Geometric Mechanics

Homework 1

Due on September 27

1. Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold with Levi-Civita connection ∇, and let

$$\langle\langle\cdot,\cdot\rangle\rangle = e^{2\rho}\langle\cdot,\cdot\rangle$$

be a metric conformally related to $\langle\cdot,\cdot\rangle$, where $\rho \in C^\infty(M)$. Show that the Levi-Civita connection $\widetilde{\nabla}$ of $\langle\langle\cdot,\cdot\rangle\rangle$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + d\rho(X)Y + d\rho(Y)X - \langle X, Y\rangle \operatorname{grad}\rho$$

for all $X, Y \in \mathfrak{X}(M)$, where the gradient is taken with respect to $\langle\cdot,\cdot\rangle$, that is, $\operatorname{grad}\rho$ is the vector field defined by

$$\langle \operatorname{grad}\rho, X\rangle = d\rho(X)$$

for all $X \in \mathfrak{X}(M)$.

(**Hint:** Use the Koszul formula).

2. Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold. A curve $c : I \subset \mathbb{R} \to M$ is said to be a **reparameterized geodesic** if $c(t) = \gamma(s(t))$ for all $t \in I$, where $\gamma : J \subset \mathbb{R} \to M$ is a geodesic and $s : I \to J$ is a diffeomorphism. Show that c is a reparameterized geodesic if and only if it satisfies

$$\frac{D\dot{c}}{dt} = f(t)\,\dot{c}$$

for some differentiable function $f : I \to \mathbb{R}$.