underset{V \to 1}{\longrightarrow} 0.$$

Since the function $r_V(\cdot, 1)$ is the uniform limit of continuous functions, it is continuous. The above, (6.169) and an application of the triangle inequality, imply that for every sequence $(u_n, V_n)_{n \in \mathbb{N}} \subset [\delta_U, U] \times [V_0, 1]$ such that $(u_n, V_n) \to (u_\infty, V_\infty) \in [\delta_U, U] \times [V_0, 1]$ as $n \to +\infty$, we have

$$|r_V(u_n, V_n) - r_V(u_\infty, V_\infty)| \underset{n \to +\infty}{\longrightarrow} 0,$$

i.e. r_V is continuous in $[\delta_U, U] \times [V_0, 1]$. We stress that $r_V(\cdot, 1) > 0$, due to lemma 6.19.

Extension of Ω^2: in light of the available estimates, we extend the function $\log \Omega^2$. Using (3.19) and (3.33), it can be verified that $\Omega_V^2(u, V) = e^{-f(v)}\Omega^2(u, v)$ in $[\delta_U, U] \times [V_0, 1]$.

Now, we use the notation

$$\rho := \frac{K_-}{K_+},$$

equation (3.9) and the estimates of proposition 6.18, to observe that there exists $a > 0$ such that, for every $(u, v) \in J^+(\gamma)$:

$$|\partial_u \partial_v \log \Omega^2|(u, v) \lesssim u^{-1+\rho\beta-\alpha} e^{-\mathcal{C}_s(u, v_\gamma(u)) - av} + \Omega^2(u, v) + u^{-1+\rho\beta-\alpha} e^{-av}.$$

If we integrate the above from the curve γ and use lemma 6.15 and the positivity of $\rho\beta - \alpha$ (which can be assumed for a suitable choice of δ and ε, see lemma 6.16), there exists $a > 0$ (possibly different from the previous one) such that

$$|\partial_v \log \Omega^2(u,v) - \partial_v \log \Omega^2(u_\gamma(v),v)| \lesssim e^{-av}, \tag{6.170}$$

for every $(u,v) \in J^+(\gamma)$.

Now, by (6.167) and (6.168), we have

$$\partial_V \log \Omega_V^2(u,V) = e^{-f(v)}\partial_v \left[\log\left(e^{-f(v)}\Omega^2\right)\right](u,v) = e^{-f(v)}\big(\partial_v \log \Omega^2(u,v) - 2K(u_\gamma(v),v)\big).$$

So, using inequality (6.108), valid along the curve γ, and (6.170):

$$\begin{aligned}
\int_{V(v)}^1 |\partial_V \log \Omega_V^2|(u,V')dV' &= \int_v^{+\infty} |\partial_v \log \Omega^2(u,v') - 2K(u_\gamma(v'),v')|dv' \\
&\leq \int_v^{+\infty} |\partial_v \log \Omega^2(u,v') - \partial_v \log \Omega^2(u_\gamma(v'),v')|dv' \\
&+ \int_v^{+\infty} |\partial_v \log \Omega^2(u_\gamma(v'),v') - 2K(u_\gamma(v'),v')|dv' \\
&\lesssim e^{-av},
\end{aligned}$$

for some $a > 0$ and for every $(u,v) \in [\delta_U, U] \times [v_0, +\infty)$. This means that, for every $V_1 \geq V_0$, the following expression admits a finite limit as $V_2 \to 1$:

$$\log \Omega_V^2(u,V_2) = \log \Omega_V^2(u,V_1) + \int_{V_1}^{V_2} \partial_V \log \Omega_V^2(u,V')dV'.$$

We define $\log \Omega_V^2(u,1) \coloneqq \lim_{V\to 1} \log \Omega_V^2(u,V)$ for every $u \in [\delta_U, U]$. Moreover:

$$\sup_{u\in[\delta_U,U]} \left|\log \Omega_V^2(u,V) - \log \Omega_V^2(u,1)\right| = \sup_{u\in[\delta_U,U]} \left|\int_V^1 \partial_V \log \Omega_V^2(u,V')dV'\right| \underset{V\to 1}{\longrightarrow} 0.$$

The reasoning used for the extension of r_V then gives an extension of $\log \Omega_V^2$ as well. Moreover, since we proved that $\log \Omega_V^2(u,1)$ is finite for every $u \in [\delta_U, U]$, it follows that $\Omega_V^2(\cdot,1) > 0$ on the same interval.

Extension of ϕ: By (6.147), there exists $a > 0$ such that:

$$\int_{V(v)}^1 |\partial_V \phi_V|(u,V')dV' = \int_v^{+\infty} |\partial_v \phi|(u,v')dv' \lesssim e^{-av},$$

for every $(u,v) \in [\delta_U, U] \times [v_0, +\infty)$. Thus, $\phi_V(u,1) \coloneqq \lim_{V\to 1} \phi_V(u,V)$ exists and is finite by the fundamental theorem of calculus. Furthermore:

$$\sup_{u\in[\delta_U,U]} |\phi_V(u,V) - \phi_V(u,1)| = \sup_{u\in[\delta_U,U]} \left|\int_V^1 \partial_V \phi_V(u,V')dV'\right| \underset{V\to 1}{\longrightarrow} 0,$$

for every $u \in [\delta_U, U]$. As for the case of r_V and Ω_V^2, this is enough to prove that ϕ_V is continuous in

$[\delta_U, U] \times [V_0, 1]$. □

Chapter 7

No-mass-inflation scenario and H^1 extensions

Heuristics and numerical work (see section 2.4) suggest the validity of the exponential Price law upper bound (3.49) along the event horizon of a solution asymptotically approaching (in the sense of section 3.2) a Reissner-Nordström-de Sitter black hole. Compared to the case $\Lambda = 0$, however, the exact asymptotics given by the constant value s in (3.49) play an even more important role, since the redshift (resp. blueshift) effect is expected to give an exponential contribution to the decay (resp. growth) of the main dynamical quantities: the value of s then determines the leading exponential contribution. This has a decisive role in determining the final fate of the dynamical black hole: a dominance of the blueshift effect is expected to cause geometric quantities such as the renormalized Hawking mass to blow up at the Cauchy horizon (this is the **mass inflation** scenario [86, 27, 22]), whereas a sufficiently fast decay of the scalar field along $\mathcal{H}^+$ may allow to extend the dynamical quantities across the Cauchy horizon in such a way that they are still weak solutions to the Einstein-Maxwell-charged-Klein-Gordon system (we have, in this case, **lack of global uniqueness**).

In [22], where the massless, real scalar field model was studied, any positive value of s was allowed and a further lower bound on the scalar field was assumed along the event horizon.[1] There, different relations between s and the quantity[2]

$$\rho \coloneqq \frac{K_-}{K_+} > 1 \tag{7.1}$$

led to different outcomes. The authors found a set in the $\rho - s$ plane for which mass inflation can be excluded for generic initial data: in this context, an H^1 extension of the solution was built and the so-called Christodoulou-Chruściel version of strong cosmic censorship was called into question. On the other hand, they found another set in the $\rho - s$ plane, disjoint from the previous one, for which mass inflation was guaranteed. This suggests that the H^1 version of the strong cosmic censorship, in this spherically symmetric toy model with $\Lambda > 0$, does hold if such initial data can be attained when starting

[1] With respect to our exponential Price law, the work [22] uses a different convention. For every $\epsilon > 0$, they choose initial data such that $e^{-(SK_+ + \epsilon)v} \lesssim |\partial_v \phi|(0, v) \lesssim e^{-(SK_+ - \epsilon)v}$ along the event horizon, for some $S > 0$.

[2] See also appendix A in [21] for a proof of the positivity of $\rho - 1$. The case $\rho = 1$ is associated to extremal black holes.

the evolution from a suitable spacelike hypersurface.

In the following, by requiring an upper bound (but no lower bound) on the scalar field along the event horizon (see (3.49)) we prove, despite the lack of many monotonicity results, that mass inflation can be avoided generically[3] also in the massive and charged case. With respect to the case of a real-valued scalar field, we extend the set of possible values of s for which H^1 extensions can be constructed (see Fig. 2.2), in agreement with the results obtained for the linear problem [18, 60].

Also in this chapter, we set U as the maximum of the ingoing null coordinate, where U is chosen suitably small, so that corollary 6.20 holds for a fixed $\tilde{\varepsilon} < \frac{r_-}{2}$.

7.1 No mass inflation

Theorem 7.1 (A sufficient condition to prevent mass inflation)
Given the characteristic IVP of section 3.1, choose initial data for which

$$s > K_- \quad \text{and} \quad \rho = \frac{K_-}{K_+} < 2 \tag{7.2}$$

are satisfied. Then, there exists a positive constant C, depending only on the initial data, such that

$$|\varpi|(u,v) \leq C, \quad \forall\, (u,v) \in J^+(\gamma),$$

provided that the value of U is sufficiently small.

Proof. We first present the main idea of the proof. Here, we choose the constants η, δ, R, Δ, ε, Y and β introduced in chapter 6 in such a way that the assumptions of proposition 6.18 are satisfied.

Using (3.33), equation (3.26) can be cast as

$$\partial_v \varpi = -\frac{|\theta|^2}{r\lambda}\varpi + \frac{\lambda}{2}m^2 r^2|\phi|^2 + Qq\,\mathrm{Im}\big(\phi\bar{\theta}\big) + \frac{|\theta|^2}{2\lambda}\left(1 + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right),$$

and thus, for every $(u,v) \in J^+(\gamma)$:

$$\begin{aligned}\varpi(u,v) = \varpi(u,v_\gamma(u))e^{-\int_{v_\gamma(u)}^{v}\frac{|\theta|^2}{r\lambda}(u,v')dv'} + \\ + \int_{v_\gamma(u)}^{v}\left[\frac{\lambda}{2}m^2r^2|\phi|^2 + Qq\,\mathrm{Im}\big(\phi\bar{\theta}\big) + \frac{|\theta|^2}{2\lambda}\left(1+\frac{Q^2}{r^2}-\frac{\Lambda}{3}r^2\right)\right](u,v')e^{-\int_{v'}^{v}\frac{|\theta|^2}{r\lambda}(u,y)dy}dv'.\end{aligned} \tag{7.3}$$

We recall that $\varpi(u,v_\gamma(u)) \to \varpi_+$ as $u \to 0$ (see proposition 6.13) and notice that the quantities inside the above square bracket, except for $\frac{|\theta|^2}{2\lambda}$, either decay or are bounded in L^∞ norm (see proposition 6.18). To control ϖ and extend it continuously to the Cauchy horizon is then sufficient to have a uniform integral bound on $\frac{|\theta|^2}{|\lambda|}$ (see also theorem 6.21, where uniform estimates were used to extend the metric). In the following, we bootstrap such an integral bound from the curve γ. This allows us to get a lower bound on

[3] Provided that ρ and s satisfy (7.2). We emphasize that, to study the admissibility of such values of ρ and s, a rigorous analysis of the exterior problem is required. Although it has been established [40, 7] that the value of s corresponding to the decay of the (real or complex) scalar field along $\mathcal{H}^+$ is determined by the spectral gap of the Laplace-Beltrami operator, a rigorous determination of such s is still generally lacking.

$|\lambda|$. We are then able to close the bootstrap argument using (6.147) and (7.2). The idea of bootstrapping such an integral bound is taken from [22], where, however, the expression for ϖ is considerably simpler and further restrictions on the values of s and β are required since less estimates are available in the region $J^+(\gamma)$. We now proceed with the detailed proof.

In the following, we repeatedly use the boundedness of r. Let $n \in \mathbb{N}_0$ be fixed and large with respect to the initial data. We denote by E the set of points $q \in J^+(\gamma)$ such that the following inequality holds for every $(u,v) \in J^-(q) \cap J^+(\gamma)$:

$$\int_{v_Y(u)}^{v} \frac{|\theta|^2}{|\lambda|}(u,v')dv' = \int_{v_Y(u)}^{v} \frac{r^2|\partial_v\phi|^2}{|\lambda|}(u,v')dv' \le \frac{1}{n}. \tag{7.4}$$

The above holds for every $(u,v) \in \gamma$, due to (6.110) and (6.132), provided that U and β are sufficiently small.

Now, let $(u,v) \in E$. Notice that $|r - r_-|$ and $|Q - Q_+|$ are small in E, provided that U is sufficiently small, due to the results of proposition 6.18 and to corollary 6.20. Moreover, the reasoning at the beginning of the current proof shows that $|\varpi - \varpi_+|$ is small in E, too. Therefore, by (3.34):

$$2K(u,v) = -2K_- + o(1), \tag{7.5}$$

where $o(1)$ refers to the limit $U \to 0$.

This gives a lower bound on $|\lambda|$ as follows. First, notice that (6.97) gives

$$0 < \frac{\nu}{1-\mu}(u,v) \le \frac{\nu}{1-\mu}(u, v_Y(u))$$

and, by construction, the same relation holds in $J^-(u,v) \cap J^+(\gamma)$. Expressions (6.129), (6.131), (3.33) and the definition of the curve γ yield

$$\begin{aligned}\int_{u_\gamma(v)}^{u} \frac{\nu}{1-\mu}(\tilde{u},v)d\tilde{u} &\le \int_{u_\gamma(v)}^{u} \frac{\nu}{1-\mu}(\tilde{u}, v_Y(\tilde{u}))d\tilde{u} \\ &\lesssim \frac{1}{C(N_{\text{i.d.}}) + o(1)} \int_{v_Y(u)}^{v_Y(u_\gamma(v))} \left(C(N_{\text{i.d.}}) + e^{-2c(s)v'}\right)dv' \\ &\le \frac{v - v_\gamma(u)}{(1+o(1))(1+\beta)},\end{aligned} \tag{7.6}$$

where the notation $o(1)$ refers to the limit $U \to 0$. Thus, if we integrate (6.137) from γ and use (6.132), (6.110), (7.5), (7.6):

$$\begin{aligned}|\lambda|(u,v) &= |\lambda|(u_\gamma(v),v)e^{\int_{u_\gamma(v)}^{u} \frac{\nu}{1-\mu}(2K - m^2 r|\phi|^2)(u',v)du'} \\ &\ge C(N_{\text{i.d.}})e^{-(2K_- + c_1\varepsilon)\frac{\beta}{1+\beta}v} e^{-(2K_- + O(\beta) + o(1))(v - v_\gamma(u))} \\ &\gtrsim C(u)e^{-(2K_- + c_1\varepsilon)(1 + O(\beta) + o(1))v},\end{aligned} \tag{7.7}$$

where we defined

$$C(u) \coloneqq e^{2K_-(1+O(\beta)+o(1))v_\gamma(u)}.$$

We now close the bootstrap argument. Using (7.7) and (6.147):

$$\int_{v_\gamma(u)}^{v} \frac{|\theta|^2}{|\lambda|}(u,v')dv' \lesssim \frac{1}{C(u)} \int_{v_\gamma(u)}^{v} e^{(-2c(s)+2K_- + C(\beta,\varepsilon)+o(1))v'}dv',$$

for some $C(\beta,\varepsilon) > 0$ that goes to zero as $\beta \to 0$ or as $\varepsilon \to 0$. Under our assumptions, the last exponent is negative. Indeed, given s and ρ as in (7.2), there exists $\delta_\varpi > 0$ and we can choose η sufficiently small in proposition 6.7 (see also remark 6.1), so that

$$s > K_- + \delta_\varpi \quad \text{and} \quad \rho < 2 - \frac{\delta_\varpi + \eta}{K_+}. \tag{7.8}$$

Therefore, by (6.27) we have:

$$c(s) > K_- + \delta_\varpi.$$

Based on the initial data, we can choose β, ε and U suitably small so that $C(\beta,\varepsilon) < \delta_\varpi$. Thus, if U is sufficiently small, there exists $a > 0$ such that:

$$\int_{v_\gamma(u)}^{v} \frac{|\theta|^2}{|\lambda|}(u,v')dv' \le C(N_{\text{i.d.}})e^{-av_\gamma(u)},$$

and bootstrap inequality (7.4) closes, possibly after choosing an even smaller value of U. □

Remark 7.2
The linear analysis performed in [18] shows the analogue of a no-mass-inflation regime in the entire region $K_- < \min\{s, 2K_+\}$. This coincides with the region of the $\rho - s$ plane given in theorem 7.1 (see also Fig. 2.2). The linear analysis in [60] presents, using different techniques, an analogous result where the regularity of the extension depends on the decay of the initial data.

Remark 7.3 (On mass inflation)
In [22], sufficient conditions to show the presence of mass inflation were found. By assuming a lower bound on the scalar field along the event horizon, namely by requiring

$$e^{-(SK_+ + \varepsilon)v} \lesssim |\partial_v\phi|(0,v) \lesssim e^{-(SK_+ - \varepsilon)v},$$

for $0 < S < \min\{\rho, 2\}$ and $\varepsilon > 0$, it was possible to show that $\lim_{v\to+\infty}|\varpi|(u,v) = +\infty$, for every $u \in]0,U]$.

In the charged scalar field case, oscillations affect this scenario and integral bounds for $|\partial_v\phi|$ are expected to be more relevant. Furthermore, the monotonicity of the quantities ϖ, θ, ζ and κ play a prominent role in the proof of the above result in the real, massless case. In particular, in [22], it was possible to prove that the mass inflation scenario is strongly related with the behaviour of the integral in (7.4)*. In particular, in the real and massless case:*

1. *The convergence of the integral in* (7.4) *implies*

$$\int_{v_\gamma(u)}^{v} \frac{|\theta|^2}{|\lambda|}(u,v')dv' \to 0, \qquad \textit{as } u \to 0,$$

whereas in the current case it is not even clear if the limit is well-defined. Even working with subsequences does not allow to immediately control the quantity ϖ.

2. *The lower bound on the scalar field can be propagated from the event horizon to $J^+(\gamma)$, due to the positivity of ζ and θ in $J^+(\mathcal{A})$. In the current case, on the other hand, the system does not include such explicit monotonicity properties (for instance we recall that ζ, θ and ϕ are complex-valued quantities).*

7.2 Construction of an H^1 extension

By prescribing initial data such that conditions (7.2) are satisfied, we proved that the solution to the Einstein-Maxwell-charged-Klein-Gordon system admits a non-trivial Cauchy horizon $\mathcal{CH}^+$, along which $|\varpi|$ remains bounded. In this context, we now prove that such a solution further admits an H^1_{loc} extension up to its Cauchy horizon, namely we show that all PDE variables of the IVP (except possibly for the ones corresponding, in a different coordinate system, to θ and κ) admit a continuous extension, that the Christoffel symbols can be extended in L^2_{loc} and that ϕ can be extended in H^1_{loc}.

Extensions in such a class represent the minimal requirement to understand the extended spacetimes as weak solutions to the Einstein equations. We stress that we required enough regularity so that the second-order PDE system under consideration is equivalent to the first-order system (3.20)–(3.33) (see also remark 4.5). It is interesting to notice that, if less regularity is demanded, it is a priori possible to construct metric extensions beyond the Cauchy horizon that solve the first-order system but not the second-order system of PDEs.

In the current case, the fact that our extensions solve the second-order system of PDEs in a weak sense beyond $\mathcal{CH}^+$ is a corollary of previous work [49], see already corollary 7.7.

To construct the H^1_{loc} extension, we first introduce the coordinate

$$\tilde{v}(v) \coloneqq r(U, v_{\mathcal{A}}(U)) - r(U, v),$$

where $v_{\mathcal{A}}(U) \coloneqq \max\{v \colon \lambda(U, v) = 0\}$ (see also corollary 6.11 for a localization and the main properties of the apparent horizon). We provide a further C^0 extension in the $(u, \tilde{v})$ coordinate system, which will be used to construct an H^1_{loc} extension. We will denote the solutions to our IVP in the coordinate system $(u, \tilde{v})$ by a tilde, e.g. $\tilde{r}(u, \tilde{v}) = r(u, v)$ and $\tilde{\lambda}(u, \tilde{v}) = \partial_{\tilde{v}} \tilde{r}(u, \tilde{v}) = -\frac{\lambda(u,v)}{\lambda(U,v)}$. In this coordinate system, the Cauchy horizon can be expressed as a level set: $\mathcal{CH}^+ = \tilde{v}^{-1}(\tilde{V})$, for some positive constant $\tilde{V}$. Moreover, we denote $\tilde{v}_0 \coloneqq \tilde{v}(v_0)$.

In order to establish an H^1_{loc} extension, we use a further coordinate $\mathring{v}$ defined by the equality

$$\mathring{\Omega}^2(U, \mathring{v}) \equiv 1, \tag{7.9}$$

where the circle superscript is used here and in the following to denote PDE variables expressed in this newly-defined coordinate. Expression (7.9), together with (3.19) and (3.33), entails, in particular, that

$$\frac{d\mathring{v}}{dv} = \Omega^2(U, v).$$

Similarly to the previous cases, we denote $\mathring{v}_0 := \mathring{v}(v_0)$.

The coordinate systems $(u, \tilde{v})$ and $(u, \mathring{v})$ were used in [22, section 12], where their C^1-compatibility up to the Cauchy horizon and the finiteness of $\mathring{V} := \mathring{v}(\tilde{V})$ were shown.[4]

Theorem 7.4

Suppose that the assumptions of theorem 7.1 hold. Then, for every $0 < \delta_U < U$, the functions $\tilde{\lambda}$, $\tilde{\varpi}$, $\tilde{\phi}$, $\tilde{Q}$, $\tilde{r}$, $\tilde{\nu}$, $\tilde{\zeta}$, $\tilde{A}_u$ (but not necessarily $\tilde{\theta}$ or $\tilde{\kappa}$) admit a continuous extension to $[\delta_U, U] \times [0, \tilde{V}]$.

Proof. As observed in the proof of theorem 6.21, to prove that a dynamical quantity $\tilde{q} \in \{\tilde{\lambda}, \tilde{\varpi}, \tilde{\phi}, \tilde{Q}, \tilde{r}, \tilde{\nu}, \tilde{\zeta}, \tilde{A}_u\}$ can be extended continuously, it will be enough to show that

$$\tilde{q}(u, \tilde{V}) := \lim_{\tilde{v} \to \tilde{V}} \tilde{q}(u, \tilde{v})$$

exists, is finite and that

$$\left| \tilde{q}(u, \tilde{v}) - \tilde{q}(u, \tilde{V}) \right| \underset{\tilde{v} \to \tilde{V}}{\longrightarrow} 0 \quad \text{uniformly in } [\delta_U, U].$$

Alternatively, it is sufficient to show that $\tilde{q}$ can be expressed in terms of quantities that can be extended.

For the sake of convenience, we define

$$f(v) := \tilde{v}(v) = r(U, v_{\mathcal{A}}(U)) - r(U, v), \tag{7.10}$$

for every $v \in [v_0, +\infty)$. To extend $\tilde{r}$, we first observe that $\tilde{r}(\cdot, \tilde{V})$ is well-defined by monotonicity. Moreover, for every $u \in [\delta_U, U]$, (6.145) implies

$$\int_{\tilde{v}}^{\tilde{V}} |\tilde{\lambda}|(u, z) dz = \int_{f^{-1}(\tilde{v})}^{+\infty} |\lambda|(u, z) dz \underset{\tilde{v} \to \tilde{V}}{\longrightarrow} 0,$$

uniformly in $[\delta_U, U]$. Due to the fundamental theorem of calculus, this entails that $\tilde{r}(\cdot, \tilde{v})$ converges to $\tilde{r}(\cdot, \tilde{V})$ uniformly in $[\delta_U, U]$.

Let us now deal with $\tilde{\varpi}$. By (7.3) and due to the estimates of proposition 6.18, it is clear that we only need to verify the convergence of $\int \frac{|\tilde{\theta}|^2}{|\tilde{\lambda}|}(u, z) dz$. In particular, using the bound (7.4), which was seen to hold in $J^+(\gamma)$, we have

$$\int_{\tilde{v}}^{\tilde{V}} \frac{|\tilde{\theta}|^2}{|\tilde{\lambda}|}(u, z) dz = \int_{f^{-1}(\tilde{v})}^{+\infty} \frac{|\theta|^2}{|\lambda|}(u, z) dz \underset{\tilde{v} \to \tilde{V}}{\longrightarrow} 0, \tag{7.11}$$

uniformly in u (in fact, $|\theta \lambda^{-1}|$ is a geometric quantity and the factor coming from $\tilde{\theta}$ cancels out with the Jacobian of the change of variables). Thus, $\tilde{\varpi}(\cdot, \tilde{V})$ is well-defined and, by the same estimate, $\tilde{\varpi}(\cdot, \tilde{v})$ converges to $\tilde{\varpi}(\cdot, \tilde{V})$ as $\tilde{v} \to \tilde{V}$, uniformly in u.

[4] In particular, every C^0 extension in the $(u, \tilde{v})$ coordinate system gives rise to a C^0 extension in the $(u, \mathring{v})$ coordinate system, and vice versa.

The charge $\tilde{Q}$ can be extended in a similar way: by (3.31) we have

$$\tilde{Q}(u,\tilde{V}) = \tilde{Q}(u,\tilde{v}) + q\int_{\tilde{v}}^{\tilde{V}} \tilde{r}\,\mathrm{Im}\Big(\tilde{\phi}\overline{\tilde{\theta}}\Big)(u,z)dz.$$

By the boundedness of $\tilde{r}$ and due to the estimates proved in proposition 6.18:

$$\int_{\tilde{v}}^{\tilde{V}} |\tilde{\phi}\tilde{\theta}|(u,z)dz = \int_{f^{-1}(\tilde{v})}^{+\infty} |\phi\theta|(u,z)dz \underset{\tilde{v}\to\tilde{V}}{\longrightarrow} 0,$$

uniformly in u. Thus, $\tilde{Q}(\cdot,\tilde{V})$ is well-defined and $\tilde{Q}$ admits a C^0 extension to the Cauchy horizon.

Analogously, we can integrate (3.23) to obtain

$$\tilde{\phi}(u,\tilde{V}) = \tilde{\phi}(u,\tilde{v}) + \int_{\tilde{v}}^{\tilde{V}} \frac{\tilde{\theta}}{\tilde{r}}(u,z)dz.$$

We can again exploit the boundedness of $\tilde{r}$ and the estimates holding in $J^+(\gamma)$ to observe that

$$\int_{\tilde{v}}^{\tilde{V}} |\tilde{\theta}|(u,z)dz = \int_{f^{-1}(\tilde{v})}^{+\infty} |\theta|(u,z)dz \underset{\tilde{v}\to\tilde{V}}{\longrightarrow} 0, \tag{7.12}$$

uniformly in $[\delta_U, U]$.

Now, $\frac{\tilde{\nu}}{1-\tilde{\mu}}(\cdot,\tilde{V})$ is well-defined in $[\delta_U, U]$, due to the monotonicity of equation (6.97) for $\tilde{v}$ large. Integrating the latter and using (7.11), we also have:

$$\sup_{u\in[\delta_U,U]} \left| \frac{\tilde{\nu}}{1-\tilde{\mu}}(u,\tilde{v}) - \frac{\tilde{\nu}}{1-\tilde{\mu}}(u,\tilde{V}) \right| \underset{\tilde{v}\to\tilde{V}}{\longrightarrow} 0, \tag{7.13}$$

i.e. $\frac{\tilde{\nu}}{1-\tilde{\mu}}$ admits a continuous extension. Moreover, equation (6.97), the finiteness of $\int \frac{|\theta|^2}{|\lambda|}$ (see (7.11)) and the estimates (6.70), (6.31) and (6.93) obtained along Γ_Y show that there exists $C > 0$ such that

$$\frac{\tilde{\nu}}{1-\tilde{\mu}}(u,\tilde{V}) \gtrsim \frac{\tilde{\nu}}{1-\tilde{\mu}}(u,\tilde{v}(v_Y(u))) = \frac{\nu}{1-\mu}(u,v_Y(u)) \geq C > 0,$$

for every u in $[\delta_U, U]$. We can use this quantity to extend $\tilde{\lambda}$. In fact, exploiting (3.33), the fact that $\tilde{\lambda}(U,\tilde{v}) = -1$ for $\tilde{v}$ large and (3.35), we have:

$$\tilde{\lambda}(u,\tilde{v}) = -\exp\left(-\int_u^U \frac{\tilde{\nu}}{1-\tilde{\mu}}\Big(2\tilde{K} - m^2 r|\tilde{\phi}|^2\Big)(u',\tilde{v})du' \right).$$

Let us now focus on the quantities under the last integral sign. By (3.34) and by the previous steps of the current proof, all these quantities admit a C^0 extension. Thus, $\tilde{\lambda}$ does as well. Moreover, $\tilde{\lambda}(\cdot,\tilde{V})$ is a negative function.

We can use the above, (3.19) and (3.33) to observe that

$$\tilde{\Omega}^2 = -4\tilde{\nu}\tilde{\kappa} = -4\tilde{\lambda}\frac{\tilde{\nu}}{1-\tilde{\mu}}$$

extends continuously and is positive on the Cauchy horizon.

To extend $\tilde{\nu}$, we observe that $\tilde{\nu}(\cdot, \tilde{V})$ is well-defined by monotonicity. Indeed, under the assumptions of theorem 7.1 and for U sufficiently small, K is sufficiently close to $-K_-$ in $J^+(\gamma)$ and thus $\partial_v \nu > 0$ by (3.35). The latter equation, together with (3.19) and the fact that $\tilde{\Omega}^2$, $\tilde{K}$ and $\tilde{\phi}$ are continuous in the compact set $[\delta_U, U] \times [\tilde{v}_0, \tilde{V}]$, gives

$$\left|\tilde{\nu}(u,\tilde{v}) - \tilde{\nu}(u,\tilde{V})\right| = \left|\int_{\tilde{v}}^{\tilde{V}} \frac{\tilde{\Omega}^2}{4}\left(2\tilde{K} - m^2\tilde{r}|\tilde{\phi}|^2\right)(u,z)dz\right| \underset{\tilde{v}\to\tilde{V}}{\longrightarrow} 0,$$

uniformly in u. An integration of (3.32), the relation $\tilde{\Omega}^2 = -4\tilde{\nu}\tilde{\kappa}$ and a similar procedure also show the continuous extension of A_u.

Finally, for every $u \in [\delta_U, U]$, we integrate (3.28) to write

$$\tilde{\zeta}(u,\tilde{V}) = \tilde{\zeta}(u,\tilde{v}) - \int_{\tilde{v}}^{\tilde{V}} \left(\frac{\tilde{\theta}}{\tilde{r}}\tilde{\nu} + \frac{\tilde{\Omega}^2\tilde{\phi}}{4\tilde{r}}\left(m^2\tilde{r}^2 + iq\tilde{Q}\right)\right)(u,z)dz.$$

The term multiplied by $\tilde{\Omega}^2$ under the last integral sign was shown to be continuous in the compact set $[\delta_U, U] \times [\tilde{v}_0, \tilde{V}]$. Moreover, $\tilde{\nu}$ and $\tilde{r}$ admit a continuous extensions, and the latter is bounded away from zero on the Cauchy horizon. To prove that $\tilde{\zeta}(\cdot, \tilde{V})$ is well-defined, we are left to prove the convergence of $\int_{\tilde{v}}^{\tilde{V}} |\tilde{\theta}|(u,z)dz$. This follows from (7.12), which is a stronger result and is also sufficient to prove the uniform convergence of $\tilde{\zeta}(\cdot,\tilde{v})$ to $\tilde{\zeta}(\cdot,\tilde{V})$, as $\tilde{v}$ goes to $\tilde{V}$. □

Theorem 7.5 (An H^1_{loc} extension)
Suppose that the assumptions of theorem 7.1 hold. Then, for every $0 < \delta_U < U$*, there exists a continuous extension of the metric* g *to the set* $\mathcal{D} = [\delta_U, U] \times [\tilde{v}_0, \tilde{V}]$*, with Christoffel symbols in* $L^2(\mathcal{D})$ *and such that* $\phi \in H^1(\mathcal{D})$.

Proof. Theorem 7.4 gives a continuous extension in the $(u, \tilde{v})$ coordinate system. Since $(u,\tilde{v})$ and $(u, \mathring{v})$ correspond to C^1–compatible local charts (see [22, section 12]), a continuous extension in the latter coordinate system promptly follows. We will work in the $(u, \mathring{v})$ system throughout the remaining part of the proof.

The Christoffel symbols are given explicitly in appendix C. Due to the fact that $\tilde{r}$ and $\tilde{\Omega}^2$ are strictly positive on the Cauchy horizon (see the proof of theorem 7.4), we only need to check that Γ^u_{uu} and $\Gamma^{\mathring{v}}_{\mathring{v}\mathring{v}}$ are in L^2_{loc}. The proof of this result is completely analogous to the proof of theorem 12.3 in [22].

To show that $\|\phi\|_{H^1(\mathcal{D})}$ is bounded, with $\mathcal{D} = [\delta_U, U] \times [\tilde{v}_0, \tilde{V}]$, we notice two key results:

- Due to the continuous extensions of $\mathring{\phi}$, $\mathring{\zeta}$ and $\mathring{A}_u$, there exists $C > 0$, depending uniquely on the initial data, such that
$$\int_{\mathring{v}_0}^{\mathring{V}} \left(|\phi|^2 + |\partial_u\phi|^2\right)(u,z)dz \le C,$$
for every $u \in [\delta_U, U]$.

- In the proof of theorem 7.1 we showed that there exists $a > 0$ such that

$$\frac{|\theta|^2}{|\lambda|}(u,v) \lesssim \frac{e^{-av}}{\tilde{C}(u)}, \quad \forall\,(u,v) \in J^+(\gamma),$$

for some constant $\tilde{C}(u) > 0$ depending on u. We obtained such a result using the lower bound (7.7). The same bound also gives

$$\frac{|\theta|^2(u,v)}{|\lambda|(U,v)} \lesssim \frac{e^{-av}}{\tilde{C}(U)}, \quad \forall\,(u,v) \in J^+(\gamma).$$

So, using the latter, using the fact that:[5]

$$\frac{\Omega^2}{|\lambda|}(U,v) = 4\frac{\nu}{1-\mu}(U,v) = 4\frac{\tilde{\nu}}{1-\tilde{\mu}}(U,\tilde{v}(v)) = \tilde{\Omega}^2(U,\tilde{v}(v)) \geq C > 0, \quad \forall\, v \geq v_0,$$

and, finally, exploiting the relation $\mathring{\theta}(u,\mathring{v}) = \Omega^{-2}(U,v(\mathring{v}))\theta(u,v(\mathring{v}))$, we have:

$$\int_{\mathring{v}_0}^{\mathring{V}} |\mathring{\theta}|^2(u,z)dz = \int_{v_0}^{+\infty} \frac{|\theta|^2(u,z)}{\Omega^2(U,z)}dz \lesssim \int_{v_0}^{+\infty} \frac{|\theta|^2(u,z)}{|\lambda|(U,z)}dz \leq C,$$

for some $C > 0$ depending uniquely on the initial data. □

Corollary 7.6
Let M_0 be the preimage of $(0,U] \times [\mathring{v}_0, \mathring{V}]$ via the coordinate functions $(u,\mathring{v})$. Then, the pull-backed scalar field and Christoffel symbols on M_0 are, respectively, in $H^1_{loc}(M_0)$ and $L^2_{loc}(M_0)$.

Proof. For every $0 < \delta_U < U$, we denote by $\mathcal{M}_\delta$ the preimage of $[\delta_U, U] \times [\mathring{v}_0, \mathring{V}]$ via the coordinate functions $(u,\mathring{v})$. The volume form on $\mathcal{M}_\delta$ is

$$dV_4 = \frac{\mathring{\Omega}^2}{2}\mathring{r}^2 du d\mathring{v}.$$

Since $\mathring{r}$ and $\mathring{\Omega}^2$ are continuous in the compact set $[\delta_U, U] \times [\mathring{v}_0, \mathring{V}]$, the results of theorem 7.5 are sufficient to conclude the proof. □

The following is a consequence of [49, Section 10], where the asymptotically flat case was considered for extremal black holes. We recall that we have defined a coordinate chart $(u,\mathring{v})$ that allows to parametrize the Cauchy horizon as $\mathcal{CH}^+ = \mathring{v}^{-1}(\mathring{V})$, with $0 < \mathring{V} < +\infty$.

Corollary 7.7 (Extensions beyond the Cauchy horizon as weak solutions)
For every $0 < \delta_U < U$, there exists $\epsilon = \epsilon(\delta_U) > 0$ such that there exist infinitely many inequivalent extensions of the metric g to the region

$$\mathcal{D} := \mathcal{P} \cup \Big([\delta_U, \delta_U + \epsilon] \times [\mathring{V}, \mathring{V} + \epsilon]\Big),$$

with Christoffel symbols in $L^2(\mathcal{D})$, $\phi \in H^1(\mathcal{D})$ and such that the extensions satisfy the second-order

[5] We notice that $|\tilde{\lambda}|(U,\cdot) \equiv 1$ and that $\tilde{\Omega}^2$ is positive up to the Cauchy horizon, see the proof of theorem 7.4.

system of PDEs in an integrated sense.

Proof. The same procedure of [49, Section 10] can be applied in our case, since the presence of the cosmological constant Λ in our second-order system (3.6)–(3.10), (3.30)–(3.32) does not represent an obstruction to the reasoning of [49], and neither does the sub-extremality condition.

To summarize, the proof requires a local well-posedness result similar to that in chapter 4, where however local existence is now proved for solutions[6] $(\phi, \Omega^2, r, A, Q)$ to the **second-order system** and the quantities $\partial_v \phi$ and $\partial_v \log \Omega^2$ are assumed to be only square-integrable in the v–direction, along the initial hypersurface. In the low-regularity local well-posedness result that we are considering, the Raychaudhuri equations and the Maxwell equations are treated as constraints. The equation (3.32) for the electromagnetic potential A is satisfied classically, whereas the wave equations for r, ϕ and $\log \Omega^2$ (see (3.8), (3.10) and (3.9)) are satisfied in an integrated sense.

The next step is to extend the outgoing null cone $\{u = \delta_U\} \subset \mathcal{P}$ up to the Cauchy horizon $\mathcal{CH}^+$. Then, a sequence of initial value problems is considered, with initial data on

$$[\delta_U, \delta_U + \epsilon] \times \{\mathring{v}_n\} \cup \{\delta_U\} \times [\mathring{v}_n, \mathring{v}_n + \epsilon],$$

where $(\mathring{v}_n)_{n \in \mathbb{N}} \subset [\mathring{v}_0, \mathring{V})$ and $\mathring{v}_n \to \mathring{V}$ as $n \to +\infty$. To prescribe the initial data for $(\mathring{\phi}, \mathring{\Omega}^2, \mathring{r}, \mathring{A})$ along the Cauchy horizon, we notice the following:

- $\mathring{\Omega}^2(\delta_U, \mathring{v})$ is continuous and locally square-integrable in the $\mathring{v}$-coordinate, and is bounded away from zero up to the Cauchy horizon (see theorem 7.5). Therefore, we can extend $\mathring{\Omega}^2$ to a subset of

  $$\{(\delta_U, \mathring{v}) \colon \mathring{v} > \mathring{V}\},$$

  in such a way that $\mathring{\Omega}^2$ is still continuous, locally square-integrable, bounded and bounded away from zero.

- the functions $\mathring{\phi}$ and $\mathring{A}$ can be extended by following a procedure analogous to that for $\mathring{\Omega}^2$. The same can be said for $\mathring{r}$, which however has to satisfy a constraint given by the Raychaudhuri equation in the $\mathring{v}$ direction.

Assume that we extended the PDE variables that play the role of the initial data beyond the Cauchy horizon, up to the value $\mathring{v} = \mathring{V} + a$, for some $a > 0$ small. The local existence theorem then provides an integrated solution in

$$[\delta_U, \delta_U + \epsilon] \times [\mathring{v}_n, \mathring{v}_n + \epsilon],$$

for every n, for a small $0 < \epsilon < a$ that depends on the L^2 and L^∞ norms of the initial data on $[\delta_U, \delta_U + \epsilon] \times \{\mathring{v}_n\} \cup \{\delta_U\} \times [\mathring{v}_n, \mathring{V} + a]$, that does not depend on the value of n.[7] For n sufficiently large, this gives a metric solution of H^1 regularity beyond the Cauchy horizon. □

[6] Here, we are considering solutions in an integrated sense, in the language of [49].

[7] Notice that the initial data in the region $\mathring{v} \leq \mathring{V}$ are given by the solution that we constructed in the previous sections. On the other hand, the norms of such solution depend on the initial data on $[0, U) \times \{v_0\} \cup \{0\} \times [v_0, +\infty)$.

Bibliography

[1] Event Horizon Telescope Collaboration et al. "First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way". In: *Astrophysical Journal Letters* 930.2 (2022), p. L12. DOI: 10.3847/2041-8213/ac6674.

[2] X. An and Z.F. Lim. "Trapped Surface Formation for Spherically Symmetric Einstein–Maxwell-Charged Scalar Field System with Double Null Foliation". In: *Ann. Henri Poincaré* 23 (2022), pp. 3159–3190. DOI: 10.1007/s00023-022-01168-y.

[3] L. Andersson and P. Blue. "Hidden symmetries and decay for the wave equation on the Kerr spacetime". In: *Ann. Math.* 182 (2015), pp. 787–853. DOI: 10.4007/annals.2015.182.3.1.

[4] Y. Angelopoulos, S. Aretakis, and D. Gajic. "Late-time asymptotics for the wave equation on spherically symmetric, stationary spacetimes". In: *Advances in Mathematics* 323 (2018), pp. 529–621. DOI: 10.1016/j.aim.2017.10.027.

[5] Y. Angelopoulos, S. Aretakis, and D. Gajic. "Price's Law and Precise Late-Time Asymptotics for Subextremal Reissner–Nordström Black Holes". In: *Ann. Henri Poincaré* 24 (2023), pp. 3215–3287. DOI: 10.1007/s00023-023-01328-8.

[6] S. Aretakis. "Stability and Instability of Extreme Reissner-Nordström Black Hole Spacetimes for Linear Scalar Perturbations I". In: *Commun. Math. Phys.* 307 (2011), pp. 17–63. DOI: 10.1007/s00220-011-1254-5.

[7] A. S. Barreto and M. Zworski. "Distribution of resonances for spherical black holes". In: *Math. Res. Lett.* 4 (1997), pp. 103–121. DOI: 10.4310/MRL.1997.v4.n1.a10.

[8] G. Benomio. *A new gauge for gravitational perturbations of Kerr spacetimes II: The linear stability of Schwarzschild revisited.* 2022. arXiv: 2211.00616 [gr-qc].

[9] V. Cardoso, J. L. Costa, K. Destounis, P. Hintz, and A. Jansen. "Quasinormal Modes and Strong Cosmic Censorship". In: *Phys. Rev. Lett.* 120 (3 Jan. 2018), p. 031103. DOI: 10.1103/PhysRevLett.120.031103.

[10] V. Cardoso, J. L. Costa, K. Destounis, P. Hintz, and A. Jansen. "Strong cosmic censorship in charged black-hole spacetimes: Still subtle". In: *Phys. Rev. D* 98 (10 Nov. 2018), p. 104007. DOI: 10.1103/PhysRevD.98.104007.

[11] M. Casals and C. I. S. Marinho. "Glimpses of violation of strong cosmic censorship in rotating black holes". In: *Phys. Rev. D* 106 (4 2022), p. 044060. DOI: 10.1103/PhysRevD.106.044060.

[12] Y. Choquet-Bruhat and R. Geroch. "Global aspects of the Cauchy problem in general relativity". In: *Commun. Math. Phys.* 14 (1969), pp. 329–335. DOI: 10.1007/BF01645389.

[13] D. Christodoulou. "On the global initial value problem and the issue of singularities". In: *Classical and Quantum Gravity* 16.12A (1999), A23–A35. ISSN: 0264-9381, 1361-6382. DOI: 10.1088/0264-9381/16/12A/302.

[14] D. Christodoulou. *The Formation of Black Holes in General Relativity.* EMS Press, 2009. DOI: 10.4171/068.

[15] P. Chruściel. "On uniqueness in the large of solutions of Einstein's equations ("strong cosmic censorship")". In: *Proceedings of the Centre for Mathematical Analysis, Australian National University* 27 (1991).

[16] P. T. Chruściel and J. Isenberg. "Nonisometric vacuum extensions of vacuum maximal globally hyperbolic spacetimes". In: *Phys. Rev. D* 48 (4 1993), pp. 1616–1628. DOI: 10.1103/PhysRevD.48.1616.

[17] P.T. Chruściel, J.L. Costa, and M. Heusler. "Stationary Black Holes: Uniqueness and Beyond". In: *Living Rev. Relativ.* 15 (7 2012). DOI: 10.12942/lrr-2012-7.

[18] J. Costa and A. T. Franzen. "Bounded Energy Waves on the Black Hole Interior of Reissner–Nordström–de Sitter". In: *Ann. Henri Poincaré* 18 (2017), pp. 3371–3398. DOI: 10.1007/s00023-017-0592-z.

[19] J. Costa, P. Girão, J. Natário, and J. Silva. "On the global uniqueness for the Einstein–Maxwell–scalar field system with a cosmological constant. Part 1: Well posedness and breakdown criterion". In: *Class. Quantum Grav.* 32.1 (2015), p. 015017. DOI: 10.1088/0264-9381/32/1/015017.

[20] J. Costa, P. Girão, J. Natário, and J. Silva. "On the global uniqueness for the Einstein–Maxwell–scalar field system with a cosmological constant. Part 2: Structure of the solutions and stability of the Cauchy horizon". In: *Commun. Math. Phys.* 339 (2015), pp. 903–947. DOI: 10.1007/s00220-015-2433-6.

[21] J. Costa, P. Girão, J. Natário, and J. Silva. "On the global uniqueness for the Einstein–Maxwell–scalar field system with a cosmological constant. Part 3: Mass inflation and extendibility of the solutions". In: *Ann. PDE* 3.8 (2017). DOI: 10.1007/s40818-017-0028-6.

[22] J. Costa, P. Girão, J. Natário, and J. Silva. "On the occurrence of mass inflation for the Einstein-Maxwell-scalar field system with a cosmological constant and an exponential Price law". In: *Commun. Math. Phys.* 361 (2018), pp. 289–341. DOI: 10.1007/s00220-018-3122-z.

[23] J.L. Costa. "The formation of trapped surfaces in the gravitational collapse of spherically symmetric scalar fields with a positive cosmological constant". In: *Class. Quantum Grav.* 37 (2020), p. 195022. DOI: 10.1088/1361-6382/abb075.

[24] J.L. Costa and P.M. Girão. "Higher order linear stability and instability of Reissner–Nordström's Cauchy horizon". In: *Anal. Math. Phys.* 10 (40 2020). DOI: 10.1007/s13324-020-00380-5.

[25] M. Dafermos. "Stability and instability of the Cauchy horizon for the spherically symmetric Einstein–Maxwell-scalar field equations". In: *Ann. Math.* 158.3 (2003), pp. 875–928. ISSN: 0003-486X. DOI: 10.4007/annals.2003.158.875.

[26] M. Dafermos. "Spherically symmetric spacetimes with a trapped surface". In: *Class. Quantum Grav.* 22.11 (2005), pp. 2221–2232. ISSN: 0264-9381, 1361-6382. DOI: 10.1088/0264-9381/22/11/019.

[27] M. Dafermos. "The interior of charged black holes and the problem of uniqueness in general relativity". In: *Commun. Pure Appl. Math.* 58.4 (2005), pp. 445–504. ISSN: 1097-0312. DOI: 10.1002/cpa.20071.

[28] M. Dafermos, G. Holzegel, and I. Rodnianski. "The linear stability of the Schwarzschild solution to gravitational perturbations". In: *Acta Math.* 222.1 (2019), pp. 1–214. DOI: 10.4310/ACTA.2019.v222.n1.a1.

[29] M. Dafermos, G. Holzegel, I. Rodnianski, and M. Taylor. *The non-linear stability of the Schwarzschild family of black holes.* 2021. arXiv: 2104.08222 [gr-qc].

[30] M. Dafermos and J. Luk. "The interior of dynamical vacuum black holes I: The C^0-stability of the Kerr Cauchy horizon". In: arXiv:1710.01722 (2017). URL: http://arxiv.org/abs/1710.01722.

[31] M. Dafermos and I. Rodnianski. "A proof of Price's law for the collapse of a self-gravitating scalar field". In: *Invent. math.* 162 (2005), pp. 381–457. DOI: 10.1007/s00222-005-0450-3.

[32] M. Dafermos and I. Rodnianski. *The wave equation on Schwarzschild-de Sitter spacetimes.* 2007. arXiv: 0709.2766 [gr-qc].

[33] M. Dafermos and Y. Shlapentokh-Rothman. "Time-Translation Invariance of Scattering Maps and Blue-Shift Instabilities on Kerr Black Hole Spacetimes". In: *Commun. Math. Phys.* 350 (2017), pp. 985–1016. DOI: 10.1007/s00220-016-2771-z.

[34] M. Dafermos and Y. Shlapentokh-Rothman. "Rough initial data and the strength of the blue-shift instability on cosmological black holes with $\Lambda > 0$". In: *Class. Quantum Grav.* 35 (2018), p. 195010. DOI: 10.1088/1361-6382/aadbcf.

[35] Alex Davey, Oscar J. C. Dias, and David Sola Gil. *Strong Cosmic Censorship in Kerr-Newman-de Sitter.* 2024. arXiv: 2404.03724 [gr-qc].

[36] O. J. C. Dias, F. C. Eperon, H. S. Reall, and J. E. Santos. "Strong cosmic censorship in de Sitter space". In: *Phys. Rev. D* 97 (10 2018), p. 104060. DOI: 10.1103/PhysRevD.97.104060.

[37] O. J. C. Dias, H. S. Reall, and J. E. Santos. "Strong cosmic censorship: taking the rough with the smooth". In: *J. High Energ. Phys.* 1 (2018). DOI: 10.1007/JHEP10(2018)001.

[38] R. Donninger, W. Schlag, and A. Soffer. "A proof of Price's Law on Schwarzschild black hole manifolds for all angular momenta". In: *Adv. Math.* 226 (1 2011), pp. 484–540. DOI: 10.1016/j.aim.2010.06.026.

[39] R. Donninger, W. Schlag, and A. Soffer. "On Pointwise Decay of Linear Waves on a Schwarzschild Black Hole Background". In: *Commun. Math. Phys.* 309 (2012), pp. 51–86. DOI: 10.1007/s00220-011-1393-8.

[40] S. Dyatlov. "Asymptotics of Linear Waves and Resonances with Applications to Black Holes". In: *Commun. Math. Phys.* 335 (2015), pp. 1445–1485. DOI: 10.1007/s00220-014-2255-y.

[41] G. F. R. Ellis and B. G. Schmidt. "Singular space-times". In: *Gen. Relat. Gravit.* 8 (1977), pp. 915–953. DOI: 10.1007/BF00759240.

[42] European Space Agency. *Science & Exploration: Space Science.* URL: https://www.esa.int/Science_Exploration/Space_Science/ (visited on 04/10/2024).

[43] European Space Agency. *White Dwarfs.* URL: https://esahubble.org/wordbank/white-dwarf/ (visited on 04/10/2024).

[44] Allen Juntao Fang. *Linear stability of the slowly-rotating Kerr-de Sitter family.* 2022. arXiv: 2207.07902 [gr-qc].

[45] Allen Juntao Fang. *Nonlinear stability of the slowly-rotating Kerr-de Sitter family.* 2022. arXiv: 2112.07183 [math.AP].

[46] Y. Fourès-Bruhat. "Théorème d'existence pour certains systèmes d'équations aux derivées partielles non linéaires". In: *Acta Mathematica* 88 (1952), pp. 141–225.

[47] D. Gajic. "Linear Waves in the Interior of Extremal Black Holes I". In: *Commun. Math. Phys.* 353 (2017), pp. 717–770. DOI: 10.1007/s00220-016-2800-y.

[48] D. Gajic. "Linear Waves in the Interior of Extremal Black Holes II". In: *Ann. Henri Poincaré* 18 (2017), pp. 4005–4081. DOI: 10.1007/s00023-017-0614-x.

[49] D. Gajic and J. Luk. "The interior of dynamical extremal black holes in spherical symmetry". In: *Pure and Applied Analysis* 1.2 (2019). arXiv:1709.09137 [gr-qc], pp. 263–326. ISSN: 2578-5885, 2578-5893. DOI: 10.2140/paa.2019.1.263.

[50] R. Geroch. "What is a singularity in general relativity?" In: *Ann. of Phys.* 48 (3 1968), pp. 526–540. DOI: 10.1016/0003-4916(68)90144-9.

[51] E. Giorgi. "The Linear Stability of Reissner–Nordström Spacetime: The Full Subextremal Range $|Q| < M$". In: *Commun. Math. Phys.* 380 (2020), pp. 1313–1360. DOI: 10.1007/s00220-020-03893-z.

[52] Elena Giorgi, Sergiu Klainerman, and Jeremie Szeftel. *Wave equations estimates and the nonlinear stability of slowly rotating Kerr black holes.* 2022. arXiv: 2205.14808 [math.AP].

[53] C. Gundlach, R. H. Price, and J. Pullin. "Late-time behavior of stellar collapse and explosions. I. Linearized perturbations". In: *Phys. Rev. D* 49.2 (1994), pp. 883–889. DOI: 10.1103/PhysRevD.49.883.

[54] Y. Gürsel, V. D. Sandberg, I. D. Novikov, and A. A. Starobinsky. "Evolution of scalar perturbations near the Cauchy horizon of a charged black hole". In: *Phys. Rev. D* 19.2 (1979), pp. 413–420.

[55] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time.* Cambridge University Press, 1973.

[56] J. I. González Hernández, R. Rebolo, L. Pasquini, G. Lo Curto, P. Molaro, E. Caffau, H.-G. Ludwig, M. Steffen, M. Esposito, A. Suárez Mascareño, B. Toledo-Padrón, R. A. Probst, T. W. Hänsch, R. Holzwarth, A. Manescau, T. Steinmetz, Th. Udem, and T. Wilken. "The solar gravitational redshift from HARPS-LFC Moon spectra – A test of the general theory of relativity". In: *Astronomy and Astrophysics* 643 (2020), A146. DOI: 10.1051/0004-6361/202038937.

[57] P. Hintz. "Non-linear Stability of the Kerr-Newman-de Sitter Family of Charged Black Holes". In: *Ann. PDE* 4.11 (2018), pp. 2199–2576. DOI: 10.1007/s40818-018-0047-y.

[58] P. Hintz. *Mode stability and shallow quasinormal modes of Kerr-de Sitter black holes away from extremality.* 2021. arXiv: 2112.14431 [gr-qc].

[59] P. Hintz. "A Sharp Version of Price's Law for Wave Decay on Asymptotically Flat Spacetimes." In: *Commun. Math. Phys.* 389 (2022), pp. 491–542. DOI: 10.1007/s00220-021-04276-8.

[60] P. Hintz and A. Vasy. "Analysis of linear waves near the Cauchy horizon of cosmological black holes". In: *J. Math. Phys.* 58.8 (2017), p. 081509. DOI: 10.1063/1.4996575.

[61] P. Hintz and A. Vasy. "The global non-linear stability of the Kerr–de Sitter family of black holes". In: *Acta Math.* 220 (2018), pp. 1–206. DOI: 10.4310/ACTA.2018.v220.n1.a1.

[62] Peter Hintz and YuQing Xie. "Quasinormal modes of small Schwarzschild–de Sitter black holes". In: *J. Math. Phys.* 63 (2022), p. 011509. DOI: 10.1063/5.0062985.

[63] G. Holzegel and J. Smulevici. "Decay Properties of Klein-Gordon Fields on Kerr-AdS Spacetimes". In: *Communications on Pure and Applied Mathematics* 66.11 (2013), pp. 1751–1802. DOI: 10.1002/cpa.21470.

[64] G. Holzegel and J. Smulevici. "Quasimodes and a lower bound on the uniform energy decay rate for Kerr–AdS spacetimes". In: *Analysis & PDE* 7.5 (2014), pp. 1057–1090. DOI: 10.2140/apde.2014.7.1057.

[65] C. Kehle. "Uniform Boundedness and Continuity at the Cauchy Horizon for Linear Waves on Reissner–Nordström–AdS Black Holes". In: *Commun. Math. Phys.* 376 (2020), pp. 146–200. DOI: 10.1007/s00220-019-03529-x.

[66] C. Kehle. "Diophantine approximation as Cosmic Censor for Kerr–AdS black holes". In: *Invent. math.* 227 (2022), pp. 1169–1321. DOI: 10.1007/s00222-021-01078-6.

[67] C. Kehle and M. Van de Moortel. *Strong Cosmic Censorship in the presence of matter: the decisive effect of horizon oscillations on the black hole interior geometry.* 2022. arXiv: 2105.04604 [gr-qc].

[68] S. Klainerman, I. Rodnianski, and J. Szeftel. "The bounded L^2 curvature conjecture". In: *Invent. math.* 202 (2015), pp. 91–216. DOI: 10.1007/s00222-014-0567-3.

[69] S. Klainerman and J. Szeftel. *Kerr stability for small angular momentum.* 2021. arXiv: 2104.11857 [math.AP].

[70] S. Klainerman and J. Szeftel. *Brief introduction to the nonlinear stability of Kerr.* 2022. arXiv: 2210.14400 [math.AP].

[71] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, Volume I.* Wiley – Interscience, 1996.

[72] J. Kommemi. "The Global Structure of Spherically Symmetric Charged Scalar Field Spacetimes". In: *Commun. Math. Phys.* 323 (2013), pp. 35–106. DOI: 10.1007/s00220-013-1759-1.

[73] J. Luk and S.-J. Oh. "Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data I. The interior of the black hole region". In: *Ann. Math.* 190.1 (2019), pp. 1–111. DOI: 10.4007/annals.2019.190.1.1.

[74] J. M. Martín–García. *xAct: efficient tensor computer algebra.* URL: http://www.xact.es (visited on 07/07/2023).

[75] G. Mavrogiannis. "Morawetz Estimates Without Relative Degeneration and Exponential Decay on Schwarzschild–de Sitter Spacetimes". In: *Ann. Henri Poincaré* (2023). DOI: 10.1007/s00023-023-01293-2.

[76] J. M. McNamara. "Instability of Black Hole Inner Horizons". In: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 358.1695 (1978), pp. 499–517.

[77] J. Metcalfe, D. Tataru, and M. Tohaneanu. "Price's law on nonstationary space–times". In: *Adv. Math.* 230.3 (2012), pp. 995–1028. DOI: 10.1016/j.aim.2012.03.010.

[78] Yuyu Mo, Yu Tian, Bin Wang, Hongbao Zhang, and Zhen Zhong. "Strong cosmic censorship for the massless charged scalar field in the Reissner-Nordstrom–de Sitter spacetime". In: *Phys. Rev. D* 98 (12 Dec. 2018), p. 124025. DOI: 10.1103/PhysRevD.98.124025.

[79] M. Van de Moortel. "Stability and Instability of the Sub-extremal Reissner–Nordström Black Hole Interior for the Einstein–Maxwell–Klein–Gordon Equations in Spherical Symmetry". In: *Commun. Math. Phys.* 360 (2018), pp. 103–168. DOI: 10.1007/s00220-017-3079-3.

[80] M. Van de Moortel. "Mass Inflation and the C^2–inextendibility of Spherically Symmetric Charged Scalar Field Dynamical Black Holes". In: *Commun. Math. Phys.* 382 (2021), pp. 1263–1341. DOI: 10.1007/s00220-020-03923-w.

[81] M. Van de Moortel. *The breakdown of weak null singularities inside black holes.* 2022. arXiv: 1912.10890 [gr-qc].

[82] J. Natário. *An Introduction to Mathematical Relativity.* Springer Cham, 2021.

[83] Ryan S. Park, William M. Folkner, Alexander S. Konopliv, James G. Williams, David E. Smith, and Maria T. Zuber. "Precession of Mercury's Perihelion from Ranging to the MESSENGER Spacecraft". In: *Astronomical Journal* 153 (2017), p. 121. DOI: 10.3847/1538-3881/aa5be2.

[84] R. Penrose. "Structure of space-time". In: *Battelle Rencontres, 1967 Lectures in mathematics and physics.* 1968, pp. 121–235.

[85] R. Penrose. "Singularities and Time-Asymmetry". In: *Hawking, S.W. and Israel, W., Eds., General Relativity: An Einstein Centenary Survey.* Cambridge University Press, 1979, pp. 581–638.

[86] E. Poisson and W. Israel. “Internal structure of black holes”. In: *Phys. Rev. D* 41 (6 Mar. 1990), pp. 1796–1809. DOI: 10.1103/PhysRevD.41.1796.

[87] R. H. Price. “Nonspherical Perturbations of Relativistic Gravitational Collapse. I. Scalar and Gravitational Perturbations”. In: *Phys. Rev. D* 5.10 (1972), pp. 2419–2438. DOI: 10.1103/PhysRevD.5.2419.

[88] T. Regge and J. A. Wheeler. “Stability of a Schwarzschild Singularity”. In: *Phys. Rev.* 108.4 (1957), pp. 1063–1069.

[89] M. Reintjes. *Strong Cosmic Censorship with Bounded Curvature*. 2023. arXiv: 2304.04444 [gr-qc].

[90] A. D. Rendall. “Reduction of the Characteristic Initial Value Problem to the Cauchy Problem and Its Applications to the Einstein Equations”. In: *Proc. R. Soc. Lond. A* 427 (1990), pp. 221–239. DOI: 10.1098/rspa.1990.0009.

[91] H. Ringström. “Strong cosmic censorship in T^3–Gowdy spacetimes”. In: *Ann. Math.* 170 (2009), pp. 1181–1240. DOI: 10.4007/annals.2009.170.1181.

[92] H. Ringström. *The Cauchy Problem in General Relativity.* EMS Press, 2009. DOI: 10.4171/053.

[93] J. Sbierski. “The C^0–inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian geometry”. In: *J. Differential Geom.* 108.2 (2018), pp. 319–378. DOI: 10.4310/jdg/1518490820.

[94] I. I. Shapiro. “New method for the detection of light deflection by solar gravity”. In: *Science* 157 (3790 1967), pp. 806–808. DOI: 10.1126/science.157.3790.806.

[95] Yakov Shlapentokh-Rothman and Rita Teixeira da Costa. *Boundedness and decay for the Teukolsky equation on Kerr in the full subextremal range $|a| < M$: frequency space analysis.* 2023. arXiv: 2007.07211 [gr-qc].

[96] Yakov Shlapentokh-Rothman and Rita Teixeira da Costa. *Boundedness and decay for the Teukolsky equation on Kerr in the full subextremal range $|a| < M$: physical space analysis.* 2023. arXiv: 2302.08916 [gr-qc].

[97] M. Simpson and R. Penrose. “Internal instability in a Reissner-Nordström black hole”. In: *Internatl. J. Theor. Phys.* 7 (1973), pp. 183–197. DOI: 10.1007/BF00792069.

[98] T. Tao. *Nonlinear Dispersive Equations: Local and Global Analysis.* American Mathematical Society, 2006.

[99] D. Tataru. “Local decay of waves on asymptotically flat stationary space-times”. In: *American Journal of Mathematics* 135.2 (2013), pp. 361–401.

Appendix A

Bootstrap

It is often the case in the realm of non-linear PDEs that the solution to an initial value problem, let it be called f, satisfies an estimate in terms of f itself, or one of its norms. Although this may seem to lead to a circular reasoning, there is a way for such an estimate to prove useful, provided that we have some form of control on f at the beginning of the dynamics and that the expression relating the f with itself allows us to improve such a control at later times.

Indeed, the *bootstrap method* or *continuity method*, which can be thought of as a continuous analogue of mathematical induction [98], is a formalization of such a reasoning. A particular formulation, which is relevant for the present work, is the following.

Proposition A.1 (A boostrap argument in $\mathbb{R}^n$)
Let $\mathcal{D} \subset \mathbb{R}^n$, for some $n \in \mathbb{N}_0$, be a connected set and let $f \colon \mathcal{D} \to \mathbb{R}$ be a continuous function. For every $C > 0$, we define the set

$$\mathcal{B}_C \coloneqq \{x \in \mathcal{D} \colon \ |f|(x) \leq C\}.$$

Consider the following two properties:

B0*: There exists $C_0 > 0$ such that $\mathcal{B}_{C_0} \neq \varnothing$,*

B1*: If $|f|(x) \leq C$ for some $x \in \mathcal{D}$ and for some $C > 0$, then there exists $0 < c < C$ such that $|f|(x) \leq c$.*

If **B0** *and* **B1** *hold, then:*

$$|f|(x) \leq C_0, \quad \forall\, x \in \mathcal{D}.$$

Proof. By continuity of f, the set $\mathcal{B}_{C_0}$ is closed. Now, due to **B1** and to the continuity of f, for every $x_0 \in \mathcal{B}_{C_0}$, there exists an open neighbourhood $A_0 \ni x_0$ such that $|f|(y) \leq C_0$ for every $y \in A_0$. By definition of $\mathcal{B}_{C_0}$, it follows that $A_0 \subset \mathcal{B}_{C_0}$. This implies that $\mathcal{B}_{C_0}$ is an open set. Since $\mathcal{B}_{C_0}$ is non-empty (by **B0**), the connectedness of $\mathcal{D}$ entails that $\mathcal{B}_{C_0} = \mathcal{D}$. □

The above proof can be easily modified for the case in which f is not a priori defined in the entire set $\mathcal{D}$, but can be extended due to some additional property, e.g. due to a local existence result.

Proposition A.2 (A bootstrap argument in $\mathbb{R}^n$ – local existence version)
Let $\mathcal{D} \subset \mathbb{R}^n$ be a connected set, and let $\mathcal{E} \subset \mathcal{D}$. Consider the function $f\colon \mathcal{E} \to \mathbb{R}$, assumed to be continuous. For every $C > 0$, we define the set

$$\mathcal{B}_C \coloneqq \{x \in \mathcal{D}\colon \ |f|(x) \leq C\}.$$

Suppose that the properties **B0** *and* **B1** *of proposition A.1 hold, and further assume that the following property is valid:*

- ***LE****: If there exists a $C > 0$ such that $\sup_{\mathcal{E}} |f| < C$, then there exists a unique continuous* ***extension*** *$\tilde{f}\colon \tilde{\mathcal{E}} \to \mathbb{R}$, with $\mathcal{E} \subset \tilde{\mathcal{E}}$, such that $\tilde{f}_{|\mathcal{E}} = f$.*

Then: there exists a unique extension $\tilde{f}$ of f defined on the entire set $\mathcal{D}$ such that

$$|\tilde{f}|(x) \leq C_0, \quad \forall\, x \in \mathcal{D}.$$

It is easy to see that the assumptions **B1** and **LE** can be replaced by more general norm bounds and local existence theorems, and the same proof follows in an analogous fashion.

In general, the assumptions on the continuity of f can also be dropped and replaced, for instance, by assuming that $f \in L^p(\mathcal{D})$ for some $1 \leq p \leq +\infty$. By defining $\mathcal{B}_C$ as a sublevel set with respect to the Lebesgue norm, the same results follow due to the fact that $\mathcal{B}_C$ is closed in the topology induced by norm itself.

Appendix B

Elements of gauge theory

In the following, M will be a smooth manifold and G a Lie group (see also [71]).

Definition B.1 (Principal G-bundle)
A principal G-bundle over M is a smooth manifold P endowed with the canonical projection $P \xrightarrow{\pi} M$ such that:

1. *There exists a right action*

$$\begin{aligned} P \times G &\to P \\ (u, g) &\mapsto u \cdot g = R_g u, \end{aligned}$$

   which is free,

2. *$M = P/G$, where $u_1 \sim u_2$ in P if there exists $g \in G$ such that $u_1 \cdot g = u_2$,*

3. *P is locally trivial, i.e. for every $x \in M$ we can find a neighbourhood $U \ni x$ such that $\pi^{-1}(U) \cong U \times G$, where the latter diffeomorphism is given by a G-equivariant map ψ: $\psi(u) = (\pi(u), \varphi(u))$, for some $\varphi \colon \pi^{-1}(U) \to G$ such that $\varphi(u \cdot g) = \varphi(u) \cdot g$ for every $u \in \pi^{-1}(U)$, $g \in G$.*

We will denote a principal G-bundle by the notation $P \xrightarrow{\pi} M$ ("P over M").

We notice that each fiber of the principal bundle is homeomorphic to G, via the right action. To each principal G-bundle, we can associate a vector bundle arising from a representation of the Lie group G.

Definition B.2 (Associated vector bundle)
Given a vector space V and a principal G-bundle $P \xrightarrow{\pi} M$, let $\rho \colon G \to AutV$ be a representation of G. Then:

$$E := {}^{(P \times V)}\!/\!_{\sim} =: P \times_G V$$

endowed with the projection map $\pi_E([p, v]) = \pi(p)$ is a vector bundle over M. Here, $\sim$ denotes the equivalence relation $E \ni (p, v) \sim (p \cdot g, \rho(g^{-1})v)$ for every $g \in G$.

An important example is given by the adjoint representation[1] $\mathrm{Ad}\colon G \to \mathrm{AutLie}(G)$, which gives the vector bundle $\mathrm{Ad}\, P = P \times_G \mathrm{Lie}(G)$. Even though principal bundles do not come equipped, in general, with any linear structure, we can define a connection by using distributions (in the sense of Frobenius's theorem).

Definition B.3 (Connection on a principal bundle)
A connection Γ on a principal G-bundle $P \xrightarrow{\pi} M$ is a (smooth) distribution

$$\Gamma\colon P \ni u \mapsto H_u \subset T_u P$$

such that

1. *For every u in P:*
$$T_u P = H_u \oplus V_u,$$
where V_u is the subspace of $T_u p$ made of vectors tangent to the fibre through u (V_u is a vertical subspace) and H_u is called a horizontal subspace,

2. *The distribution Γ is G-invariant, i.e.*
$$H_{u\cdot g} = (R_g)_* H_u$$
for every $g \in G$, $u \in P$.

In particular, for every vector field $X \in \mathcal{P}$, we can decompose it as $X = X_h \oplus X_v$, with $X_h(u) \in H_u$ and $X_v(u) \in V_u$ for every $u \in U$.

It is interesting to notice that there is a Lie algebra isomorphism

$$\begin{aligned} \mathrm{Lie}(G) &\to \mathcal{P} \\ A &\mapsto X_A, \end{aligned}$$

with X_A fundamental vector field (i.e. infinitesimal generator) corresponding to A, and such a homomorphism is determined by the G-action. Moreover, it can be showed that the vertical subspace of each tangent space of a principal bundle is given by the fundamental vector fields:

$$V_u = \{X_A(u) \,|\, A \in \mathrm{Lie}(G)\} \quad \forall\, u \in P.$$

Definition B.4 (Connection form)
Given a connection Γ on a G-principal bundle P, we define the (global) Lie(G)-valued 1-form ω by the condition of "killing the horizontal part of the vector fields", i.e.

$$\omega(X)_{|v} = A \in \mathit{Lie}(G) \quad \forall\, v \in P,$$

[1]We recall that, if $G = GL(n)$ for some integer $n > 0$, the map Ad just gives a matrix conjugation: $\mathrm{Ad}(A)B = ABA^{-1}$ for every $A, B \in GL(n)$.

where $X_v = X_a \oplus X_h$.

Proposition B.5 (Relation between connections and connection forms)
Let ω *be a connection form for the connection* Γ *defined on the principal* G*-bundle* P. *Then:*

1. $\omega(X_A) = A \in Lie(G)$ *for every* $A \in Lie(G)$,

2. $R_g^*\omega = Ad(g^{-1})\omega$ *for every* $g \in G$.

Conversely, any $\omega \in \Omega^1(P, Lie(G))$ *satisfying (1) and (2) defines a connection* Γ *on* P.

This connection Γ can be used to define cohomology classes that describe topologically-invariant properties of the principal bundle. A prerequisite for this is a suitable exterior derivative, which is also covariant (i.e. allows us to compare quantities defined on different fibers of the principal bundle).

Definition B.6 (Exterior covariant derivative, Curvature)
The exterior covariant derivative associated to Γ *is*

$$
\begin{aligned}
D\colon \Omega^k(P, Lie(G)) &\to \Omega^{k+1}(P, Lie(G)) \\
\theta &\mapsto d \circ h(\theta),
\end{aligned}
$$

where d *is the usual exterior derivative and* h *is the horizontal projection (i.e.* $h(X) = X_h$ *for every vector field* X*). Alternatively, we can write*

$$D\theta(X_1, \ldots, X_{k+1}) = d\theta(h(X_1), \ldots, h(X_{k+1})), \quad X_j \in \mathcal{P}.$$

The curvature 2-form $\Omega \in \Omega^2(P, Lie(G))$ *of* Γ *is defined as* $\Omega := D\omega$.

We notice that property (2) of connection forms in proposition B.5 is also satisfied by the curvature:

$$R_g^*\Omega = \mathrm{Ad}(g^{-1})\Omega.$$

We will now consider a covering $\{U_\alpha\}_\alpha$ of our manifold M and choose local sections $\{s_\alpha\colon M \to P\}_\alpha$. The local trivializations of the principal bundle determine the transitions functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to M$ that relate two different sections on the intersection of their domains:

$$s_\beta = s_\alpha g_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta.$$

Let us choose a local section s_α (a local gauge) for some α. We define the local connection and curvature forms as $\omega_\alpha := s_\alpha^*\omega$ and $\Omega_\alpha := s_\alpha^*\Omega$ on U_α.

Proposition B.7
On $U_\alpha \cap U_\beta$, *we have*

$$
\begin{cases}
\omega_\beta = Ad(g_{\alpha\beta}^{-1})\omega_\alpha + g_{\alpha\beta}^{-1} dg_{\alpha\beta}, \\
\Omega_\beta = Ad(g_{\alpha\beta}^{-1})\Omega_\alpha.
\end{cases}
$$

Similarly, we can construct local connections and curvature forms on the vector bundle associated to a principal bundle, by considering $\{\rho_*(\omega_\alpha)\}_\alpha$ and $\{\rho_*(\Omega_\alpha)\}_\alpha$, where ρ is the respective representation of the Lie group.

Theorem B.8 (Structure equations and Bianchi identity)
We have:

$$\begin{cases} \Omega = d\omega + \frac{1}{2}[\omega, \omega] & \textit{(Structure equations)} \\ D\Omega = 0 & \textit{(Bianchi identity).} \end{cases}$$

This entire machinery is able to describe the particles of the Standard Model of modern physics. In particular, given a manifold M, suitable to describe a spacetime, we can relate particles to sections of the vector bundle associated (with respect to some representation ρ of G) to the G-principal bundle $P \to M$, where G is the Lie group describing the symmetries of the particle. We will now analyse the case $G = U(1)$, that is of relevance for the Einstein-Maxwell-Klein-Gordon model.

Let us consider a 4-dimensional spacetime (M, g) as the base space of a principal $U(1)$-bundle $P \xrightarrow{\pi} M$. We can define a connection Γ on the principal bundle, and associate to it the global $\mathfrak{u}(1)$-valued connection ω. We denote the exterior covariant derivative associated to Γ by $d_\Gamma \equiv d$. The 2-form $\mathcal{F} = d\omega$ is the corresponding curvature form. If we consider a local section (local gauge) $s\colon U \subset M \to \pi^{-1}(U)$, we can define on U the $\mathfrak{u}(1)$-valued forms $\tilde{A} = s^*\omega$ and $\tilde{F} = s^*\mathcal{F} = d\tilde{A}$. Then

$$F = -i\tilde{F}$$

is a global, $\mathbb{R}$-valued 2-form. The former property follows from the fact that $U(1)$ is Abelian and thus $\mathcal{F}$ is "invariant under gauge transformations" by proposition B.7. The latter is implied by the isomorphism $\mathfrak{u}(1) \cong i\mathbb{R}$. The form F satisfies two of the Maxwell equations since it is a closed form. It turns out that it also satisfies the remaining two Maxwell equations stemming from the Yang-Mills expression $d \star F = \star J$, for a current J given by the Lagrangian of a charged scalar field. For this reason, we will identify F with the electromagnetic field strength permeating the spacetime manifold.

A scalar field ϕ with charge $q \in \mathbb{Z}$ (the integer nature of the charge follows from Dirac quantization or, in the mathematical formalism, can be deduced by the fact that the Chern numbers associated to our principal bundle are integers) is a map $\phi\colon P \to \mathbb{C}$ which is $U(1)$-equivariant i.e.

$$\phi(p \cdot g) = \rho(g)^{-1}\phi(p) \quad \forall\, p \in P,\, g \in U(1),$$

for a given representation $\rho\colon U(1) \to GL(1, \mathbb{C})$ defined as $\rho(g) = g^q$ for every $g \in U(1)$. This scalar field can also be seen as a global section to the associated vector bundle $P \times_\rho \mathbb{C}$. As we have already seen, we can define a connection form on this associated complex line bundle by setting $\omega_\rho := \rho_*\omega$. This 1-form is $\mathfrak{u}(1)$-valued. If we choose the local gauge $s\colon U \subset M \to \pi_\rho^{-1}(U)$, we can define the 1-form

$\tilde{A}_\rho \coloneqq s^*\omega_\rho = s^*\rho_*\omega = iqA$. It can be verified that the exterior covariant derivative on $P \times_\rho \mathbb{C}$ is

$$D = d + iqA.$$

Appendix C

Useful expressions

Let us consider the metric for a spherically symmetric spacetime. It is always possible to locally express such a metric using null coordinates (u, v) as

$$g = -\Omega^2(u,v)dudv + r^2(u,v)\sigma_{\mathbb{S}^2}. \tag{C.1}$$

The respective Christoffel symbols are

$$\begin{aligned}
\Gamma^u_{uu} &= \partial_u \log \Omega^2, & \Gamma^u_{\theta\theta} &= \frac{\Gamma^u_{\varphi\varphi}}{\sin^2\theta} = \frac{2r}{\Omega^2}\partial_v r,\\
\Gamma^v_{vv} &= \partial_v \log \Omega^2, & \Gamma^v_{\theta\theta} &= \frac{\Gamma^v_{\varphi\varphi}}{\sin^2\theta} = \frac{2r}{\Omega^2}\partial_u r,\\
\Gamma^\theta_{u\theta} &= \Gamma^\varphi_{u\varphi} = \frac{1}{r}\partial_u r, & \Gamma^\theta_{v\theta} &= \Gamma^\varphi_{v\varphi} = \frac{1}{r}\partial_v r\\
\Gamma^\theta_{\varphi\varphi} &= -\cos\theta\sin\theta, & \Gamma^\varphi_{\theta\varphi} &= \cot\theta,
\end{aligned}$$

where the remaining symbols are either zero or can be obtained by symmetries. We used the xAct package suite [74] in Mathematica for the following computations. The non-zero components of the Ricci tensor are:

$$\begin{aligned}
R_{uu} &= \frac{2}{r\Omega}\left(2\partial_u r\partial_u\Omega - \Omega\partial_u^2 r\right),\\
R_{uv} = R_{vu} &= -\frac{2\partial_u\partial_v r}{r} + \frac{2}{\Omega^2}\left(\partial_v\Omega\partial_u\Omega - \Omega\partial_u\partial_v\Omega\right),\\
R_{vv} &= \frac{2}{r\Omega}\left(2\partial_v r\partial_v\Omega - \Omega\partial_v^2 r\right),\\
R_{\theta\theta} &= 1 + \frac{4}{\Omega^2}\left(\partial_v r\partial_u r + r\partial_u\partial_v r\right),\\
R_{\varphi\varphi} &= \sin^2\theta R_{\theta\theta},
\end{aligned}$$

and the Ricci scalar is given by

$$R = \frac{2}{r^2} - \frac{8\partial_v\Omega\partial_u\Omega}{\Omega^4} + \frac{8}{r^2\Omega^2}\left(\partial_v r\partial_u r + 2r\partial_u\partial_v r\right) + \frac{8}{\Omega^3}\partial_u\partial_v\Omega.$$

Finally, the non-zero components of the Einstein tensor are

$$
\begin{aligned}
G_{uu} &= \frac{4\partial_u r \partial_u \Omega}{r\Omega} - \frac{2}{r}\partial_u^2 r, \\
G_{uv} = G_{vu} &= \frac{\Omega^2}{2r^2} + \frac{2\partial_v r \partial_u r}{r^2} + \frac{2\partial_u \partial_v r}{r}, \\
G_{vv} &= \frac{4}{r\Omega}\partial_v r \partial_v \Omega - \frac{2}{r}\partial_v^2 r, \\
G_{\theta\theta} &= -\frac{4r}{\Omega^2}\partial_u \partial_v r + \frac{4r^2}{\Omega^4}\partial_v \Omega \partial_u \Omega - \frac{4r^2}{\Omega^3}\partial_u \partial_v \Omega, \\
G_{\varphi\varphi} &= \sin^2\theta G_{\theta\theta}.
\end{aligned}
$$

To study the system of equations (2.1), we also report the components of the energy-momentum tensor T^{EM}:

$$
\begin{aligned}
T^{\mathrm{EM}}_{uu} &= T^{\mathrm{EM}}_{vv} = 0, \\
T^{\mathrm{EM}}_{uv} &= \frac{\Omega^2 Q^2}{4r^4}, \\
T^{\mathrm{EM}}_{\theta\theta} &= \frac{T^{\mathrm{EM}}_{\varphi\varphi}}{\sin^2\theta} = \frac{Q^2}{2r^2},
\end{aligned}
$$

and of T^{ϕ}:

$$
\begin{aligned}
T^{\phi}_{uu} &= |D_u\phi|^2, & T^{\phi}_{vv} &= |D_v\phi|^2, \\
T^{\phi}_{uv} &= \frac{m^2\Omega^2}{4}|\phi|^2, & T^{\phi}_{\theta\theta} &= \frac{T^{\phi}_{\varphi\varphi}}{\sin^2\theta} = \frac{2r^2}{\Omega^2}\,\mathrm{Re}\left(D_u\phi\overline{D_v\phi}\right) - \frac{1}{2}m^2 r^2|\phi|^2.
\end{aligned}
$$

A short computation also gives (see (3.2) and (3.4)):

$$
\star F = -Q(u,v)\sin\theta d\theta \wedge d\varphi
$$

and

$$
\star J = q r^2 \sin\theta\big(\mathrm{Im}\big(\phi\overline{D_u\phi}\big)du \wedge d\theta \wedge d\varphi + \mathrm{Im}\big(\phi\overline{D_v\phi}\big)dv \wedge d\theta \wedge d\varphi\big).
$$

Appendix D

The Reissner-Nordström-de Sitter solution and the choice of coordinates

In this short section we review the main properties of spherically symmetric charged black holes. At the same time, we establish the relation between the null coordinates employed in this work and some Eddington-Finkelstein coordinates commonly adopted for the Reissner-Nordström-de Sitter solution.

The (sub-extremal, or non-degenerate) Reissner-Nordström-de Sitter metric is the unique static, spherically symmetric solution of the electro-vacuum Einstein equations

$$\mathrm{Ric}_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} + \Lambda g_{\mu\nu} = 2g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} - \frac{1}{2}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta},$$

for a positive cosmological constant Λ. Here, the strength field tensor F satisfies the Maxwell equations $dF = 0$ and $d \star F = 0$. This solution is denoted by the three real constants $\varpi_+ > 0$ (mass parameter), Q_+ (charge parameter) and Λ.[1] To such values, we associate the function

$$1 - f(r) \coloneqq 1 - \frac{2\varpi_+}{r} + \frac{Q_+^2}{r^2} - \frac{\Lambda}{3}r^2,$$

required to have three distinct real roots r_-, r_+ and r_C, with $r_- < r_+ < r_C$ (**sub-extremality assumption**). In general, $1 - f$ may possess a fourth distinct (negative) root, which we denote by r_n. The Reissner-Nordström-de Sitter metric, able to describe a black hole spacetime in an expanding cosmological scenario, can be written in coordinates $(t, r) \in \mathbb{R} \times (r_-, r_+)$ as

$$g_{\mathrm{RNdS}} = -(1 - f(r))dt^2 + \frac{1}{1 - f(r)}dr^2 + r^2\sigma_{S^2}, \tag{D.1}$$

where σ_{S^2} is the standard metric on the unit round sphere. We refer the reader to [18, 24] for an overview

[1]This notation for the mass and charge is not standard, but we adopt it here to draw a comparison with our work on the Einstein-Maxwell-charged-Klein-Gordon system.

on this spacetime, on the different coordinate systems to resolve the coordinate singularities at $r = r_+$ (corresponding to the event horizon $\mathcal{H}^+_{\text{RNdS}}$) and $r = r_-$ (corresponding to the Cauchy horizon $\mathcal{CH}^+_{\text{RNdS}}$), and on the failure of determinism inherent in this non-generic solution.

We now show that the outgoing null coordinate chosen in section 3.2 for our Einstein-Maxwell-charged-Klein-Gordon system is strictly related to the outgoing Eddington-Finkelstein coordinate

$$\tilde{v} = \frac{r^* + t}{2}, \quad \text{with } r^* := \int \frac{dr}{1 - f(r)}, \tag{D.2}$$

that can be used to describe regions of the Reissner-Nordström-de Sitter spacetime close to the event horizon $\mathcal{H}^+_{\text{RNdS}}$. As we will see, this result implies that **the v coordinate in the exponential Price law upper bound** (3.49) **is half of the outgoing Eddington-Finkelstein coordinate adopted for the linear analyses [18] and [60]**. This is of particular interest in the cosmological scenario, since different coordinate choices could lead to different exponential behaviours of the scalar field along the event horizon $\mathcal{H}^+$ of the dynamical black hole.

In particular, using the Eddington-Finkelstein coordinates

$$\tilde{u} = \frac{r^* - t}{2}, \qquad \tilde{v} = \frac{r^* + t}{2},$$

and then the coordinate

$$u = \frac{1}{2K_+} e^{2K_+ \tilde{u}},$$

where K_+ is the surface gravity of $\mathcal{H}^+_{\text{RNdS}}$, we can write (D.1) as

$$\begin{aligned} g_{\text{RNdS}} &= 4\big(1 - f(r(\tilde{u}, \tilde{v}))\big) d\tilde{u} d\tilde{v} + r^2(\tilde{u}, \tilde{v}) \sigma_{S^2} \\ &= 4e^{-2K_+ \tilde{u}(u)} \big(1 - f(r(u, \tilde{v}))\big) du d\tilde{v} + r^2(u, \tilde{v}) \sigma_{S^2} \\ &=: -\Omega^2_{\text{RNdS}}(u, \tilde{v})^2 du d\tilde{v} + r^2(u, \tilde{v}) \sigma_{S^2}. \end{aligned}$$

Notice that $1 - f(r(u, \tilde{v}))$ is non-positive for $r(u, \tilde{v}) \in [r_-, r_+]$ (see [20, section 3]), and that $\mathcal{H}^+_{\text{RNdS}} = \{\tilde{u} = -\infty\} = \{u = 0\}$. We observe that:

$$\tilde{v} + \tilde{u} = r^* = \frac{1}{2K_+} \log |r - r_+| + O(1),$$

for r close to r_+. So, by decomposing $1 - f$ into the product of its roots, we have

$$1 - f(r) \sim -e^{2K_+(\tilde{v} + \tilde{u})}, \quad \text{as } r \nearrow r_+.$$

Therefore:

$$\Omega^2_{\text{RNdS}} \sim e^{2K_+ \tilde{v}}, \quad \text{as } r \nearrow r_+. \tag{D.3}$$

We now compare this expression with (3.3). Assumption **(A)** in section 3.2, (3.19) and (6.4) give

$$\Omega^2(0,v) = -4\nu(0,v) \sim e^{2K_+ v}, \quad \forall\, v \geq v_0. \tag{D.4}$$

This result holds for all initial data satisfying the assumptions of section 3.2 and also for Reissner-Nordström-de Sitter initial data[2] for the characteristic IVP of section 3.1. Indeed, the proof of proposition 6.2 can be easily extended to the case $\lambda|_{\mathcal{H}} \equiv 0$. In particular, we have $v = \tilde{v}$.

[2]Namely the case in which $\varpi_0 \equiv \varpi_+$, $r(0,v) \equiv r_+$, $Q_0 \equiv Q_+$, $\lambda_0 \equiv 0$, $\kappa_0 \equiv 1$ and $\Omega^2(0,v) = 4e^{2K_+ v}$. See also [19].