Geometric Mechanics

Homework 2

Due on October 1

1. Recall that the hyperbolic plane is the upper half plane

$$H = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\}$$

with the Riemannian metric

$$\langle \cdot, \cdot \rangle = \frac{1}{y^2} \left(dx \otimes dx + dy \otimes dy \right).$$

   (a) Use the local coordinate expression of Newton's law to write the geodesic equations.

   (b) Determine the Christoffel symbols for the Levi-Civita connection of $(H, \langle \cdot, \cdot \rangle)$ in the coordinates (x, y).

   (c) Show that the images of geodesics are either vertical half-lines or half-circles centered on the x-axis. (**Hint:** Use the conservation of mechanical energy).

2. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold with Levi-Civita connection $\widetilde{\nabla}$, and let $(N, \langle\langle \cdot, \cdot \rangle\rangle)$ be a submanifold endowed with the induced metric, with Levi-Civita connection ∇. Let $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(M)$ be local extensions of $X, Y \in \mathfrak{X}(N)$.

   (a) Show that

$$\nabla_X Y = \left(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} \right)^{\top},$$

   where $^{\top} : TM|_N \to TN$ is the orthogonal projection. (**Hint:** Use the Koszul formula).

   (b) The **second fundamental form** of N is the map $B : T_pN \times T_pN \to (T_pN)^{\perp}$ defined at each point $p \in N$ by

$$B(X_p, Y_p) := \left(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} \right)_p - \left(\nabla_X Y \right)_p = \left(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} \right)_p^{\perp}.$$

   Show that B is well defined, symmetric and bilinear.