Geometric Mechanics

Homework 5

Due on October 25

1. Using the Frobenius theorem, show that an $(n-1)$-dimensional distribution Σ on an n-manifold M is integrable if and only if

$$d\omega \wedge \omega = 0$$

for all locally defined differential forms ω whose kernels determine Σ.

2. Recall that our model for an ice skate is given by the non-holonomic constraint Σ defined on $\mathbb{R}^2 \times S^1$ by the kernel of the 1-form $\omega = -\sin\theta dx + \cos\theta dy$.

   (a) Show that the ice skate can access all points in the configuration space: given two points $p, q \in \mathbb{R}^2 \times S^1$ there exists a piecewise smooth curve $c : [0,1] \to \mathbb{R}^2 \times S^1$ compatible with Σ such that $c(0) = p$ and $c(1) = q$. Why does this show that Σ is non-integrable?

   (b) Assuming that the kinetic energy of the skate is

$$K = \frac{M}{2}\left((v^x)^2 + (v^y)^2\right) + \frac{I}{2}\left(v^\theta\right)^2$$

   and that the reaction force is perfect, show that the skate moves with constant speed along straight lines or circles. What is the physical interpretation of the reaction force?