Algebraic and Geometric Methods in Engineering and Physics

Abbreviated lecture notes

1. Set Theory

1. A **relation** on a set A is a subset $\mathcal{R} \subset A \times A$. We say that $x \in A$ is **related to** $y \in A$ if $(x, y) \in \mathcal{R}$, and we write $x\mathcal{R}y$.
2. An **equivalence relation** is a relation $\mathcal{R}$ on a set A satisfying the following three properties:
   (i) **Reflexivity:** $x\mathcal{R}x$ for all $x \in A$;
   (ii) **Symmetry:** If $x\mathcal{R}y$ the $y\mathcal{R}x$ for all $x, y \in A$;
   (iii) **Transitivity:** If $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$ for all $x, y, z \in A$;

   If $\mathcal{R}$ is an equivalence relation then we write $x \sim y$ to mean $x\mathcal{R}y$.
3. If $\sim$ is an equivalence relation on A and $x \in A$ then the **equivalence class** of x is the set
$$[x] = \{y \in A : y \sim x\} \subset A.$$
4. A **partition** of a set A is a family $\{A_i\}_{i \in I}$ of subsets of A such that:
   (i) $\bigcup_{i \in I} A_i = A$;
   (ii) If $i \neq j$ then $A_i \cap A_j = \varnothing$.
5. The equivalence classes of an equivalence relation on A form a partition of A; conversely, given a partition of A there exists an equivalence relation on A whose equivalence classes are precisely the subsets of the partition.
6. The **quotient set** of A by an equivalence relation $\sim$ is the set $A/\sim$ of its equivalence classes. The **quotient map**, or **canonical projection**, is the map $\pi : A/\sim \to A$ defined as $\pi(x) = [x]$ for all $x \in A$.

2. Groups

1. A **binary operation** on a set A is a map $f : A \times A \to A$. We often write $f(x, y) = x \cdot y = xy$ for $x, y \in A$.
2. A **group** is a pair $(G, \cdot)$, where G is a set and $\cdot : G \times G \to G$ is a binary operation satisfying:
   (i) **Associativity:** $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in G$;
   (ii) **Existence of identity:** There exists $e \in G$ such that $e \cdot x = x \cdot e = x$ for all $x \in G$;
   (iii) **Existence of inverses:** For each $x \in G$ there exists $y \in G$ such that $y \cdot x = x \cdot y = e$ for all $x, y \in G$.

3. If $(G, \cdot)$ is a group then:
   (i) The identity element $e \in G$ is unique;
   (ii) Each element $x \in G$ has a unique inverse $x^{-1} \in G$.
4. A group $(G, \cdot)$ is called **abelian** if the group operation $\cdot$ is commutative, that is, if $x \cdot y = y \cdot x$ for all $x, y \in G$. In this case we often write $x \cdot y = x + y$, $e = 0$ and $x^{-1} = -x$.
5. The **order** of a group G is the number of elements in G, and is represented as $|G|$.
6. If A is a set the its **permutation group** is the set

$$\mathrm{Sym}(A) = \{f : A \to A : f \text{ is bijective } \}$$

   with the composition operation.
7. The **symmetric group on $n \in \mathbb{N}$ elements** is the group $S_n = \mathrm{Sym}(\{1, \ldots, n\})$. It is nonabelian for $n > 2$, and $|S_n| = n!$.
8. A subset $H \subset G$ of a group G is called a **subgoup** if H is itself a group under the group operation of G, or, equivalently, if:
   (i) $xy \in H$ for all $x, y \in H$;
   (ii) $e \in H$;
   (iii) $x^{-1} \in H$ for all $x \in H$.
9. If $g \in G$ and H is a subgroup of G then the **left coset** determined by g and H is the set

$$gH = \{gh : h \in H\}.$$

   The set G/H of all left cosets forms a partition of G, and the corresponding equivalence relation is given by

$$g_1 \sim g_2 \Leftrightarrow g_1^{-1} g_2 \in H.$$

10. **Lagrange's Theorem:** If G is a finite group and $H \subset G$ is a subgroup then $|H|$ is a divisor of $|G|$.
11. The **index** of a subgroup H on a group G is the number $[G\!:\!H]$ of left cosets. If G is finite then $[G\!:\!H] = |G|/|H|$.
12. A subgroup H of a group G is called a **normal subgroup** if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$. In that case, the set G/H of cosets forms a group under the operation $(g_1 H)(g_2 H) = (g_1 g_2)H$ for all $g_1, g_2 \in G$ (called the **quotient group**).
13. For each $d \in \mathbb{N}$ the group of **integers mod d** is $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$. We write $a \equiv b \pmod{d}$ to signify that $a, b \in \mathbb{Z}$ satisfy $[a] = [b]$ in $\mathbb{Z}_d$.
14. A map $\varphi : G \to H$ between two groups is called a **homomorphism** if $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$ for all $g_1, g_2 \in G$. An **isomorphism** is a bijective homomorphism. If $\varphi : G \to H$ is an isomorphism then G and H are said to be **isomorphic**.
15. If $\varphi : G \to H$ is a homomorphism then:
   (i) $\varphi(e_G) = (e_H)$;
   (ii) $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$.
16. The composition of homomorphisms is again a homomorphism.
17. **Isomorphism Theorem:** If $\varphi : G \to H$ is a homomorphism then:

(i) $\operatorname{Im}\varphi$ is a subgroup of H;

(ii) $\ker\varphi$ is a normal subgroup of G;

(iii) The map $\tilde{\varphi} : G/\ker\varphi \to \operatorname{Im}\varphi$ defined by $\tilde{\varphi}([g]) = \varphi(g)$ is an isomorphism.

18. If G is a group and $g \in G$ then the **subgroup generated by g** is

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$$

(where g^n has the obvious meaning). The **order** of g is $\operatorname{ord}(g) = |\langle g \rangle|$. If $\operatorname{ord}(g) = d$ then $\langle g \rangle \cong \mathbb{Z}_d$, and if $\operatorname{ord}(g) = \infty$ then $\langle g \rangle \cong \mathbb{Z}$. If $G = \langle g \rangle$ for some $g \in G$ then G is said to be **cyclic**.

19. If G and H are groups then $G \times H$ is also a group with the operation defined by the formula $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$.

3. Rings and number theory

1. A **ring** $(R, +, \cdot)$ is a set equipped with two binary operations such that $(R, +)$ is an abelian group and $\cdot$ is associative, has an identity 1 and is **distributive** with respect to $+$, that is, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$.
2. A ring $(R, +, \cdot)$ is called **commutative** if $\cdot$ is commutative.
3. On a ring $(R, +, \cdot)$ we have $a \cdot 0 = 0 \cdot a = 0$, for any $a \in R$. If $a, b \in R \setminus \{0\}$ are such that $ab = 0$ then a and b are called **zero divisors**.
4. An element $a \in R$ on a ring $(R, +, \cdot)$ is called **invertible** if there exists $b \in R$ such that $ab = ba = 1$, in which case we write $b = a^{-1}$. The set R^* of all invertible elements is a group for the multiplication.
5. A **field** is a commutative ring $(R, +, \cdot)$ such that $R^* = R \setminus \{0\}$.
6. $(\mathbb{Z}_d, +, \cdot)$ is a commutative ring for the multiplication given by $[m] \cdot [n] = [mn]$.
7. $\mathbb{Z}_d^* = \{[n] : \gcd(n, d) = 1\}$. In particular, $\mathbb{Z}_d$ is a field if and only if d is prime.
8. The **Euler function** $\varphi : \mathbb{N} \to \mathbb{N}$ is defined as

$$\varphi(n) = |\{m \in \mathbb{N} : m \leq n \text{ and } \gcd(m, n) = 1\}|,$$

so that $|\mathbb{Z}_d^*| = \varphi(d)$. If $\gcd(m, n) = 1$ then $\varphi(m, n) = \varphi(m)\varphi(n)$, and so if $n = p_1^{r_1} \cdots p_k^{r_k}$ with $p_1, \ldots, p_k$ distinct prime numbers then $\varphi(n) = (p_1^{r_1} - p_1^{r_1 - 1}) \cdots (p_k^{r_k} - p_1^{r_k - 1})$.

9. **Euler's Theorem:** If $\gcd(a, d) = 1$ then $a^{\varphi(d)} \equiv 1 \pmod{d}$.
10. **Chinese Remainder Theorem:** If $n_1, \ldots, n_k$ are coprime and $N = n_1 \cdot n_k$ then the map

$$\mathbb{Z}_N \ni [m] \mapsto ([m], \ldots, [m]) \in \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$$

is an isomorphism.

11. **Classification of finite abelian groups:** If G is a finite abelian group of order $|G| = p_1^{r_1} \cdots p_k^{r_k}$ with $p_1, \ldots, p_k$ distinct prime numbers then

$$G \cong G_1 \times \cdots \times G_k,$$

where $|G_i| = p_i^{r_i}$ for each $i \in \{1, \ldots, k\}$. Moreover, there exists a unique nondecreasing sequence $a_i^1, \ldots, a_i^\ell \in \mathbb{N}$ with $a_i^1 + \cdots + a_i^\ell = r_i$ such that

$$G_i \cong \mathbb{Z}_{a_i^1} \times \cdots \times \mathbb{Z}_{a_i^\ell}.$$

12. **RSA public key encryption:** Two (large) distinct prime numbers p and q are chosen, and the number $N = pq$ is published, along with an **encryption exponent** e, chosen such that $\gcd(\varphi(N), e) = 1$. To encrypt a message $[X] \in \mathbb{Z}_N$ the sender simply computes $[X]^e$. To decrypt the message the receiver uses the fact that $([X]^e)^d = [X]$, where the **decryption exponent** d is such that $[d] = [e]^{-1}$ in $\mathbb{Z}_{\varphi(N)}$. The security of this system relies on the fact that obtaining the descryption exponent requires knowledge of $\varphi(N) = (p-1)(q-1)$, which would involve factorizing N, a very hard task for p and q large enough.

4. Group actions

1. An **action** $G \overset{\varphi}{\curvearrowright} M$ of a group G on a set M is a homomorphism $\varphi : G \to \mathrm{Sym}(M)$ (we write $\varphi(g) = \varphi_g$). The action is called **effective** if $\ker(\varphi) = \{e\}$.
2. If $G \overset{\varphi}{\curvearrowright} M$ is an action of a group G on a set M then:
   (i) The **orbit** of a point $x \in M$ is the set
   $$\varphi_G(x) = \{\varphi_g(x) : g \in G\}.$$
   The set of all orbits is denoted M/G, and is a partition of M.
   (ii) The **stabilizer** of a set $X \subset M$ is the subgroup of G defined by
   $$G_X = \{g \in G : \varphi_g(X) = X\}.$$
   The **isotropy subgroup** of $x \in M$ is just $G_x = G_{\{x\}}$.
   (iii) A point $x \in M$ is called a **fixed point** of the action if $G_x = G$. The set of all fixed points of the action is denoted by M^G.
3. The **dihedral group on order n** is the group D_n defined by two generators r and s together with the relations
   $$r^n = e, \qquad s^2 = e, \qquad rs = sr^{-1}.$$
   We have $|D_n| = 2n$, since
   $$D_n = \{e, r, \dots, r^{n-1}, s, sr, \dots, sr^{n-1}\}.$$
4. An action $G \overset{\varphi}{\curvearrowright} M$ is called:
   (i) **Free** if $G_x = \{e\}$ for all $x \in M$;
   (ii) **Transitive** if $\varphi_G(x) = M$ for some (thus all) $x \in M$.
5. The **canonical action** of a group G on a space of left cosets G/H is the action $G \overset{\varphi^{\mathrm{can}}}{\curvearrowright} G/H$ defined by $\varphi^{\mathrm{can}}_g([g_1]) = [gg_1]$.
6. If $G \overset{\varphi}{\curvearrowright} M$ and $G \overset{\varphi'}{\curvearrowright} M'$ are G-actions then a map $T : M \to M'$ is called:
   (i) **G-equivariant** if $T(\varphi_g(x)) = \varphi'_g(T(x))$ for all $g \in G$ and $x \in M$.
   (ii) An **equivalence of G-actions** if it is a G-equivariant bijection.
7. Any transitive action $G \overset{\varphi}{\curvearrowright} M$ is equivalent to $G \overset{\varphi^{\mathrm{can}}}{\curvearrowright} G/G_{x_0}$, where G_{x_0} is the isotropy subgroup of any point $x_0 \in M$ and $T_{x_0} : G/G_{x_0} \to M$ given by $T_{x_0}([g]) = \varphi_g(x_0)$ is an equivalence of G-actions.

8. The **conjugation action** $G \overset{\varphi^c}{\curvearrowright} G$ is the action of G on itself given by $\varphi^c_g(h) = ghg^{-1}$. The orbits of this action are called **conjugacy classes**. The set $Z(G)$ of fixed points of this action is called the **center** of G, and is a subgroup (subgroup of the elements in G which commute with every other element). The isotropy subgroup $Z(g)$ of a given element $g \in G$ is called the **centralizer** of g (largest subgroup of G which has g in its center). The stabilizer $N(H)$ of a subgroup $H \subset G$ is called the **normalizer** of H (largest subgroup of G which contains H as a normal subgroup).
9. If $\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$ then $\sigma = (j_1 \cdots j_k) \in S_n$ (called a ***k*-cycle**) represents the permutation defined by $\sigma(j_r) = j_{r+1}$ for $r \in \{1, \ldots, k-1\}$, $\sigma(j_k) = j_1$ and $\sigma(j) = j$ for $j \notin \{j_1, \ldots, j_k\}$. This representation is not unique, as $(j_1\, j_2 \cdots j_k) = (j_k\, j_1 \cdots j_{k-1})$. If $\{j_1, \ldots, j_k\} \cap \{j_k, \ldots, j_r\} = \{j_k\}$ then $(j_1 \cdots j_k) \circ (j_k \cdots j_r) = (j_1 \cdots j_r)$. Every permutation is the product of unique disjoint cycles.
10. A 2-cycle is called a **transposition**. A transposition (ij) is called **simple** if $|i-j| = 1$. The set of simple transpositions generates S_n.
11. S_n is isomorphic to the group with $n-1$ generators $s_1, \ldots, s_{n-1}$ subject to the following relations:
    (i) $s_i^2 = e$ for $i = 1, \ldots, n-1$;
    (ii) $s_i s_j = s_j s_i$ if $|i-j| \geq 2$;
    (iii) $s_i s_{i+1}$ has order 3 for $i = 1, \ldots, n-2$.
12. The **sign** of a perturbation $\sigma \in S_n$ is $\operatorname{sgn}(\sigma) = (-1)^{n(\sigma)}$, where $n(\sigma)$ is the number of transpositions on any decomposition of σ as a product of transpositions. The map $\operatorname{sgn} : S_n \to \mathbb{Z}^*$ is a group homomorphism, and its kernel A_n is called the **alternating group** (normal subgroup of order $n!/2$).
13. A **partition** of $n \in \mathbb{N}$ is a k-tuple $(i_1, \ldots, i_k) \in \mathbb{N}^k$ where $i_1 \geq \ldots \geq i_k$ and $i_1 + \ldots + i_k = n$.
14. The conjugacy classes of S_n are in 1 to 1 correspondence with the partitions of n via

$$D_{(i_1,\ldots,i_k)} = \{\sigma_1 \circ \cdots \circ \sigma_k : \sigma_1, \ldots, \sigma_k \text{ are disjoint cycles of lengths } i_1, \ldots, i_k\}.$$

15. **Burnside's Counting Theorem:** Let $G \overset{\varphi}{\curvearrowright} M$ be an action of a finite group G on a finite set M. Then the number of orbits is

$$|M/G| = \frac{1}{|G|} \sum_{g \in G} |M^g|,$$

where M^g is the set of points in M that are fixed by g:

$$M^g = \{x \in M : \varphi_g(x) = x\}.$$

5. Representations of finite groups

1. A **representation** of a group G is an action $G \overset{\varphi}{\curvearrowright} V$ on a complex vector space V by linear maps. The **degree** of the representation is $\deg \varphi = \dim V$. If V is finite-dimensional and has an inner product then the representation is called **unitary** if the maps φ_g are unitary for all $g \in G$.

2. An **intertwining map** bewteen two representations $G \overset{\varphi}{\curvearrowright} V$ and $G \overset{\varphi'}{\curvearrowright} V'$ is a linear equivariant map $T : V \to V'$. If T is invertible then the representations are said to be **equivalent**.
3. An **invariant subspace** for a representation $G \overset{\varphi}{\curvearrowright} V$ is a subspace $W \subset V$ such that $\varphi_g(W) \subset W$ for all $g \in G$. If the representation is unitary and W is an invariant subspace then $W^\perp$ is also an invariant subspace.
4. The **direct sum** of two given representations $G \overset{\varphi}{\curvearrowright} V$ and $G \overset{\varphi'}{\curvearrowright} V'$ is the representation $G \overset{\varphi\oplus\varphi'}{\curvearrowright} V \oplus V'$ defined as

$$(\varphi \oplus \varphi')_g(v, v') = (\varphi_g(v), \varphi'_g(v')).$$

5. A representation $G \overset{\varphi}{\curvearrowright} V$ is said to be **irreducible** if its only invariant subspaces are $\{0\}$ and V.
6. **Maschke's Theorem:** Every finite-dimensional representation of a finite group is equivalent to a direct sum of irreducible representations.
7. **Schur's Lemma:** Let $G \overset{\varphi}{\curvearrowright} V$ and $G \overset{\psi}{\curvearrowright} W$ be two irreducible finite-dimensional representations of the finite group G, and let $T : V \to W$ be an intertwining map. Then either $T = 0$ ot T is invertible. Moreover, if the two representations coincide then $T = \lambda\, \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.
8. Every irreducible finite-dimensional representation of a finite abelian group has degree 1.
9. The **regular representation** of a finite group G on $L(G) = \{f : G \to \mathbb{C}\}$ is defined as

$$(\varphi_g^{(r)}(f))(h) = f(g^{-1}h)$$

for all $g, h \in G$. This representation is unitary for the inner product

$$\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

10. **Schur's orthogonality relations:** Let G be a finite group and let $\varphi : G \to U(n)$ and $\rho : G \to U(m)$ be two inequivalent irreducible representations. Then:
    (i) $\langle \varphi_{ij}, \rho_{kl}\rangle = 0$.
    (ii) $\langle \varphi_{ij}, \varphi_{kl}\rangle = \frac{1}{n}\delta_{ik}\delta_{jl}$.
11. The **character** of a finite-dimensional representation $G \overset{\varphi}{\curvearrowright} V$ of a finite group G is the map $\chi_\varphi \in L(G)$ defined by $\chi_\varphi(g) = \mathrm{tr}(\varphi_g)$. In particular, $\chi_\varphi(e) = \deg(\varphi)$.
12. The character χ_φ of a finite-dimensional representation $G \overset{\varphi}{\curvearrowright} V$ of a finite group is a **class function**, that is, $\chi_\varphi(hgh^{-1}) = \chi_\varphi(g)$ for all $g, h \in G$. We represent by $Z(L(G)) \subset L(G)$ the set of class functions.
13. If $G \overset{\varphi}{\curvearrowright} V$ and $G \overset{\psi}{\curvearrowright} W$ are irreducible finite-dimensional representations of a finite group G then

$$\langle \chi_\varphi, \chi_\psi\rangle = \begin{cases} 1 \text{ if } \varphi \sim \psi\,, \\ 0 \text{ if } \varphi \not\sim \psi\,. \end{cases}$$

14. Let $[\varphi^{(1)}], \ldots, [\varphi^{(s)}]$ be the equivalence classes of irreducible finite-dimensional representations of a finite group G, and let $G \overset{\rho}{\curvearrowright} V$ be a finite-dimensional representation of G,

$$\rho \sim m_1\varphi^{(1)} \oplus \ldots \oplus m_s\varphi^{(s)},$$

where

$$m_i\varphi^{(i)} = \overbrace{\varphi^{(i)} \oplus \ldots \oplus \varphi^{(i)}}^{m_i}.$$

Then

$$m_i = \langle \chi_\rho, \chi_{\varphi^{(i)}} \rangle.$$

15. Let $[\varphi^{(1)}], \ldots, [\varphi^{(s)}]$ be the equivalence classes of irreducible finite-dimensional representations of a finite group G, and let $d_i = \deg(\varphi^{(i)})$. Then

$$d_1^2 + \ldots + d_s^2 = |G|.$$

16. Let $[\varphi^{(1)}], \ldots, [\varphi^{(s)}]$ be the equivalence classes of irreducible finite-dimensional representations of a finite group G. Then $\{\chi_{\varphi^{(1)}}, \ldots, \chi_{\varphi^{(s)}}\}$ is an orthonormal basis of the space $Z(L(G))$ of class functions. In particular, there are as many irreducible representations as conjugacy classes.

17. If $G \overset{\varphi}{\curvearrowright} V$ is a finite-dimensional representations of the finite group G and $f \in L(G)$ then

$$\varphi_f = \sum_{g \in G} f(g)\varphi_g.$$

18. Let $[\varphi^{(1)}], \ldots, [\varphi^{(s)}]$ be the equivalence classes of irreducible finite-dimensional representations of a finite group G, and let $d_i = \deg(\varphi^{(i)})$ and $\chi_i = \chi_{\varphi^{(i)}}$. If a given finite-dimensional representation $G \overset{\varphi}{\curvearrowright} V$ decomposes as

$$\varphi \sim m_1\varphi^{(1)} \oplus \ldots \oplus m_s\varphi^{(s)},$$

corresponding to the orthogonal decomposition

$$V = V_1 \oplus \ldots \oplus V_s,$$

then the orthogonal projection $P_i : V \to V_i$ is

$$P_i = \frac{d_i}{|G|}\varphi_{\overline{\chi}_i}.$$

19. A structure with n degrees of freedom near equilibrium is described by the symmetric positive (semi-)definite $n \times n$ **mass matrix** M and the symmetric $n \times n$ **stiffness matrix** K. The frequencies ω of small oscillations of the structure satisfy

$$\omega^2 Mu = Ku$$

for some $u \in \mathbb{R}^n \setminus \{0\}$. If the structure admits a symmetry group acting linearly, $G \overset{\varphi}{\curvearrowright} \mathbb{R}^n$, then M and K are intertwiners for φ, and so they are block diagonal with respect to the decomposition of φ into irreducible representations. Moreover, if a given irreducible representation occurs only once then M and K are multiples of the identity on the corresponding block (but this is not true if the irreducible representation occurs more than once).

6. Topology

1. A **distance function** (or **metric**) on a set M is a function $d : M \times M \to \mathbb{R}$ satisfying:
   (i) **Positivity:** $d(x, y) \geq 0$ for all $x, y \in M$, and $d(x, y) = 0 \Rightarrow x = y$;
   (ii) **Symmetry:** $d(x, y) = d(y, x)$ for all $x, y \in M$;
   (iii) **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

   If d is a distance then (M, d) is said to be a **metric space**.
2. If (M, d) is a metric space and $x \in M$ then the **open ball** with center x and radius $r > 0$ is the set
$$B_r(x) = \{y \in M : d(x, y) < r\}.$$
3. If (M, d) is a metric space then a subset $U \subset M$ is called **open** if for each $x \in U$ there exists $r > 0$ such that $B_r(x) \subset U$. A subset $F \subset M$ is called **closed** if $M \setminus F$ is open.
4. Open balls are open sets.
5. The family $\mathcal{T}_d$ of open sets on a metric space (M, d) satisfies the following properties:
   (i) $\varnothing, M \in \mathcal{T}_d$;
   (ii) If $\{U_\alpha\}_{\alpha \in A} \subset \mathcal{T}_d$ then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_d$;
   (iii) If $U_1, \ldots, U_n \in \mathcal{T}_d$ then $\bigcap_{i=1}^n U_i \in \mathcal{T}_d$.
6. A **topology** on a set M is a family $\mathcal{T}$ of subsets of M satisfying the following properties:
   (i) $\varnothing, M \in \mathcal{T}$;
   (ii) If $\{U_\alpha\}_{\alpha \in A} \subset \mathcal{T}$ then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$;
   (iii) If $U_1, \ldots, U_n \in \mathcal{T}$ then $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

   The sets $U \in \mathcal{T}$ are called **open sets**, and the sets $F \subset M$ such that $M \setminus F \in \mathcal{T}$ are called **closed sets**. An open set containing $x \in M$ is called a **neighborhood** of x. If $\mathcal{T}$ is a topology on M the $(M, \mathcal{T})$ is said to be a **topological space**.
7. A topological space $(M, \mathcal{T})$ is said to be **Hausdorff** if given $x, y \in M$ with $x \neq y$ there exist $U, V \in \mathcal{T}$ with $U \cap V = \varnothing$ such that $x \in U$ and $y \in V$.
8. A subfamily $\mathcal{B} \subset \mathcal{T}$ of open sets is said to be a **basis** for the topology $\mathcal{T}$ on M is it satisfies any of the following equivalent properties:
   (i) For each $U \in \mathcal{T}$ there exists $\mathcal{B}_U \subset \mathcal{B}$ such that $U = \bigcup_{V \in \mathcal{B}_U} V$.
   (ii) For each $U \in \mathcal{T}$ and each $x \in U$ there exists $U_x \in \mathcal{B}$ such that $x \in U_x \subset U$.
9. If $(M, \mathcal{T})$ is a topological space and $N \subset M$ then $\mathcal{T}_N = \{U \cap N : U \in \mathcal{T}\}$ is a topology in N (**subspace topology**).
10. A map $f : M \to N$ between two topological spaces $(M, \mathcal{T}_M)$ and $(N, \mathcal{T}_N)$ is said to be **continuous** if $f^{-1}(U) \in \mathcal{T}_M$ for all $U \in \mathcal{T}_N$.
11. The composition of continuous maps is continuous.
12. A **homeomorphism** between two topological spaces $(M, \mathcal{T}_M)$ and $(N, \mathcal{T}_N)$ is a continuous bijection $f : M \to N$ with continuous inverse. If such a map exists then $(M, \mathcal{T}_M)$ and $(N, \mathcal{T}_N)$ are said to be **homeomorphic**.
13. If $(M, \mathcal{T})$ is a topological space, $\sim$ is an equivalence relation on M and $\pi : M \to M/\!\sim$ is the canonical projection then the **quotient topology** on $M/\!\sim$ is
$$\mathcal{T}_\pi = \{U \in M/\!\sim\, : \pi^{-1}(U) \in \mathcal{T}\}.$$

14. An **open cover** for a subset $N \subset M$ of a topological space $(M, \mathcal{T})$ is a collection $\mathcal{O} \subset \mathcal{T}$ such that
$$N \subset \bigcup_{U \in \mathcal{O}} U.$$
A **subcover** is a subcollection $\mathcal{O}' \subset \mathcal{O}$ which is still an open cover of N. A set N is said to be **compact** if every open cover of N admits a finite subcover.
15. If $(M, \mathcal{T})$ is Hausdorff and $N \subset M$ is compact then N is closed.
16. **Heine-Borel theorem:** a subset $K \subset \mathbb{R}^n$ is compact (for the usual topology) if and only if it is closed and bounded.
17. Continuous maps carry compact sets to compact sets.
18. A topological space is said to be **connected** if the only subsets of M which are simultaneously open and closed are $\varnothing$ and M. A subset $N \subset M$ is said to be a **connected subset** if it is a connected topological space for the subspace topology.
19. A subset of $\mathbb{R}$ with the usual topology is connected if and only if it is an interval.
20. Continuous maps carry connected sets to connected sets.

7. Differential geometry

1. A **topological manifold of dimension n** is a topological space $(M, \mathcal{T})$ such that:
   (i) $\mathcal{T}$ is Hausdorff.
   (ii) $\mathcal{T}$ has a countable basis.
   (iii) For each $x \in M$ there exists $U \in \mathcal{T}$ with $x \in U$ and a homeomorphism $\varphi : U \to V$ for some open set $V \subset \mathbb{R}^n$.

   The pair (U, φ) is called a **local chart**.
2. Two local charts (U, φ) and (V, ψ) on an n-dimensional topological manifold M are said to be **compatible** if the maps $\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset \mathbb{R}^n \to \psi(U \cap V) \subset \mathbb{R}^n$ and $\varphi \circ \psi^{-1} : \psi(U \cap V) \subset \mathbb{R}^n \to \varphi(U \cap V) \subset \mathbb{R}^n$ are smooth. An **atlas** for M is a family $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of compatible local charts such that $M = \bigcup_{\alpha \in A} U_\alpha$. Two atlases $\mathcal{A}$ and $\mathcal{A}'$ are said to be **equivalent** if $\mathcal{A} \cup \mathcal{A}'$ is still an atlas. A **differentiable structure** on M is a choice of an equivalence class of atlases. Finally, a **differentiable manifold of dimension n** is a topological manifold of dimension n with a choice of difrerentiable structure.
3. A map $f : M \to N$ between differentiable manifolds is said to be **differentiable**, or **smooth**, if for any choices of local charts (U, φ) on an atlas for M and (V, ψ) on an atlas for N the map $\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \subset \mathbb{R}^m \to \psi(V) \subset \mathbb{R}^n$ is smooth. The set of differentiable maps between the differentiable manifolds M and N is represented as $C^\infty(M, N)$, and one usually writes $C^\infty(M) = C^\infty(M, \mathbb{R})$.
4. A **diffeomorphism** between two differentiable manifolds M and N is a smooth bijection $f : M \to N$ with a smooth inverse. A **local diffeomorphism** between two differentiable manifolds M and N is a smooth map $f : M \to N$ such that for each $x \in M$ there exist open sets $U \ni x$ and $V \ni f(x)$ such that $f|_U : U \to V$ is a diffeomorphism.

5. A **tangent vector** to a differentiable manifold M at a point $p \in M$ is a differential operator $\dot{c}(0) : C^\infty(M) \to \mathbb{R}$ of the form

$$\dot{c}(0) \cdot f = \frac{d}{dt}_{|t=0} f(c(t)),$$

where $c : \mathbb{R} \to M$ is a differentiable curve with $c(0) = p$.

6. The set T_pM of all tangent vectors to the n-dimensional differentiable manifold M at the point $p \in M$ is a vector space of dimension n, called the **tangent space** to M at p. If (U, φ) is a local chart with $p \in U$, corresponding to the local coordinates $(x^1, \ldots, x^n)$, a basis for T_pM is

$$\left\{ \frac{\partial}{\partial x^1}_{|p}, \ldots, \frac{\partial}{\partial x^n}_{|p} \right\},$$

where

$$\frac{\partial}{\partial x^i}_{|p} = \dot{c}_i(0), \qquad c_i(t) = \varphi^{-1}(x^1(p), \ldots, x^i(p) + t, \ldots, x^n(p)).$$

If $\dot{c}(0) \in T_pM$ is a tangent vector with $\varphi(c(t)) = (x^1(t), \ldots, x^n(t))$ then

$$\dot{c}(0) = \sum_{i=1}^{n} \dot{x}^i(0) \frac{\partial}{\partial x^i}_{|p}.$$

7. If $f : M \to N$ is a differentiable map then the **derivative** of f at the point $p \in M$ is the linear map $(df)_p : T_pM \to T_{f(p)}N$ given by

$$(df)_p(\dot{c}(0)) = \frac{d}{dt}_{|t=0} f(c(t)).$$

If (U, φ) is a local chart on M with $p \in U$, corresponding to the local coordinates $(x^1, \ldots, x^n)$, and (V, ψ) is a local chart on N with $f(p) \in V$, corresponding to the local coordinates $(y^1, \ldots, y^m)$, then

$$(df)_p \left(\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}_{|p} \right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \frac{\partial \hat{f}^j}{\partial x^i} v^i \right) \frac{\partial}{\partial y^j}_{|f(p)},$$

where $\hat{f} = \psi \circ f \circ \varphi^{-1}$ is the local representation of f in the local coordinates $(x^1, \ldots, x^n)$ on M and $(y^1, \ldots, y^m)$ on N.

8. The **cotangent space** to an n-dimensional differentiable manifold M at as point $p \in M$ is

$$T_p^*M = (T_pM)^* = \{\alpha : T_pM \to \mathbb{R} : \alpha \text{ is linear } \}.$$

T_p^*M is itself an n-dimensional vector space, with basis

$$\{(dx^1)_p, \ldots, (dx^n)_p\}$$

satisfying

$$(dx^i)_p \left(\frac{\partial}{\partial x^j}_{|p} \right) = \delta_{ij}$$

(**dual basis**).