Geometric Mechanics

Homework 6

Due on November 2

1. The upper half-plane $H = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ can be identified with the Lie group consisting of the matrices of the form
$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}.$$
Show that the hyperbolic plane metric
$$\langle \cdot, \cdot \rangle = \frac{1}{y^2}\left(dx \otimes dx + dy \otimes dy\right)$$
is left-invariant for this Lie group structure.

2. (a) Prove that there exists a linear isomorphism $\Omega : \mathfrak{so}(3) \to \mathbb{R}^3$ such that
$$A\xi = \Omega(A) \times \xi$$
for all $\xi \in \mathbb{R}^3$ and $A \in \mathfrak{so}(3)$.

   (b) Check that if $A, B \in \mathfrak{so}(3)$ then $[A,B] \in \mathfrak{so}(3)$, where $[A,B] = AB - BA$. Moreover, show that $(\mathfrak{so}(3), [\cdot,\cdot])$ is a **Lie algebra**, that is, show that the operation $[\cdot,\cdot]$ is antisymmetric, bilinear and satisfies the **Jacobi identity**
$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$
for all $A, B, C \in \mathfrak{so}(3)$.

   (c) Show that
$$\Omega([A,B]) = \Omega(A) \times \Omega(B)$$
for all $A, B \in \mathfrak{so}(3)$, that is, Ω is a Lie algebra isomorphism between $(\mathfrak{so}(3), [\cdot,\cdot])$ and $(\mathbb{R}^3, \times)$.