## Algebraic and Geometric Methods in Engineering and Physics

# Abbreviated lecture notes

## 1. Set Theory

- 1. A relation on a set A is a subset  $\mathcal{R} \subset A \times A$ . We say that  $x \in A$  is related to  $y \in A$  if  $(x, y) \in \mathcal{R}$ , and we write  $x\mathcal{R}y$ .
- 2. An equivalence relation is a relation  $\mathcal{R}$  on a set A satisfying the following three properties:
  - (i) **Reflexivity:**  $x\mathcal{R}x$  for all  $x \in A$ ;
  - (ii) Symmetry: If  $x\mathcal{R}y$  the  $y\mathcal{R}x$  for all  $x, y \in A$ ;
  - (iii) **Transitivity:** If  $x\mathcal{R}y$  and  $y\mathcal{R}z$  then  $x\mathcal{R}z$  for all  $x, y, z \in A$ ;
  - If  $\mathcal{R}$  is an equivalence relation then we write  $x \sim y$  to mean  $x\mathcal{R}y$ .
- 3. If  $\sim$  is an equivalence relation on A and  $x \in A$  then the **equivalence class** of x is the set

$$[x] = \{y \in A : y \sim x\} \subset A.$$

- 4. A partition of a set A is a family  $\{A_i\}_{i \in I}$  of subsets of A such that:
  - (i)  $\bigcup_{i \in I} A_i = A;$
  - (ii) If  $i \neq j$  then  $A_i \cap A_j = \emptyset$ .
- 5. The equivalence classes of an equivalence relation on A form a partition of A; conversely, given a partition of A there exists an equivalence relation on A whose equivalence classes are precisely the subsets of the partition.
- 6. The quotient set of A by an equivalence relation ~ is the set A/~ of its equivalence classes. The quotient map, or canonical projection, is the map π : A → A/~ defined as π(x) = [x] for all x ∈ A.

## 2. Groups

- 1. A binary operation on a set A is a map  $f : A \times A \to A$ . We often write  $f(x, y) = x \cdot y = xy$  for  $x, y \in A$ .
- 2. A group is a pair  $(G, \cdot)$ , where G is a set and  $\cdot : G \times G \to G$  is a binary operation satisfying:
  - (i) Associativity:  $x \cdot (y \cdot z) = (x \cdot y) \cdot z$  for all  $x, y, z \in G$ ;
  - (ii) **Existence of identity:** There exists  $e \in G$  such that  $e \cdot x = x \cdot e = x$  for all  $x \in G$ ;
  - (iii) **Existence of inverses:** For each  $x \in G$  there exists  $y \in G$  such that  $y \cdot x = x \cdot y = e$  for all  $x, y \in G$ .
- 3. If  $(G, \cdot)$  is a group then:
  - (i) The identity element  $e \in G$  is unique;
  - (ii) Each element  $x \in G$  has a unique inverse  $x^{-1} \in G$ .
- 4. A group  $(G, \cdot)$  is called **abelian** if the group operation  $\cdot$  is commutative, that is, if  $x \cdot y = y \cdot x$  for all  $x, y \in G$ . In this case we often write  $x \cdot y = x + y$ , e = 0 and  $x^{-1} = -x$ .
- 5. The **order** of a group G is the number of elements in G, and is represented as |G|.
- 6. If A is a set the its **permutation group** is the set

$$Sym(A) = \{f : A \to A : f \text{ is bijective } \}$$

with the composition operation.

- 7. The symmetric group on  $n \in \mathbb{N}$  elements is the group  $S_n = \text{Sym}(\{1, \ldots, n\})$ . It is nonabelian for n > 2, and  $|S_n| = n!$ .
- 8. A subset  $H \subset G$  of a group G is called a **subgoup** if H is itself a group under the group operation of G, or, equivalently, if:
  - (i)  $xy \in H$  for all  $x, y \in H$ ;
  - (ii)  $e \in H$ ;
  - (iii)  $x^{-1} \in H$  for all  $x \in H$ .
- 9. If  $g \in G$  and H is a subgroup of G then the **left coset** determined by g and H is the set

$$gH = \{gh : h \in H\}.$$

The set G/H of all left cosets forms a partition of G, and the corresponding equivalence relation is given by

$$g_1 \sim g_2 \Leftrightarrow g_1^{-1} g_2 \in H.$$

- 10. Lagrange's Theorem: If G is a finite group and  $H \subset G$  is a subgroup then |H| is a divisor of |G|.
- 11. The **index** of a subgroup H on a group G is the number [G:H] of left cosets. If G is finite then [G:H] = |G|/|H|.
- 12. A subgroup H of a group G is called a **normal subgroup** if  $ghg^{-1} \in H$  for all  $g \in G$ and  $h \in H$ . In that case, the set G/H of cosets forms a group under the operation  $(g_1H)(g_2H) = (g_1g_2)H$  for all  $g_1, g_2 \in G$  (called the **quotient group**).
- 13. For each  $d \in \mathbb{N}$  the group of integers mod d is  $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$ . We write  $a \equiv b \pmod{d}$  to signify that  $a, b \in \mathbb{Z}$  satisfy [a] = [b] in  $\mathbb{Z}_d$ .

- 14. A map  $\varphi: G \to H$  between two groups is called a **homomorphism** if  $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$ for all  $g_1, g_2 \in G$ . An **isomorphism** is a bijective homomorphism. If  $\varphi: G \to H$  is an isomorphism then G and H are said to be **isomorphic**.
- 15. If  $\varphi: G \to H$  is a homomorphism then:
  - (i)  $\varphi(e_G) = (e_H);$
  - (ii)  $\varphi(g^{-1}) = \varphi(g)^{-1}$  for all  $g \in G$ .
- 16. The composition of homomorphisms is again a homomorphism.
- 17. Isomorphism Theorem: If  $\varphi: G \to H$  is a homomorphism then:
  - (i)  $\operatorname{im} \varphi$  is a subgroup of *H*;
  - (ii) ker  $\varphi$  is a normal subgroup of G;
  - (iii) The map  $\tilde{\varphi}: G/\ker \varphi \to \operatorname{im} \varphi$  defined by  $\tilde{\varphi}([g]) = \varphi(g)$  is an isomorphism.
- 18. If G is a group and  $g \in G$  then the subgroup generated by g is

$$\langle g \rangle = \{ g^n : n \in \mathbb{Z} \}$$

(where  $g^n$  has the obvious meaning). The **order** of g is  $\operatorname{ord}(g) = |\langle g \rangle|$ . If  $\operatorname{ord}(g) = d$  then  $\langle g \rangle \cong \mathbb{Z}_d$ , and if  $\operatorname{ord}(g) = \infty$  then  $\langle g \rangle \cong \mathbb{Z}$ . If  $G = \langle g \rangle$  for some  $g \in G$  then G is said to be **cyclic**.

19. If G and H are groups then  $G \times H$  is also a group with the operation defined by the formula  $(g_1, h_1) \cdot (g_2, h_2) = (g_1g_2, h_1h_2).$ 

#### 3. Rings and number theory

- 1. A ring  $(R, +, \cdot)$  is a set equipped with two binary operations such that (R, +) is an abelian group and  $\cdot$  is associative, has an identity 1 and is **distributive** with respect to +, that is,  $a \cdot (b + c) = a \cdot b + a \cdot c$  and  $(a + b) \cdot c = a \cdot c + b \cdot c$  for all  $a, b, c \in R$ .
- 2. A ring  $(R, +, \cdot)$  is called **commutative** if  $\cdot$  is commutative.
- 3. On a ring  $(R, +, \cdot)$  we have  $a \cdot 0 = 0 \cdot a = 0$ , for any  $a \in R$ . If  $a, b \in R \setminus \{0\}$  are such that ab = 0 then a and b are called **zero divisors**.
- 4. An element  $a \in R$  on a ring  $(R, +, \cdot)$  is called **invertible** if there exists  $b \in R$  such that ab = ba = 1, in which case we write  $b = a^{-1}$ . The set  $R^*$  of all invertible elements is a group for the multiplication.
- 5. A field is a commutative ring  $(R, +, \cdot)$  such that  $R^* = R \setminus \{0\}$ .
- 6.  $(\mathbb{Z}_d, +, \cdot)$  is a commutative ring for the multiplication given by  $[m] \cdot [n] = [mn]$ .
- 7.  $\mathbb{Z}_d^* = \{[n] : gcd(n,d) = 1\}$ . In particular,  $\mathbb{Z}_d$  is a field if and only if d is prime.
- 8. The **Euler function**  $\varphi : \mathbb{N} \to \mathbb{N}$  is defined as

$$\varphi(n) = |\{m \in \mathbb{N} : m \le n \text{ and } \gcd(m, n) = 1\}|,$$

so that  $|\mathbb{Z}_d^*| = \varphi(d)$ . If gcd(m, n) = 1 then  $\varphi(m, n) = \varphi(m)\varphi(n)$ , and so if  $n = p_1^{r_1} \cdots p_k^{r_k}$  with  $p_1, \ldots, p_k$  distinct prime numbers then  $\varphi(n) = (p_1^{r_1} - p_1^{r_1-1}) \cdots (p_k^{r_k} - p_1^{r_k-1})$ .

- 9. Euler's Theorem: If gcd(a, d) = 1 then  $a^{\varphi(d)} \equiv 1 \pmod{d}$ .
- 10. Chinese Remainder Theorem: If  $n_1, \ldots, n_k$  are coprime and  $N = n_1 \cdot n_k$  then the map

$$\mathbb{Z}_N \ni [m] \mapsto ([m], \dots, [m]) \in \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}$$

is an isomorphism.

11. Classification of finite abelian groups: If G is a finite abelian group of order  $|G| = p_1^{r_1} \cdots p_k^{r_k}$  with  $p_1, \ldots, p_k$  distinct prime numbers then

$$G \cong G_1 \times \cdots \times G_k,$$

where  $|G_i| = p_i^{r_i}$  for each  $i \in \{1, \ldots, k\}$ . Moreover, there exists a unique nondecreasing sequence  $a_i^1, \ldots, a_i^\ell \in \mathbb{N}$  with  $a_i^1 + \cdots + a_i^\ell = r_i$  such that

$$G_i \cong \mathbb{Z}_{a_i^1} \times \cdots \times \mathbb{Z}_{a_i^\ell}.$$

12. **RSA public key encryption:** Two (large) distinct prime numbers p and q are chosen, and the number N = pq is published, along with an **encryption exponent** e, chosen such that  $gcd(\varphi(N), e) = 1$ . To encrypt a message  $[X] \in \mathbb{Z}_N$  the sender simply computes  $[X]^e$ . To decrypt the message the receiver uses the fact that  $([X]^e)^d = [X]$ , where the **decryption exponent** d is such that  $[d] = [e]^{-1}$  in  $\mathbb{Z}_{\varphi(N)}$ . The security of this system relies on the fact that obtaining the descryption exponent requires knowledge of  $\varphi(N) = (p-1)(q-1)$ , which would involve factorizing N, a very hard task for p and q large enough.

## 4. Group actions

- 1. An action  $G \stackrel{\varphi}{\curvearrowright} M$  of a group G on a set M is a homomorphism  $\varphi : G \to \text{Sym}(M)$  (we write  $\varphi(g) = \varphi_g$ ). The action is called **effective** if  $\text{ker}(\varphi) = \{e\}$ .
- 2. If  $G \stackrel{\varphi}{\sim} M$  is an action of a group G on a set M then:
  - (i) The **orbit** of a point  $x \in M$  is the set

$$\varphi_G(x) = \{\varphi_g(x) : g \in G\}.$$

The set of all orbits is denoted M/G, and is a partition of M.

(ii) The **stabilizer** of a set  $X \subset M$  is the subgroup of G defined by

$$G_X = \{g \in G : \varphi_g(X) = X\}.$$

The isotropy subgroup of  $x \in M$  is just  $G_x = G_{\{x\}}$ .

- (iii) A point  $x \in M$  is called a **fixed point** of the action if  $G_x = G$ . The set of all fixed points of the action is denoted by  $M^G$ .
- 3. The **dihedral group on order** n is the group  $D_n$  defined by two generators r and s together with the relations

$$r^n = e, \qquad s^2 = e, \qquad rs = sr^{-1}$$

We have  $|D_n| = 2n$ , since

$$D_n = \{e, r, \dots, r^{n-1}, s, sr, \dots, sr^{n-1}\}.$$

- 4. An action  $G \stackrel{\varphi}{\sim} M$  is called:
  - (i) Free if  $G_x = \{e\}$  for all  $x \in M$ ;
  - (ii) **Transitive** if  $\varphi_G(x) = M$  for some (thus all)  $x \in M$ .
- 5. The **canonical action** of a group G on a space of left cosets G/H is the action  $G \overset{\varphi^{\text{can}}}{\curvearrowright} G/H$  defined by  $\varphi^{\text{can}}_{g}([g_1]) = [gg_1]$ .
- 6. If  $G \stackrel{\varphi}{\sim} M$  and  $G \stackrel{\varphi'}{\sim} M'$  are G-actions then a map  $T: M \to M'$  is called:
  - (i) G-equivariant if  $T(\varphi_g(x)) = \varphi'_g(T(x))$  for all  $g \in G$  and  $x \in M$ .
  - (ii) An equivalence of *G*-actions if it is a *G*-equivariant bijection.
- 7. Any transitive action  $G \stackrel{\varphi}{\curvearrowright} M$  is equivalent to  $G \stackrel{\varphi^{\operatorname{can}}}{\curvearrowleft} G/G_{x_0}$ , where  $G_{x_0}$  is the isotropy subgroup of any point  $x_0 \in M$  and  $T_{x_0} : G/G_{x_0} \to M$  given by  $T_{x_0}([g]) = \varphi_g(x_0)$  is an equivalence of G-actions.
- 8. The conjugation action  $G \curvearrowright^{\varphi^c} G$  is the action of G on itself given by  $\varphi_g^c(h) = ghg^{-1}$ . The orbits of this action are called conjugacy classes. The set Z(G) of fixed points of this action is called the center of G, and is a subgroup (subgroup of the elements in G which commute with every other element). The isotropy subgroup Z(g) of a given element  $g \in G$  is called the centralizer of g (largest subgroup of G which has g in its center). The stabilizer N(H) of a subgroup  $H \subset G$  is called the normalizer of H (largest subgroup of G which contains H as a normal subgroup).

- 9. If  $\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$  then  $\sigma = (j_1 \cdots j_k) \in S_n$  (called a *k*-cycle) represents the permutation defined by  $\sigma(j_r) = j_{r+1}$  for  $r \in \{1, \ldots, k-1\}$ ,  $\sigma(j_k) = j_1$  and  $\sigma(j) = j$  for  $j \notin \{j_1, \ldots, j_k\}$ . This representation is not unique, as  $(j_1 j_2 \cdots j_k) = (j_k j_1 \cdots j_{k-1})$ . If  $\{j_1, \ldots, j_k\} \cap \{j_k, \ldots, j_r\} = \{j_k\}$  then  $(j_1 \cdots j_k) \circ (j_k \cdots j_r) = (j_1 \cdots j_r)$ . Every permutation is the product of unique disjoint cycles.
- 10. A 2-cycle is called a **transposition**. A transposition (ij) is called **simple** if |i-j| = 1. The set of simple transpositions generates  $S_n$ .
- 11.  $S_n$  is isomorphic to the group with n-1 generators  $s_1, \ldots, s_{n-1}$  subject to the following relations:
  - (i)  $s_i^2 = e$  for i = 1, ..., n 1;
  - (ii)  $s_i s_j = s_j s_i$  if  $|i j| \ge 2$ ;
  - (iii)  $s_i s_{i+1}$  has order 3 for  $i = 1, \ldots, n-2$ .
- 12. The sign of a perturbation  $\sigma \in S_n$  is  $\operatorname{sgn}(\sigma) = (-1)^{n(\sigma)}$ , where  $n(\sigma)$  is the number of transpositions on any decomposition of  $\sigma$  as a product of transpositions. The map  $\operatorname{sgn} : S_n \to \mathbb{Z}^*$  is a group homomorphism, and its kernel  $A_n$  is called the alternating group (normal subgroup of order n!/2).
- 13. A partition of  $n \in \mathbb{N}$  is a k-tuple  $(i_1, \ldots, i_k) \in \mathbb{N}^k$  where  $i_1 \geq \ldots \geq i_k$  and  $i_1 + \ldots + i_k = n$ .
- 14. The conjugacy classes of  $S_n$  are in 1 to 1 correspondence with the partitions of n via

 $D_{(i_1,\ldots,i_k)} = \{\sigma_1 \circ \cdots \circ \sigma_k : \sigma_1, \ldots, \sigma_k \text{ are disjoint cycles of lengths } i_1, \ldots, i_k\}.$ 

15. Burnside's Counting Theorem: Let  $G \stackrel{\varphi}{\sim} M$  be an action of a finite group G on a finite set M. Then the number of orbits is

$$|M/G| = \frac{1}{|G|} \sum_{g \in G} |M^g|,$$

where  $M^g$  is the set of points in M that are fixed by g:

$$M^g = \{ x \in M : \varphi_q(x) = x \} .$$

## 5. Representations of finite groups

- 1. A **representation** of a group G is an action  $G \stackrel{\varphi}{\sim} V$  on a complex vector space V by linear maps. The **degree** of the representation is deg  $\varphi = \dim V$ . If V is finite-dimensional and has an inner product then the representation is called **unitary** if the maps  $\varphi_g$  are unitary for all  $g \in G$ .
- 2. An **intertwining map** bewteen two representations  $G \stackrel{\varphi}{\sim} V$  and  $G \stackrel{\varphi'}{\sim} V'$  is a linear equivariant map  $T: V \to V'$ . If T is invertible then the representations are said to be **equivalent**.
- 3. An **invariant subspace** for a representation  $G \stackrel{\varphi}{\sim} V$  is a subspace  $W \subset V$  such that  $\varphi_g(W) \subset W$  for all  $g \in G$ . If the representation is unitary and W is an invariant subspace then  $W^{\perp}$  is also an invariant subspace.
- 4. The **direct sum** of two given representations  $G \stackrel{\varphi}{\sim} V$  and  $G \stackrel{\varphi'}{\sim} V'$  is the representation  $G \stackrel{\varphi \oplus \varphi'}{\sim} V \oplus V'$  defined as

$$(\varphi \oplus \varphi')_g(v, v') = (\varphi_g(v), \varphi'_q(v')).$$

- 5. A representation  $G \stackrel{\varphi}{\sim} V$  is said to be **irreducible** if its only invariant subspaces are  $\{0\}$  and V.
- 6. **Maschke's Theorem:** Every finite-dimensional representation of a finite group is equivalent to a direct sum of irreducible representations.
- 7. Schur's Lemma: Let  $G \stackrel{\varphi}{\sim} V$  and  $G \stackrel{\psi}{\sim} W$  be two irreducible finite-dimensional representations of the finite group G, and let  $T : V \to W$  be an intertwining map. Then either T = 0 ot T is invertible. Moreover, if the two representations coincide then  $T = \lambda \operatorname{id}_V$  for some  $\lambda \in \mathbb{C}$ .
- 8. Every irreducible finite-dimensional representation of a finite abelian group has degree 1.
- 9. The regular representation of a finite group G on  $L(G) = \{f : G \to \mathbb{C}\}$  is defined as

$$(\varphi_q^{(r)}(f))(h) = f(g^{-1}h)$$

for all  $g, h \in G$ . This representation is unitary for the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

10. Schur's orthogonality relations: Let G be a finite group and let  $\varphi : G \to U(n)$  and  $\rho : G \to U(m)$  be two inequivalent irreducible representations. Then:

(i) 
$$\langle \varphi_{ij}, \rho_{kl} \rangle = 0.$$

(ii) 
$$\langle \varphi_{ij}, \varphi_{kl} \rangle = \frac{1}{n} \delta_{ik} \delta_{jl}.$$

- 11. The **character** of a finite-dimensional representation  $G \stackrel{\varphi}{\sim} V$  of a finite group G is the map  $\chi_{\varphi} \in L(G)$  defined by  $\chi_{\varphi}(g) = \operatorname{tr}(\varphi_g)$ . In particular,  $\chi_{\varphi}(e) = \operatorname{deg}(\varphi)$ .
- 12. The character  $\chi_{\varphi}$  of a finite-dimensional representation  $G \stackrel{\varphi}{\frown} V$  of a finite group is a **class** function, that is,  $\chi_{\varphi}(hgh^{-1}) = \chi_{\varphi}(g)$  for all  $g, h \in G$ . We represent by  $Z(L(G)) \subset L(G)$  the set of class functions.

13. If  $G \stackrel{\varphi}{\sim} V$  and  $G \stackrel{\psi}{\sim} W$  are irreducible finite-dimensional representations of a finite group G then

$$\langle \chi_{\varphi}, \chi_{\psi} \rangle = \begin{cases} 1 \text{ if } \varphi \sim \psi \,, \\ 0 \text{ if } \varphi \not\sim \psi \,. \end{cases}$$

14. Let  $[\varphi^{(1)}], \ldots, [\varphi^{(s)}]$  be the equivalence classes of irreducible finite-dimensional representations of a finite group G, and let  $G \stackrel{\rho}{\sim} V$  be a finite-dimensional representation of G,

$$\rho \sim m_1 \varphi^{(1)} \oplus \ldots \oplus m_s \varphi^{(s)},$$

where

$$m_i \varphi^{(i)} = \overbrace{\varphi^{(i)} \oplus \ldots \oplus \varphi^{(i)}}^{m_i}.$$

Then

$$m_i = \langle \chi_\rho, \chi_{\varphi^{(i)}} \rangle.$$

15. Let  $[\varphi^{(1)}], \ldots, [\varphi^{(s)}]$  be the equivalence classes of irreducible finite-dimensional representations of a finite group G, and let  $d_i = \deg(\varphi^{(i)})$ . Then

$$d_1^2 + \ldots + d_s^2 = |G|.$$

- 16. Let  $[\varphi^{(1)}], \ldots, [\varphi^{(s)}]$  be the equivalence classes of irreducible finite-dimensional representations of a finite group G. Then  $\{\chi_{\varphi^{(1)}}, \ldots, \chi_{\varphi^{(s)}}\}$  is an orthonormal basis of the space Z(L(G)) of class functions. In particular, there are as many irreducible representations as conjugacy classes.
- 17. If  $G \stackrel{\varphi}{\frown} V$  is a finite-dimensional representations of the finite group G and  $f \in L(G)$  then

$$\varphi_f = \sum_{g \in G} f(g) \varphi_g$$

18. Let  $[\varphi^{(1)}], \ldots, [\varphi^{(s)}]$  be the equivalence classes of irreducible finite-dimensional representations of a finite group G, and let  $d_i = \deg(\varphi^{(i)})$  and  $\chi_i = \chi_{\varphi^{(i)}}$ . If a given finite-dimensional representation  $G \stackrel{\varphi}{\sim} V$  decomposes as

$$\varphi \sim m_1 \varphi^{(1)} \oplus \ldots \oplus m_s \varphi^{(s)},$$

corresponding to the orthogonal decomposition

$$V = V_1 \oplus \ldots \oplus V_s,$$

then the orthogonal projection  $P_i: V \to V_i$  is

$$P_i = \frac{d_i}{|G|} \varphi_{\overline{\chi}_i}$$

19. A structure with n degrees of freedom near equilibrium is described by the symmetric positive (semi-)definite  $n \times n$  mass matrix M and the symmetric  $n \times n$  stiffness matrix K. The frequencies  $\omega$  of small oscillations of the structure satisfy

$$\omega^2 M u = K u$$

for some  $u \in \mathbb{R}^n \setminus \{0\}$ . If the structure admits a symmetry group acting linearly,  $G \stackrel{\varphi}{\sim} \mathbb{R}^n$ , then M and K are intertwiners for  $\varphi$ , and so they are block diagonal with respect to the decomposition of  $\varphi$  into irreducible representations. Moreover, if a given irreducible representation occurs only once then M and K are multiples of the identity on the corresponding block (but this is not true if the irreducible representation occurs more than once).

#### 6. Topology

- 1. A distance function (or metric) on a set M is a function  $d: M \times M \to \mathbb{R}$  satisfying:
  - (i) **Positivity:**  $d(x,y) \ge 0$  for all  $x, y \in M$ , and  $d(x,y) = 0 \Rightarrow x = y$ ;
  - (ii) Symmetry: d(x, y) = d(y, x) for all  $x, y \in M$ ;
  - (iii) Triangle inequality:  $d(x,z) \le d(x,y) + d(y,z)$  for all  $x, y, z \in M$ .
  - If d is a distance then (M, d) is said to be a **metric space**.
- 2. If (M, d) is a metric space and  $x \in M$  then the **open ball** with center x and radius r > 0 is the set

$$B_r(x) = \{ y \in M : d(x, y) < r \}.$$

- 3. If (M, d) is a metric space then a subset  $U \subset M$  is called **open** if for each  $x \in U$  there exists r > 0 such that  $B_r(x) \subset U$ . A subset  $F \subset M$  is called **closed** if  $M \setminus F$  is open.
- 4. Open balls are open sets.
- 5. The family  $\mathcal{T}_d$  of open sets on a metric space (M, d) satisfies the following properties:
  - (i)  $\emptyset, M \in \mathcal{T}_d$ ;
  - (ii) If  $\{U_{\alpha}\}_{\alpha \in A} \subset \mathcal{T}_d$  then  $\bigcup_{\alpha \in A} U_{\alpha} \in \mathcal{T}_d$ ;
  - (iii) If  $U_1, \ldots, U_n \in \mathcal{T}_d$  then  $\bigcap_{i=1}^n U_i \in \mathcal{T}_d$ .

## 6. A **topology** on a set M is a family $\mathcal{T}$ of subsets of M satisfying the following properties:

- (i)  $\emptyset, M \in \mathcal{T};$
- (ii) If  $\{U_{\alpha}\}_{\alpha\in A}\subset \mathcal{T}$  then  $\bigcup_{\alpha\in A}U_{\alpha}\in \mathcal{T}$ ;
- (iii) If  $U_1, \ldots, U_n \in \mathcal{T}$  then  $\bigcap_{i=1}^n U_i \in \mathcal{T}$ .

The sets  $U \in \mathcal{T}$  are called **open sets**, and the sets  $F \subset M$  such that  $M \setminus F \in \mathcal{T}$  are called **closed sets**. An open set containing  $x \in M$  is called a **neighborhood** of x. If  $\mathcal{T}$  is a topology on M the  $(M, \mathcal{T})$  is said to be a **topological space**.

- 7. A topological space  $(M, \mathcal{T})$  is said to be **Hausdorff** if given  $x, y \in M$  with  $x \neq y$  there exist  $U, V \in \mathcal{T}$  with  $U \cap V = \emptyset$  such that  $x \in U$  and  $y \in V$ .
- 8. A subfamily  $\mathcal{B} \subset \mathcal{T}$  of open sets is said to be a **basis** for the topology  $\mathcal{T}$  on M is it satisfies any of the following equivalent properties:
  - (i) For each  $U \in \mathcal{T}$  there exists  $\mathcal{B}_U \subset \mathcal{B}$  such that  $U = \bigcup_{V \in \mathcal{B}_U} V$ .
  - (ii) For each  $U \in \mathcal{T}$  and each  $x \in U$  there exists  $U_x \in \mathcal{B}$  such that  $x \in U_x \subset U$ .
- 9. If  $(M, \mathcal{T})$  is a topological space and  $N \subset M$  then  $\mathcal{T}_N = \{U \cap N : U \in \mathcal{T}\}$  is a topology in N (subspace topology).
- 10. A map  $f: M \to N$  between two topological spaces  $(M, \mathcal{T}_M)$  and  $(N, \mathcal{T}_N)$  is said to be **continuous** if  $f^{-1}(U) \in \mathcal{T}_M$  for all  $U \in \mathcal{T}_N$ .
- 11. The composition of continuous maps is continuous.
- 12. A homeomorphism between two topological spaces  $(M, \mathcal{T}_M)$  and  $(N, \mathcal{T}_N)$  is a continuous bijection  $f : M \to N$  with continuous inverse. If such a map exists then  $(M, \mathcal{T}_M)$  and  $(N, \mathcal{T}_N)$  are said to be homeomorphic.
- 13. If  $(M, \mathcal{T})$  is a topological space,  $\sim$  is an equivalence relation on M and  $\pi : M \to M/\sim$  is the canonical projection then the **quotient topology** on  $M/\sim$  is

$$\mathcal{T}_{\pi} = \{ U \in M / \sim : \pi^{-1}(U) \in \mathcal{T} \}$$

14. An **open cover** for a subset  $N \subset M$  of a topological space  $(M, \mathcal{T})$  is a collection  $\mathcal{O} \subset \mathcal{T}$  such that

$$N \subset \bigcup_{U \in \mathcal{O}} U.$$

A subcover is a subcollection  $\mathcal{O}' \subset \mathcal{O}$  which is still an open cover of N. A set N is said to be **compact** if every open cover of N admits a finite subcover.

- 15. If  $(M, \mathcal{T})$  is Hausdorff and  $N \subset M$  is compact then N is closed.
- 16. Heine-Borel theorem: a subset  $K \subset \mathbb{R}^n$  is compact (for the usual topology) if and only if it is closed and bounded.
- 17. Continuous maps carry compact sets to compact sets.
- 18. A topological space is said to be **connected** if the only subsets of M which are simultaneously open and closed are  $\emptyset$  and M. A subset  $N \subset M$  is said to be a **connected subset** if it is a connected topological space for the subspace topology.
- 19. A subset of  $\mathbb{R}$  with the usual topology is connected if and only if it is an interval.
- 20. Continuous maps carry connected sets to connected sets.

#### 7. Differential geometry

- 1. A topological manifold of dimension n is a topological space  $(M, \mathcal{T})$  such that:
  - (i)  $\mathcal{T}$  is Hausdorff.
  - (ii)  $\mathcal{T}$  has a countable basis.
  - (iii) For each  $x \in M$  there exists  $U \in \mathcal{T}$  with  $x \in U$  and a homeomorphism  $\varphi : U \to V$  for some open set  $V \subset \mathbb{R}^n$ .

The pair  $(U, \varphi)$  is called a **local chart**.

- 2. Two local charts  $(U, \varphi)$  and  $(V, \psi)$  on an *n*-dimensional topological manifold M are said to be **compatible** if the maps  $\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset \mathbb{R}^n \to \psi(U \cap V) \subset \mathbb{R}^n$  and  $\varphi \circ \psi^{-1} : \psi(U \cap V) \subset \mathbb{R}^n \to \varphi(U \cap V) \subset \mathbb{R}^n$  are smooth. An **atlas** for M is a family  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$  of compatible local charts such that  $M = \bigcup_{\alpha \in A} U_\alpha$ . Two atlases  $\mathcal{A}$ and  $\mathcal{A}'$  are said to be **equivalent** if  $\mathcal{A} \cup \mathcal{A}'$  is still an atlas. A **differentiable structure** on M is a choice of an equivalence class of atlases. Finally, a **differentiable manifold of dimension** n is a topological manifold of dimension n with a choice of differentiable structure.
- 3. A map  $f: M \to N$  between differentiable manifolds is said to be **differentiable**, or **smooth**, if for any choices of local charts  $(U, \varphi)$  on an atlas for M and  $(V, \psi)$  on an atlas for N the map  $\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \subset \mathbb{R}^m \to \psi(V) \subset \mathbb{R}^n$  is smooth. The set of differentiable maps between the differentiable manifolds M and N is represented as  $C^{\infty}(M, N)$ , and one usually writes  $C^{\infty}(M) = C^{\infty}(M, \mathbb{R})$ .
- 4. A diffeomorphism between two differentiable manifolds M and N is a smooth bijection f : M → N with a smooth inverse. A local diffeomorphism between two differentiable manifolds M and N is a smooth map f : M → N such that for each x ∈ M there exist open sets U ∋ x and V ∋ f(x) such that f|U : U → V is a diffeomorphism.
- 5. A tangent vector to a differentiable manifold M at a point  $p \in M$  is a differential operator  $\dot{c}(0): C^{\infty}(M) \to \mathbb{R}$  of the form

$$\dot{c}(0) \cdot f = \frac{d}{dt}_{|_{t=0}} f(c(t)),$$

where  $c : \mathbb{R} \to M$  is a differentiable curve with c(0) = p.

6. The set  $T_pM$  of all tangent vectors to the *n*-dimensional differentiable manifold M at the point  $p \in M$  is a vector space of dimension n, called the **tangent space** to M at p. If  $(U, \varphi)$  is a local chart with  $p \in U$ , corresponding to the local coordinates  $(x^1, \ldots, x^n)$ , a basis for  $T_pM$  is

$$\left\{\frac{\partial}{\partial x^1}_{|_p},\ldots,\frac{\partial}{\partial x^n}_{|_p}\right\},\,$$

where

$$\frac{\partial}{\partial x^i}|_p = \dot{c}_i(0), \qquad c_i(t) = \varphi^{-1}(x^1(p), \dots, x^i(p) + t, \dots, x^n(p)).$$

If  $\dot{c}(0) \in T_pM$  is a tangent vector with  $\varphi(c(t)) = (x^1(t), \dots, x^n(t))$  then

$$\dot{c}(0) = \sum_{i=1}^{n} \dot{x}^{i}(0) \frac{\partial}{\partial x^{i}}_{|_{p}}.$$

7. If  $f: M \to N$  is a differentiable map then the **derivative** of f at the point  $p \in M$  is the linear map  $(df)_p: T_pM \to T_{f(p)}N$  given by

$$(df)_p(\dot{c}(0)) = \frac{d}{dt}_{|_{t=0}} f(c(t)).$$

If  $(U,\varphi)$  is a local chart on M with  $p \in U$ , corresponding to the local coordinates  $(x^1,\ldots,x^n)$ , and  $(V,\psi)$  is a local chart on N with  $f(p) \in V$ , corresponding to the local coordinates  $(y^1,\ldots,y^m)$ , then

$$(df)_p\left(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i}_{|_p}\right) = \sum_{j=1}^m \left(\sum_{i=1}^n \frac{\partial \hat{f}^j}{\partial x^i} v^i\right) \frac{\partial}{\partial y^j}_{|_{f(p)}}$$

where  $\hat{f} = \psi \circ f \circ \varphi^{-1}$  is the local representation of f in the local coordinates  $(x^1, \ldots, x^n)$  on M and  $(y^1, \ldots, y^m)$  on N.

8. The **cotangent space** to an *n*-dimensional differentiable manifold M at as point  $p \in M$  is

$$T_p^*M = (T_pM)^* = \{\alpha : T_pM \to \mathbb{R} : \alpha \text{ is linear } \}.$$

The set  $T_p^*M$  is itself an *n*-dimensional vector space, with basis

$$\left\{(dx^1)_p,\ldots,(dx^n)_p\right\}$$

satisfying

$$(dx^i)_p \left(\frac{\partial}{\partial x^j}_{|_p}\right) = \delta_{ij}$$

(dual basis).

9. The tangent bundle of an *n*-dimensional differentiable manifold M is  $TM = \bigcup_{p \in M} T_p M$ . The natural projection is the map  $\pi : TM \to M$  defined as  $\pi(v) = p$  for  $v \in T_p M$ . For each local chart  $(U, \varphi)$  of M we define the local chart  $(\pi^{-1}(U), \tilde{\varphi})$  of TM by

$$\tilde{\varphi}(x^1,\ldots,x^n,v^1,\ldots,v^n) = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}_{|_{\varphi^{-1}(x^1,\ldots,x^n)}}$$

These charts fom an atlas for TM (with the appropriate topology) giving it the structure of a 2n-dimensional manifold.

- 10. A vector field on M is a smooth map  $X : M \to TM$  such that  $X_p \equiv X(p) \in T_pM$  for all  $p \in M$ , that is, such that  $\pi \circ X = id_M$  (such maps are called sections of the tangent bundle). The set of vector fields on M is denoted by  $\mathfrak{X}(M)$ .
- 11. The **cotangent bundle** of an *n*-dimensional differentiable manifold M is  $T^*M = \bigcup_{p \in M} T_p^*M$ . The **natural projection** is the map  $\pi : T^*M \to M$  defined as  $\pi(\alpha) = p$  for  $\alpha \in T_p^*M$ . For each local chart  $(U, \varphi)$  of M we define the local chart  $(\pi^{-1}(U), \tilde{\varphi})$  of  $T^*M$  by

$$\tilde{\varphi}(x^1,\ldots,x^n,p_1,\ldots,p_n) = \sum_{i=1}^n p_i(dx^i)_{\varphi^{-1}(x^1,\ldots,x^n)}$$

These charts fom an atlas for  $T^*M$  (with the appropriate topology) giving it the structure of a 2n-dimensional manifold.

- 12. A covector field on M is a smooth map  $\alpha : M \to T^*M$  such that  $\alpha_p \equiv \alpha(p) \in T_p^*M$  for all  $p \in M$ , that is, such that  $\pi \circ \alpha = \mathrm{id}_M$  (such maps are called sections of the cotangent bundle).
- 13. A covariant k-tensor on  $T_pM$  is a multilinear map  $T: (T_pM)^k \to \mathbb{R}$ .
- 14. If T is a covariant k-tensor on  $T_pM$  and S is a covariant l-tensor on  $T_pM$  then their **tensor** product is the covariant (k + l)-tensor  $T \otimes S$  defined by

$$(T \otimes S)(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) = T(v_1, \ldots, v_k)S(v_{k+1}, \ldots, v_{k+l}).$$

15. If M is an n-dimensional differentiable manifold then the vector space of covariant k-tensors on  $T_pM$  has dimension  $n^k$ , and a basis is

$$\left\{ (dx^{i_1})_p \otimes \cdots \otimes (dx^{i_k})_p \right\}_{i_1,\dots,i_k=1}^n$$

If T is a covariant k-tensor on  $T_pM$  then

$$T = \sum_{i_1,\dots,i_k=1}^n T_{i_1\dots i_k} (dx^{i_1})_p \otimes \dots \otimes (dx^{i_k})_p,$$

where

$$T_{i_1\dots i_k} = T\left(\frac{\partial}{\partial x^{i_1}}_{|_p}, \dots, \frac{\partial}{\partial x^{i_k}}_{|_p}\right)$$

16. There is a canonical identification  $(T_p^*M)^* \cong T_pM$  through

$$v(\alpha) = \alpha(v)$$

for all  $v \in T_pM$  and all  $\alpha \in T_p^*M$ .

17. A mixed tensor of type (k, l) (or k times covariant and l times contravariant) on  $T_pM$ is a multilinear map  $T : (T_pM)^k \times (T_p^*M)^l \to \mathbb{R}$ . If M is an n-dimensional differentiable manifold then the vector space  $T_p^{(k,l)}M$  of tensors of type (k, l) on  $T_pM$  has dimension  $n^{k+l}$ , and a basis is

$$\left\{ (dx^{i_1})_p \otimes \cdots \otimes (dx^{i_k})_p \otimes \frac{\partial}{\partial x^{j_1}}_{|_p} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_l}}_{|_p} \right\}_{i_1, \dots, i_k, j_1, \dots, j_l = 1}^n.$$

If T is a mixed tensor of type (k, l) on  $T_pM$  then

$$T = \sum_{i_1,\dots,i_k,j_1,\dots,j_l=1}^n T^{j_1,\dots,j_l}_{i_1\dots i_k} (dx^{i_1})_p \otimes \dots \otimes (dx^{i_k})_p \otimes \frac{\partial}{\partial x^{j_1}}_{|_p} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}}_{|_p},$$

where

$$T_{i_1...,i_k}^{j_1,...,j_l} = T\left(\frac{\partial}{\partial x^{i_1}}_{|_p},\ldots,\frac{\partial}{\partial x^{i_k}}_{|_p},(dx^{j_1})_p,\ldots(dx^{j_l})_p\right)$$

- 18. The **bundle of mixed tensors of type** (k, l) on an *n*-dimensional differentiable manifold is the set  $T^{(k,l)}M = \bigcup_{p \in M} T_p^{(k,l)}M$ . This set is a differentiable manifold of dimension  $n + n^{k+l}$  in the obvious way.
- 19. A mixed tensor field of type (k, l) on M is a section of  $T^{(k,l)}M$ .

- 20. A **Riemannian metric** on a differentiable manifold M is a covariant 2-tensor field g satisfying:
  - (i) Symmetry:  $g_p(v, w) = g_p(w, v)$  for all  $v, w \in T_pM$  and all  $p \in M$ .
  - (ii) **Positivity:**  $g_p(v,v) > 0$  for all  $v \in T_pM \setminus \{0\}$  and all  $p \in M$ .

The **length** of a curve  $c : [t_0, t_1] \to M$  is then defined as

$$l(c) = \int_{t_0}^{t_1} \sqrt{g(\dot{c}(t), \dot{c}(t))} dt.$$

- 21. An integral curve of a vector field  $X \in \mathfrak{X}(M)$  is a smooth curve  $c : (-\varepsilon, \varepsilon) \to M$  satisfying  $\dot{c}(t) = X_{c(t)}$  for all  $t \in (-\varepsilon, \varepsilon)$ .
- 22. Given  $X \in \mathfrak{X}(M)$  and  $p \in M$  there exists an open neighborhood  $U \ni p$  and a smooth map  $\phi^X : (-\varepsilon, \varepsilon) \times U \to M$  satisfying

$$\begin{cases} \frac{\partial \phi^X}{\partial t}(t,q) = X_{\phi^X(t,q)} \text{ for all } t \in (-\varepsilon,\varepsilon) \text{ and } q \in U, \\ \phi^X(0,q) = q \text{ for all } q \in U, \end{cases}$$

called the **local flow** of X. The map  $\phi_t^X : U \to \phi_t^X(U)$  defined as  $\phi_t^X(q) = \phi^X(t,q)$  is a diffeomorphism for all  $t \in (-\varepsilon, \varepsilon)$ . If  $\phi^X$  is defined on  $\mathbb{R} \times M$  the X is said to be **complete**.

23. Any vector field  $X \in \mathfrak{X}(M)$  can be interpreted as a linear operator  $X : C^{\infty}(M) \to C^{\infty}(M)$ bu defining  $(X \cdot f)(p) = X_p \cdot f$ . If  $X, Y \in \mathfrak{X}(M)$  then their commutator [X, Y] as linear operators, called their **Lie bracket**, is still a vector field. If  $(U, \varphi)$  is a local chart and

$$X_p = \sum_{i=1}^n X^i(p) \frac{\partial}{\partial x^i}_{|_p}, \qquad Y_p = \sum_{i=1}^n Y^i(p) \frac{\partial}{\partial x^i}_{|_p}$$

for smooth functions  $X^i,Y^i:U\to \mathbb{R}$  then

$$[X,Y]_p = \sum_{i=1}^n (X_p \cdot Y^i - Y_p \cdot X^i) \frac{\partial}{\partial x^i}_{|_p}.$$

#### 8. Lie groups and Lie algebras

- A Lie algebra over a field K (usually R or C) is a pair (A, [·, ·]), where A is a vector space over K and [·, ·] : A × A → A is a binary operation satisfying:
  - (i) Antisymmetry: [X, Y] = -[Y, X] for all  $X, Y \in A$ .
  - (ii) **Bilinearity:** [aX + bY, Z] = a[X, Z] + b[Y, Z] for all  $X, Y \in A$  and  $a, b \in \mathbb{K}$ .
  - (iii) Jacobi identity: [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 for all  $X, Y, Z \in A$ .
- 2. If  $(A, \circ)$  is an **associative algebra**, that is, a vector space with a bilinear binary operation  $\circ : A \times A \to A$  such that  $(X \circ Y) \circ Z = X \circ (Y \circ Z)$  for all  $X, Y, Z \in A$  then  $(A, [\cdot, \cdot])$  is a Lie algebra for  $[A, B] = A \circ B B \circ A$ .
- 3. A linear map  $\phi : \mathfrak{g} \to \mathfrak{h}$  between two Lie algebras  $\mathfrak{g}$  and  $\mathfrak{h}$  is called a Lie algebra homomorphism if

$$\phi([X,Y]) = [\phi(X),\phi(Y)]$$

for all  $X, Y \in \mathfrak{g}$ . If  $\phi$  is a bijection then it is called a **Lie algebra isomorphism**.

- 4. A subspace  $\mathfrak{h} \subset \mathfrak{g}$  of a Lie algebra  $\mathfrak{g}$  is called a Lie subalgebra if  $[X, Y] \in \mathfrak{h}$  for all  $X, Y \in \mathfrak{h}$ . A subalgebra  $\mathfrak{i} \subset \mathfrak{g}$  is called an **ideal** if  $[X, Y] \in \mathfrak{i}$  for all  $X \in \mathfrak{i}$  and  $Y \in \mathfrak{g}$ .
- 5. If  $\mathfrak{g}$  is a Lie algebra and  $\mathfrak{i} \subset \mathfrak{g}$  is an ideal the the quotient  $\mathfrak{g}/\mathfrak{i}$  (as an abelian group) is a Lie algebra for the multiplication by scalars defined as

$$a\pi(X) = \pi(aX)$$

and the Lie bracket

$$[\pi(X), \pi(Y)] = \pi([X, Y])$$

where  $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$  is the quotient map.

- 6. Isomorphism Theorem: If  $\phi : \mathfrak{g} \to \mathfrak{h}$  is a Lie algebra homomorphism then:
  - (i)  $im(\phi) \subset \mathfrak{h}$  is a subalgebra.
  - (ii)  $\ker(\phi) \subset \mathfrak{g}$  is an ideal.
  - (iii) The map  $\tilde{\phi} : \mathfrak{g}/\ker(\phi) \to \operatorname{im}(\phi)$  given by  $\tilde{\phi}(\pi(X)) = \phi(X)$  is a Lie algebra isomorphism.
- 7. A **Lie group** is a smooth manifold G which is also a group such that the maps  $\mu : G \times G \to G$ and  $\iota : G \to G$  defined as  $\mu(g, h) = gh$  and  $\iota(g) = g^{-1}$  are smooth.
- 8. Any topologically closed subgroup  $G \subset GL(n, \mathbb{R})$  is a Lie group. Examples are:
  - (i)  $GL(n,\mathbb{R})$ .
  - (ii)  $SL(n,\mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) : \det A = 1\}.$
  - (iii)  $O(n) = \{A \in M_{n \times n}(\mathbb{R}) : A^t A = I\}.$
  - (iv)  $SO(n) = \{A \in O(n) : \det A = 1\}.$
  - (v)  $U(n) = \{A \in M_{n \times n}(\mathbb{C}) : A^*A = I\}.$
  - (vi)  $SU(n) = \{A \in U(n) : \det A = 1\}.$
- 9. A Lie group homomorphism is a homomorphism  $\varphi : G \to H$  between Lie groups which is also a diffeomorphism. A Lie group isomorphism is a bijective Lie group homomorphism.
- 10. If  $f: M \to N$  is a diffeomorphism and  $X \in \mathfrak{X}(M)$  then the **pushforward** of X by f is the vector field  $f_*X \in \mathfrak{X}(N)$  defined by  $(f_*X)_p = (df)_{f^{-1}(p)}X_{f^{-1}(p)}$ . If  $X, Y \in \mathfrak{X}(M)$  then  $f_*[X,Y] = [f_*X, f_*Y]$ .

- 11. Let G be a Lie group. A vector field  $X \in \mathfrak{X}(G)$  is said to be **left-invariant** if  $(L_g)_*X = X$  for all  $g \in G$ . For each  $V \in T_eG$ , the vector field  $X^V$  defined as  $X_g = (dL_g)_eV$  is left-invariant, and in fact the map  $V \mapsto X^V$  is a linear isomorphism between  $T_eG$  and the space  $X^L(G)$  of left-invariant vector fields.
- 12. Given a Lie group G, the subspace  $\mathfrak{X}^{L}(G) \subset \mathfrak{X}(G)$  is a Lie subalgebra. The Lie algebra of G is  $\mathfrak{g} = T_e G$  with the induced Lie bracket  $[V, W] = [X^V, X^W]_e$ .
- 13. We have  $\mathfrak{gl}(n,\mathbb{R}) = M_{n\times n}(\mathbb{R})$  with the bracket [A,B] = AB BA. The same bracket works for any matrix Lie group  $G \subset GL(n,\mathbb{R})$ .
- 14. If  $V \in \mathfrak{g}$  then  $X^V \in X^L(G)$  is complete. We define the **exponential map**  $\exp : \mathfrak{g} \to G$  as  $\exp(V) = \phi_1^{X^V}(e)$ , where  $\phi_t^{X^V} : G \to G$  is the flow of  $X^V$ .
- 15. The map  $\exp : \mathfrak{gl}(n,\mathbb{R}) \to GL(n,\mathbb{R})$  is given by  $\exp(A) = e^A = \sum_{k=0}^{+\infty} \frac{1}{k!} A^k$ . The same formula works for any matrix Lie group  $G \subset GL(n,\mathbb{R})$ .
- 16. If  $G \subset GL(n, \mathbb{R})$  is a matrix Lie group with Lie algebra  $\mathfrak{g}$  then  $A \in \mathfrak{g}$  if and only if  $e^{tA} \in G$  for all  $t \in \mathbb{R}$ . In particular:
  - (i)  $\mathfrak{gl}(n,\mathbb{R}) = M_{n \times n}(\mathbb{R}).$
  - (ii)  $\mathfrak{sl}(n,\mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) : \operatorname{tr} A = 0\}.$
  - (iii)  $\mathfrak{o}(n) = \{A \in M_{n \times n}(\mathbb{R}) : A^t = -A\}.$
  - (iv)  $\mathfrak{so}(n) = \mathfrak{o}(n)$ .
  - (v)  $\mathfrak{u}(n) = \{A \in M_{n \times n}(\mathbb{C}) : A^* = -A\}.$
  - (vi)  $\mathfrak{su}(n) = \{A \in \mathfrak{u}(n) : \operatorname{tr} A = 0\}.$
- 17. Let M be a topological space and  $x_0 \in M$ . The **fundamental group** of M with base point  $x_0$  is

 $\pi_1(M, x_0) = \{c : [0, 1] \to M \text{ continuous with } c(0) = c(1) = x_0\} / \sim,$ 

where  $c_0 \sim c_1$  if and only if they are **homotopic with base point**  $x_0$ , that is, if there exists a continuous map  $H : [0,1] \times [0,1] \rightarrow M$  such that  $H(t,0) = c_0(t)$ ,  $H(t,1) = c_1(t)$  and  $H(0,s) = H(1,s) = x_0$  for all  $t, s \in [0,1]$ . The group operation is  $[c_0] \cdot [c_1] = [c_0 * c_1]$ , where

$$c_0 * c_1 = \begin{cases} c_0(2t) \text{ for } t \in [0, \frac{1}{2}] \\ c_1(2t-1) \text{ for } t \in [\frac{1}{2}, 1] \end{cases}$$

If M is **path-connected**, that is, if for every  $x, y \in M$  there exists a continuous path  $c : [0,1] \to M$  such that c(0) = x and c(1) = y, then the fundamental group does not depend on the base point. Finally, M is said to be **simply connected** if it is path-connected and  $\pi_1(M) = \{e\}$ .

18. Lie's Theorem: Given a finite-dimensional real Lie algebra  $\mathfrak{g}$ , there exists a unique simply connected Lie group  $\widetilde{G}$  with Lie algebra  $\mathfrak{g}$ . If G is any other Lie group with Lie algebra  $\mathfrak{g}$ , then there exists a discrete subgroup D of  $Z(\widetilde{G})$  (the center of  $\widetilde{G}$ ) such that  $G \cong \widetilde{G}/D$ , and  $\pi_1(G) \cong D$  (where  $\widetilde{G}/D$  is given the natural differentiable structure).

#### 9. Representations of Lie algebras

- 1. A Lie algebra  $\mathfrak{g}$  is called **simple** if dim  $\mathfrak{g} > 1$  and  $\mathfrak{g}$  does not have nontrivial ideals. A Lie algebra  $\mathfrak{g}$  is called **semisimple** if  $\mathfrak{g} = \mathfrak{g}_1 \otimes \ldots \otimes \mathfrak{g}_s$  for simple Lie algebras  $\mathfrak{g}_1, \ldots, \mathfrak{g}_s$  (where the bracket it the direct sum is defined in the obvious way).
- 2. A representation of the Lie algebra  $\mathfrak{g}$  on a vector space V is a Lie algebra homomorphism  $\pi : \mathfrak{g} \to L(V)$ , where L(V) is the space of linear transformations of V with the commutator. The representation is called **irreducible** if V does not contain nontrivial invariant subspaces.
- 3. The adjoint representation  $ad : \mathfrak{g} \to L(\mathfrak{g})$  is defined as  $ad_X(Y) = [X, Y]$  for all  $X, Y \in \mathfrak{g}$ .
- The Killing form on a Lie algebra g over K (where K = R or K = C) is the bilinear form
   B: g × g → K given by B(X, Y) = tr(ad<sub>X</sub> ad<sub>Y</sub>).
- 5. Cartan's Theorem: A Lie algebra g is semisimple if and only if its Killing form B is nondegenerate (that is, if and only if B(X,Y) = 0 for all Y ∈ g implies X = 0). Moreover, if g is a real Lie algebra then B is negative definite if and only if g is semisimple and is the Lie algebra of a compact Lie group.
- 6. The complexification of a real Lie algebra g is the complex vector space g<sub>C</sub> consisting of elements of the form X + iY with X, Y ∈ g with the obvious operations. A real subalgebra h ⊂ g is called a real form of the complex Lie algebra g if g ≃ h<sub>C</sub>.
- 7. The structure constants associated to a basis  $\{X_1, \ldots, X_n\}$  of a Lie algebra  $\mathfrak{g}$  are the scalars  $C_{ij}^k$  such that

$$[X_i, X_j] = \sum_{k=1}^n C_{ij}^k X_k.$$

Finding a real form of a complex Lie algebra  $\mathfrak{g}$  amounts to finding a basis with real structure constants.

- 8. Any complex semisimple Lie algebra g contains a unique (up to isomorphism) real form t whose Killing form is definite negative (called a **compact real form**). Moreover, there is a one-to-one correspondence between finite-dimensional irreducible representations of g and unitary irreducible representations of the simply connected compact group K whose Lie algebra is the compact real form t.
- 9. Every complex semisimple algebra g has a Cartan subalgebra, that is, a maximal abelian subalgebra h ⊂ g such that ad<sub>H</sub> are simultaneously diagonalizable for all H ∈ h. Therefore we have the root decomposition

$$\mathfrak{g} = \mathfrak{h} \otimes \mathfrak{g}_{\alpha_1} \otimes \ldots \oplus \mathfrak{g}_{\alpha_s},$$

where the covectors  $\alpha_1, \ldots, \alpha_s \in \mathfrak{h}^*$ , called the **roots**, satisfy

$$\operatorname{ad}_H(X) = \alpha_i(H)X$$

for all  $X \in \mathfrak{g}_{\alpha_i}$ . The **rank** of  $\mathfrak{g}$  is  $\operatorname{rank}(\mathfrak{g}) = \dim(\mathfrak{h})$ .

10. A Cartan subalgebra h ⊂ g always contains a real subspace h<sub>ℝ</sub>, with real dimension equal to rank(g), such that the roots can be seen as elements of h<sub>ℝ</sub><sup>\*</sup>. Moreover, the Killing form B restricts to a real inner product on h<sub>ℝ</sub>, an so the linear isomorphism Φ : h<sub>ℝ</sub> → h<sub>ℝ</sub><sup>\*</sup> defined by Φ(X)(Y) = B(X,Y) induces an inner product on h<sub>ℝ</sub><sup>\*</sup>. It turns out that the geometry of the roots relative to this inner product is highly constrained.

- 11. The finite-dimensional simple complex Lie algebras are:
  - (i)  $\mathfrak{sl}(n+1,\mathbb{C})$  for  $n \ge 1$ ;
  - (ii)  $\mathfrak{so}(2n+1,\mathbb{C})$  for  $n \geq 2$ ;
  - (iii)  $\mathfrak{sp}(n,\mathbb{C})$  for  $n \geq 3$  (the symplectic Lie algebras);
  - (iv)  $\mathfrak{so}(2n,\mathbb{C})$  for  $n \geq 4$ ;
  - (v)  $\mathfrak{e}_6$ ,  $\mathfrak{e}_7$ ,  $\mathfrak{e}_8$ ,  $\mathfrak{f}_4$  and  $\mathfrak{g}_2$  (the exceptional Lie algebras).
- 12. There exist  $r = \operatorname{rank}(\mathfrak{g})$  roots  $\alpha_1, \ldots, \alpha_r$ , called **simple roots**, such that any root  $\alpha$  can be written as

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i$$

with all  $n_i \in \mathbb{Z}_0^-$  or with all  $n_i \in \mathbb{Z}_0^+$ .

The fundamental weights associated to a given choice α<sub>1</sub>,..., α<sub>r</sub> of simple roots are the covectors λ<sub>1</sub>,..., λ<sub>r</sub> ∈ β<sup>\*</sup><sub>ℝ</sub> defined by

$$2\frac{\langle\lambda_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} = \delta_{ij}$$

for i, j = 1, ..., r. The weight lattice is  $\Lambda^w = \operatorname{span}_{\mathbb{Z}} \{\lambda_1, ..., \lambda_r\}$ , and the dominant weights are the elements of  $\Lambda^w_+ = \operatorname{span}_{\mathbb{Z}^+} \{\lambda_1, ..., \lambda_r\}$ .

- 14. Highest weight theorem: If  $\pi : \mathfrak{g} \to L(V)$  is an irreducible finite-dimensional representation of the complex semisimple Lie algebra  $\mathfrak{g}$  then:
  - (i) The linear transformations in

$$\pi(\mathfrak{h}) = \{\pi(H) : H \in \mathfrak{h}\}$$

are simultaneously diagonalizable, so that

$$V = \bigoplus_{\lambda \in \Lambda^w} V_\lambda$$

with  $\pi(H)v = \lambda(H)v$  for all  $H \in \mathfrak{h}$  and  $v \in V_{\lambda}$ . Here  $V_{\lambda} = \{0\}$  except for finitely many  $\lambda \in \Lambda^w$ , called the **weights** of the representation.

- (ii) There exists a **highest weight**  $\lambda_{\max} \in \Lambda^w$  such that:
  - 1. dim  $V_{\lambda_{\max}} = 1$ .
  - 2.  $\pi(X_{\alpha})v = 0$  for all  $v \in V_{\lambda_{\max}}$  and  $X_{\alpha} \in \mathfrak{g}_{\alpha}$  with  $\alpha$  a positive root.
  - 3.  $V = \operatorname{span}_{\mathbb{C}} \left\{ \pi(X_{-\alpha_{i_1}}) \circ \cdots \circ \pi(X_{-\alpha_{i_s}}) v_{\lambda_{\max}} : s \in \mathbb{N}, i_1, \dots, i_s \in \{1, \dots, r\} \right\},$ where  $X_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$  and  $v_{\lambda_{\max}} \in V_{\lambda_{\max}} \setminus \{0\}.$
- (iii) All weights of  $\pi$  are of the form  $\lambda_{\max} m_1 \alpha_1 \ldots m_r \alpha_r$  with  $m_1, \ldots, m_r \in \mathbb{Z}_0^+$ .
- (iv) For every dominant weight  $\lambda \in \Lambda^w_+$  there exists a unique (up to equivalence) irreducible finite-dimensional representation of  $\mathfrak{g}$  with highest weight  $\lambda$ .
- 15. If  $G \stackrel{\varphi}{\sim} V$  and  $G \stackrel{\psi}{\sim} W$  are finite-dimensional representations of a group G then their **tensor product** is the representation  $G \stackrel{\varphi \otimes \psi}{\sim} V \otimes W$  determined by

$$(\varphi \otimes \psi)_g(v \otimes w) = \varphi_g(v) \otimes \psi_g(w)$$

for all  $g \in G$ ,  $v \in V$  and  $w \in W$ .

16. If  $\pi : \mathfrak{g} \to L(V)$  and  $\rho : \mathfrak{g} \to L(W)$  are finite-dimensional representations of a Lie algebra  $\mathfrak{g}$  then their **tensor product** is the representation  $\pi \otimes \rho : \mathfrak{g} \to L(V \otimes W)$  determined by

 $(\pi \otimes \rho)(X)(v \otimes w) = \pi(X)(v) \otimes w + v \otimes \rho(X)(w)$ 

for all  $X \in \mathfrak{g}$ ,  $v \in V$  and  $w \in W$ .