Geometric Mechanics

Homework 9

Due on November 19

1. Let (M, ω) be a symplectic manifold. Show that:
   (a) $\omega = \sum_{i=1}^{n} dp_i \wedge dx^i$ if and only if $\{x^i, x^j\} = \{p_i, p_j\} = 0$ and $\{p_i, x^j\} = \delta_{ij}$ for $i, j = 1, \ldots, n$;
   (b) M is orientable;
   (c) If M is compact then ω cannot be exact;
   (d) The only sphere that admits a symplectic structure is S^2.

2. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold, B the Poisson bivector and $(x^1, \ldots, x^n)$ local coordinates on M. Show that:
   (a) B can be written in these local coordinates as

   $$B = \sum_{i,j=1}^{n} B^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j},$$

   where $B^{ij} = \{x^i, x^j\}$ for $i, j = 1, \ldots, n$;
   (b) The Hamiltonian vector field generated by $F \in C^\infty(M)$ can be written as

   $$X_F = \sum_{i,j=1}^{p} B^{ij} \frac{\partial F}{\partial x^i} \frac{\partial}{\partial x^j};$$

   (c) The components of B must satisfy

   $$\sum_{l=1}^{n} \left(B^{il} \frac{\partial B^{jk}}{\partial x^l} + B^{jl} \frac{\partial B^{ki}}{\partial x^l} + B^{kl} \frac{\partial B^{ij}}{\partial x^l} \right) = 0$$

   for all $i, j, k = 1, \ldots, n$;
   (d) If $\{\cdot, \cdot\}$ arises from a symplectic form ω then $(B^{ij}) = -(\omega_{ij})^{-1}$;
   (e) If B is nondegenerate then it arises from a symplectic form.