Geometric Mechanics

Homework 13

Due on December 18

1. Show that if we allow for a **cosmological constant** $\Lambda \in \mathbb{R}$, i.e. for an Einstein equation of the form
$$Ric = 4\pi\rho(2\nu \otimes \nu + g) + \Lambda g$$
then the equations for the FLRW models become
$$\begin{cases} \frac{\dot{a}^2}{2} - \frac{\alpha}{a} - \frac{\Lambda}{6}a^2 = -\frac{k}{2} \\ \\ \frac{4\pi}{3}a^3\rho = \alpha \end{cases}$$
Analyze the possible behaviors of the function $a(t)$. (**Remark:** It is currently thought that there exists indeed a positive cosmological constant, also known as **dark energy**. The model favored by experimental observations seems to be $k = 0$, $\Lambda > 0$).

2. Consider two galaxies in a FLRW model, whose spatial locations can be assumed to be $r = 0$ and $(r, \theta, \varphi) = (r_1, \theta_1, \varphi_1)$.
   (a) Show that the family (reparameterized) null geodesics connecting the first galaxy to the second galaxy can be written as
   $$(t, r, \theta, \varphi) = (t(r, t_0), r, \theta_1, \varphi_1) \qquad (0 < r < r_1),$$
   where $t(r, t_0)$ is the solution of
   $$\begin{cases} \frac{dt}{dr} = \frac{a(t)}{\sqrt{1 - kr^2}} \\ \\ t(0, t_0) = t_0 \end{cases}.$$
   (b) Prove that $\frac{\partial t}{\partial t_0}(r_1, t_0) = \frac{a(t_1)}{a(t_0)}$, where $t_1 = t(r_1, t_0)$.
   (c) The **redshift** of the light propagating from the first galaxy to the second galaxy is defined as
   $$z = \frac{\partial t}{\partial t_0}(r_1, t_0) - 1 = \frac{a(t_1)}{a(t_0)} - 1.$$
   This light is spread over a sphere of radius $R = a(t_1)r_1$, and so its brightness is inversely proportional to R^2. Compute R as a function of z for the following FLRW models:

(i) **Milne universe** ($k = -1$, $\alpha = \Lambda = 0$), for which $a(t) = t$;

(ii) **Flat de Sitter universe** ($k = \alpha = 0$, $\Lambda = 3H^2$), for which $a(t) = e^{Ht}$;

(iii) **Einstein-de Sitter universe** ($k = \Lambda = 0$, $\alpha = 2/9{t_1}^2$), for which $a(t) = (t/t_1)^{2/3}$.

(**Remark:** The brightness of distant galaxies is further reduced by a factor of $(1+z)^2$, since each photon has frequency, hence energy, $(1+z)$ times smaller at reception, and the rate of detection of photons is $(1+z)$ times smaller than the rate of emission; with this correction, R can be deduced from the observed brightness for galaxies of known luminosity, and the correct FLRW model can be chosen as the one whose curve $R = R(z)$ best fits observations).